Pricing and Hedging in Affine Models with Possibility of Default

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Affine Models with Default

1. Model

2. Pricing

3. Hedging
Outline

1 Model

2 Pricing

3 Hedging
Motivation

Main risk factors for equity derivatives pricing:
- stock returns
- risk of default (of the underlying)
- interest rates
- volatility.

Assets needed for a “perfect” hedge of an equity derivative:
- stock
- corporate bonds (or CDS)
- government bonds
- liquid vanilla options.
Notation

- Conservative affine process \((X_t)_{t \geq 0}\) on \(D = \mathbb{R}_+^m \times \mathbb{R}^n\), \(N = m + n\)
- \(\mathcal{I} = \{1, \ldots, m\}\), \(\mathcal{J} = \{m + 1, \ldots, N\}\)
- Standard Poisson Process \((N_t)_{t \geq 0}\) independent from \((X_t)_{t \geq 0}\)
- \(\mathbb{P}^x[\cdot]\): risk-neutral measure (conditional on \(X_0 = x \in D\))
- \(\mathbb{E}^x[\cdot]\): conditional expectation \(\mathbb{E}[\cdot|X_0 = x]\)
- \(\langle \cdot, \cdot \rangle\): Euclidean scalar product on \(\mathbb{C}^N\), i.e.

\[
\forall x, y \in \mathbb{C}^N : \langle x, y \rangle = \sum_{i=1}^{N} x_i y_i.
\]
Stock, interest rates and default

- Stock price:
  \[ S_t = \exp(s_t + R_t + \Lambda_t)1_{\{t<\tau\}} \]

- Log stock evolution (before interest rates and adjustment for default): 
  \[ s_t = e + \langle \varepsilon, X_t \rangle \]

- Interest rates: 
  \[ r_t = d + \langle \delta, X_t, \mathcal{I} \rangle, \quad (d, \delta) \in \mathbb{R}_+ \times \mathbb{R}_+^m, \]
  \[ R_t = \int_0^t r_s ds \]

- Default intensity (intensity of Poisson jump to default): 
  \[ \lambda_t = c + \langle \gamma, X_t, \mathcal{I} \rangle, \quad (c, \gamma) \in \mathbb{R} \times \mathbb{R}_+^m, \quad \Lambda_t = \int_0^t \lambda_s ds \]

- Default: 
  \[ \tau = \inf\{t > 0 : N_{\Lambda_t} = 1\} \]
Affine Models with Default

Literature

- Reduced form affine models of credit default: Lando (1998)
Regular Affine Processes

Definition

A Markov process \((X_t)_{t \geq 0}\) is regular affine if there exist functions \(\phi(t, u)\) and \(\psi(t, u)\) such that the characteristic function is given by

\[
\mathbb{E}^x [\exp (\langle u, X_t \rangle)] = \exp (\phi(t, u) + \langle \psi(t, u), x \rangle), \quad u \in \mathbb{C}^m \times i\mathbb{R}^n
\]

and for all \(t \geq 0\), a.s. \(X_s \to X_t\) as \(s \to t\).

- A regular affine process can be described by its characteristic octet \((a, \alpha, b, \beta, c, \gamma, \nu, \mu)\).
- Functions \(\phi\) and \(\psi\) can be calculated as solutions of a system of generalized Riccati equations.
Infinitesimal generator: diffusion, drift, jumps

\[
\mathcal{G} f(x) = \sum_{k,l=1}^{N} (a_{kl} + \langle \alpha_{kl}^{\mathcal{I}}, x_{\mathcal{I}} \rangle) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \\
+ \langle b + \beta x, \nabla f(x) \rangle - (c + \langle \gamma, x_{\mathcal{I}} \rangle) \\
+ \int_{D \backslash \{0\}} (f(x + \xi) - f(x) - \langle \nabla \mathcal{J} f(x), \chi_{\mathcal{J}}(\xi) \rangle) \nu(d\xi) \\
+ \sum_{i=1}^{m} \int_{D \backslash \{0\}} (f(x + \xi) - f(x) - \\
- \langle \nabla \mathcal{J} \cup \{i\} f(x), \chi_{\mathcal{J} \cup \{i\}} f(\xi) \rangle) x_i \mu_i(d\xi).
\]
Example: Heston model with stochastic interest rates and jump to default

\[
\begin{align*}
    dX_t^1 &= \kappa_1 (\theta_1 - X_t^1) \, dt + \eta_1 \sqrt{X_t^1} \, dW_t^1 \\
    dX_t^2 &= \kappa_2 (\theta_2 - X_t^2) \, dt + \eta_2 \sqrt{X_t^2} \, dW_t^2 \\
    dX_t^3 &= -\frac{1}{2}X_t^1 \, dt + \sqrt{X_t^1} \, dW_t^3
\end{align*}
\]

with correlation matrix of the Brownian motions

\[
\begin{pmatrix}
    1 & 0 & \rho \\
    0 & 1 & 0 \\
    \rho & 0 & 1
\end{pmatrix}
\]
$s_t, r_t, \lambda_t$

- Interest rates: $r_t = d + \delta_1 X^1_t + \delta_2 X^2_t$
- Default intensity: $\lambda_t = c + \gamma_1 X^1_t + \gamma_2 X^2_t$
- Pure log returns: $s_t = X^3_t$
We want to price European options on $S_T$ with payoff function $\varphi$.

Cases of particular interest include:

- Government bonds $\varphi \equiv 1$,
- Corporate bonds $\varphi(S) = 1_{\{S>0\}}$,
- Call options $\varphi(S) = (S - K)^+$,
- Power payoffs $\varphi(S) = S^p 1_{\{S>0\}}$.  


Generalized discounted moments

We define the \textit{generalized discounted moments}:

\[ h_{t,x}(z) = \mathbb{E}^x \left[ \exp \left( -R_t \right) S_t^z 1_{\{\tau > t\}} \right]. \]

for all \( z \in U_{t,x} \) with

\[ U_{t,x} = \{ z \in \mathbb{C} : h_{t,x}(\text{Re}(z)) < \infty \}. \]

One can show that \( U_{t,x} \) is an open vertical strip, an open vertical half-space or all of \( \mathbb{C} \).
Expansion of state space

\[ \hat{D} = \mathbb{R}_+^{m+2} \times \mathbb{R}^n \cup \{\Delta\} \]

\[ Y_t = \begin{cases} (X_t, R_t, \Lambda_t) & \text{if } t < \tau \\ \Delta & \text{otherwise} \end{cases} \]

\((Y_t)_{t\geq 0}\) is an affine process

\(h_{t,x}(z)\) can be calculated for \(z \in i\mathbb{R}\) using Riccati equations for the characteristic function of \((Y_t)_{t\geq 0}\)
Affine Models with Default Pricing

Riccati equations

\[
\begin{align*}
\partial_t A(t, u, v, w) &= F(B(t, u, v, w), v, w) \\
\partial_t B_I(t, u, v, w) &= G(B(t, u, v, w), v, w) \\
B_J(t, u, v, w) &= \exp(\beta_J^T t) u_J \\
A(0, u, v, w) &= 0, \\
B_I(0, u, v, w) &= u_I,
\end{align*}
\]
Riccati equations (2)

\[
F(u, v, w) = \langle au, u \rangle + \langle b, u \rangle + dv + c(w - 1) \\
+ \int_{D\setminus\{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle u_J, \chi_J(\xi) \rangle \right) \nu(d\xi)
\]

\[
G_i(u, v, w) = \langle \alpha_i u, u \rangle + \sum_{k=1}^{d} \beta_{ki}u_k + \delta_i v + \gamma_i(w - 1) \\
+ \int_{D\setminus\{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle u_{J \cup \{i\}}, \chi_{J \cup \{i\}}(\xi) \rangle \right) \mu_i(d\xi) \quad \text{for } i \in \mathcal{I}.
\]
Affine Models with Default Pricing

Preparations for the main result

- \( V_t := \{ z \in \mathbb{C} : B_i(t, z \varepsilon, z - 1, z) \text{ is finite for all } i \in \mathcal{I} \} \).
- For all \( z \in V_t \):
  \[
  l_{t,x}(z) = \exp(ze + A(t, z \varepsilon, z - 1, z) + \langle B(t, z \varepsilon, z - 1, z), x \rangle)
  \]
- By extension of state space: \( h_{t,x}(iy) = l_{t,x}(iy) \) for all \( y \in \mathbb{R} \)
- \( I_t \): largest interval around 0 contained in \( V_t \cap \mathbb{R} \)
- \( V_t^0 \): connected component of \( V_t \) containing 0
Main result

**Theorem**

For all \((t, x) \in \mathbb{R}_+ \times D\), \(U_{t,x}\) is an open subset of \(\mathbb{C}\) containing \(\{z \in \mathbb{C} : \text{Re}(z) \in I_t\}\) and \(h_{t,x}(z) = l_{t,x}(z)\) for each \(z \in U_{t,x} \cap V_t^0\).

Idea of the proof:

1. Show that \(h_{t,x}\) is an analytic characteristic function (under a different measure)
2. Show that \(l_{t,x}\) is analytic on \(V_t^0\)
Applications of the main result

Main result yields:

- Pricing formulas for power payoffs, corporate and government bonds
- The following corollary:

Corollary

The condition

\[ F(\varepsilon, 0, 1) = 0, \quad G(\varepsilon, 0, 1) = 0 \quad \text{and} \quad \beta \mathcal{J} \mathcal{J} = 0, \quad (1) \]

is sufficient for the discounted stock price \( \exp(s_t + \Lambda_t)1_{\{t<\tau\}} \) to be a martingale with respect to all \( \mathbb{P}^x, \, x \in D \). If all components of \( \varepsilon \mathcal{J} \) are different from 0, then (1) is also necessary.
Call option with log strike $k$:

$$c_{t,x}(k) = \mathbb{E}^x \left[e^{-R_t} (S_t - e^k)^+ \right].$$

Pricing formula: Let $p > 0$ such that $p + 1 \in U_{t,x}$. Then,

$$c_{t,x}(k) = \frac{e^{-pk}}{2\pi} \int_{\mathbb{R}} e^{-iyk} g_c(y) dy = \frac{e^{-pk}}{\pi} \int_0^\infty \text{Re} \left(e^{-iyk} g_c(y)\right) dy,$$

where

$$g_c(y) = \frac{h_{t,x}(p + 1 + iy)}{p^2 + p - y^2 + iy(2p + 1)}.$$
Pricing in Heston model with stochastic interest rates and jump to default

\[ h_{t,x}(z) = \exp(A(t, (0, 0, z), z - 1, z) + \langle B(t, (0, 0, z), z - 1, z), x \rangle) \]
\[ =: \exp \left( \tilde{A}(t, z) + \tilde{B}_1(t, z)x_1 + \tilde{B}_2(t, z)x_2 + zx_3 \right), \]

where

\[
\begin{cases}
\partial_t \tilde{A}(t, z) = \kappa_1 \theta_1 \tilde{B}_1(t, z) + \kappa_2 \theta_2 \tilde{B}_2(t, z) + (c + d)(z - 1) \\
\partial_t \tilde{B}_1(t, z) = \frac{1}{2} \eta_1^2 \tilde{B}_1^2(t, z) + (\rho \eta_1 z - \kappa_1) \tilde{B}_1(t, z) \\
+ (\frac{1}{2} z + \gamma_1 + \delta_1)(z - 1) \\
\partial_t \tilde{B}_2(t, z) = \frac{1}{2} \eta_2^2 \tilde{B}_2^2(t, z) - \kappa_2 \tilde{B}_2(t, z) + (\gamma_2 + \delta_2)(z - 1) \\
\tilde{A}(0, x) = \tilde{B}_1(0, z) = \tilde{B}_2(0, z) = 0.
\end{cases}
\]
Implied volatility surface: with and without default
Pricing of European options with arbitrary payoff $\varphi$

Integrability condition

$$L_{t,x} = \left\{ \varphi : \mathbb{R}_+ \to \mathbb{R} | \mathbb{E}_x^{\varphi} \exp(-R_t) | \varphi(S_t)| < \infty \right\}.$$ 

Procedure:

1. Let $\varphi \in L_{t,x}$.
2. Take a set $\mathcal{K}$ of strikes of European calls.
3. Take a set $\mathcal{P}$ of powers of power payoffs in $L_{t,x}$.
4. Use regression weighted by the heuristic density of $S_T$ in order to find the best approximation. For better numerical performance use Gram-Schmidt in order to orthogonalize the power payoffs.
Application: truncated log payoff

- Payoff: $\varphi(S) = \log(S) \vee k$.
- Example: $S = 1, k = -1$.
- Approximating assets:
  1. call options with strikes $\mathcal{K} = \{0.02, 0.04, \ldots, 2\}$ ($\mathcal{P} = \emptyset$)
  2. power payoffs of powers $\mathcal{P} = \{0, 0.05, \ldots, 4.95\}$ ($\mathcal{K} = \emptyset$),
     where $p = 0$ is a government bond
  3. using $\mathcal{K} = \{0.02, 0.06, \ldots, 1.98\}$ and
     $\mathcal{P} = \{0, 0.1, \ldots, 4.9\}$, where $p = 0$ is a government bond
- Heuristic density for $S_T$:

  $\rho(S) = \begin{cases} 
  \exp(-10S) & S < 0.5 \\
  \exp(-10|S - 1|) & 0.5 \leq S \leq 1.5 \\
  \exp(-5) & S > 1.5.
\end{cases}$
Comparison of different approximation methods

Approximation using power payoffs only

Approximation using calls only

Approximation using both

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Hedging of European Options

- Number of assets must match the number of sources of risk and the sources of risk must be “hedgeable”
- Jumps can only have discrete size:
  - \( \nu = \sum_{k=1}^{M} r_k \delta x_k \) for some \( r_j \in \mathbb{R} \setminus \{0\} \) and \( x_j \in D \setminus \{0\} \)
  - \( \mu_i = \sum_{j=1}^{M_i} r_{jk} \delta x_{jk} \) for all \( 1 \leq i \leq m \) with \( r_{ik} \in \mathbb{R} \setminus \{0\} \) and \( x_{ik} \in D \setminus \{0\} \).
- \( L = N + M + \sum_{i=1}^{m} M_i + 1 \) instruments needed
- Portfolio of basic instruments (European options):
  \( \mathcal{B} = \{ \varphi_1, \ldots, \varphi_L \} \)
Hedging parameters

- Greeks: For $i = 1, \ldots, N$:
  \[ G_{t,x}^i = \frac{\partial}{\partial x_i} \mathbb{E}^x [\exp(-R_t) \varphi(S_T)] \]

- Sensitivity to jumps (described by jump measure $\nu$): For all $k \in \{1, \ldots, M\}$:
  \[ H_{t,x}^k = \mathbb{E}^{x+x_k} [\exp(-R_t) \varphi(S_t)] - \mathbb{E}^x [\exp(-R_t) \varphi(S_t)] \]

- Sensitivity to jumps (described by jump measures $\mu_i$): For all $i \in \mathcal{I}, k \in \{1, \ldots, M^i\}$:
  \[ H_{t,x}^{ik} = \mathbb{E}^{x+x_{ik}} [\exp(-R_t) \varphi(S_t)] - \mathbb{E}^x [\exp(-R_t) \varphi(S_t)] \]

- Impact of jump to default:
  \[ D_{t,x} = \mathbb{E}^x [\exp(-R_t) \varphi(0)] - \mathbb{E}^x [\exp(-R_t) \varphi(S_t)]. \]
Hedging ratios $\xi$:

$$
G_{t-s,x}^l = \sum_{l=1}^{L} \varphi^l(t - s, x) G_{t_l-s,x}^l,
$$

$$
H_{t-s,x}^{11} = \sum_{l=1}^{L} \varphi^l(t - s, x) H_{t_l,x}^{l,11},
$$

$$
G_{t-s,x}^N = \sum_{l=1}^{L} \varphi^l(t - s, x) G_{t_l-s,x}^d,
$$

$$
H_{t-s,x}^{1M_1} = \sum_{l=1}^{L} \varphi^l(t - s, x) H_{t_l,x}^{l,1M_1},
$$

$$
H_{t-s,x}^{N_1} = \sum_{l=1}^{L} \varphi^l(t - s, x) H_{t_l,x}^{1,1},
$$

$$
H_{t-s,x}^{N_1} = \sum_{l=1}^{L} \varphi^l(t - s, x) H_{t_l,x}^{1,1},
$$

$$
H_{t-s,x}^{N_1} = \sum_{l=1}^{L} \varphi^l(t - s, x) H_{t_l,x}^{1,1},
$$

$$
D_{t-s,x} = \sum_{l=1}^{L} \varphi^l(t - s, x) D_{t_l,x},
$$
Hedging of European options

- All European options can be hedged iff the system of linear equations defined on the previous chart has a unique solution for all $\varphi \in L_{t,x}$.

- In the Heston model with stochastic interest rates we can hedge all European options if we trade in the stock, a government bond, a corporate bond (same company) and a liquid vanilla option and the following technical condition is fulfilled for all $0 \leq s \leq t$:

$$e^{x_3}P^0_{s,x}P_{s,x} \{-\partial_{x_1} c_{s,x}(k)\tilde{B}^0_2(s,0) + \partial_{x_2} c_{s,x}(k)\tilde{B}^0_1(s,0)$$

$$+ [\tilde{B}^0_1(s,0)\tilde{B}_2(s,0) - \tilde{B}^0_2(s,0)\tilde{B}_1(s,0)] \times$$

$$[\partial_{x_3} c_{s,x}(k) - c_{s,x}(k)] \} \neq 0$$

$(\tilde{B}^0$ is $\tilde{B}$ in the case with no default)$
Summary

- General affine model for equity derivatives that incorporates stochastic volatility, stochastic interest rates and jump to default
- Notion of discounted moments for cases when $\log S_T = -\infty$ with positive probability
- Pricing in semi-closed form for most common European equity derivatives; otherwise: approximation
- Under additional assumptions: hedging
Thank you!

THE END