Concavity of the Consumption Function with Recursive Preferences

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Abstract

Carroll and Kimball (1996) show that the consumption function for an agent with time-separable, isoelastic preferences is concave in the presence of income uncertainty. In this paper I show that concavity breaks down if we abandon time-separability. Namely, if an agent maximizing an isoelastic recursive utility has preferences for early resolution of uncertainty, there always exists a distribution of income risk such that consumption function is not concave in wealth. I also derive sufficient conditions guaranteeing that the consumption function is concave if the agent has preferences for late resolution of uncertainty.

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1 Introduction

The theory of the consumption function (defined as the map between current wealth and current consumption) is one of the cornerstones of modern macroeconomics. Individual agents’ consumption and saving decisions aggregate in the economy and influence business cycle fluctuations and growth. Therefore, it is important to understand the origins of these decisions.

One of the useful and intuitive properties of the consumption function is known as the “decreasing marginal propensity to consume,” which refers to the fact that marginal propensity to consume (MPC) out of wealth declines with the level of wealth or transitory income. In mathematical terms, this means that the consumption function is concave in wealth. Important implications of this property of concavity have been discussed in the literature at least since Keynes (1935). For example, concavity generates persistence in the consumption growth, and implies that the cross-sectional distribution of wealth is an important variable for the aggregate dynamics.\(^1\) Carroll (1997) shows that the concavity of the consumption function is crucial for the theory of “buffer-stock” target saving behavior. Kimball (1990) shows that the concavity of the consumption function implies that the effective risk aversion of an agent’s value function is higher than the risk aversion of the utility function. As a result, financial risk-taking decisions become wealth-dependent. More generally, the behaviour of MPC is intimately linked to the theory of precautionary savings. See, for example, Kimball (1990), Carroll and Samwick (1997), Huggett and Ospina (2001), Huggett (2004), Parker and Preston (2005), and Carroll (2009). There is also some empirical evidence (Browning and Lusardi, 1996; Souleles, 1999; Souleles, Parker, and Johnson, 2006) suggesting that the MPC is indeed decreasing in the level of wealth.

Zeldes (1989) was the first to show (through numerical experiments) that uninsurable income shocks make the consumption function concave.\(^2\) In an important paper, Carroll and Kimball (1996) rigorously proved that the consumption function is indeed concave, assuming (i) time-separable Hyperbolic Absolute Risk Aversion (HARA) preferences and (ii) a positive precautionary saving motive. In particular, their result covers the most important case of isoelastic (Constant Relative Risk Aversion, CRRA) utility function.

While the time-separable CRRA utility specification remains an important element of many macro-models, it has a major drawback, in that Elasticity of Intertemporal Substitution (EIS) is given by the reciprocal of risk aversion. This is quite inconvenient, especially because the sensitivity of the rate of consumption growth to changes in real rate is given by EIS, and it is not clear why it ought to be directly linked to risk preferences. The creation of the recursive utility theory by Kreps and Porteus (1978), Epstein and Zin (1989, 1991), and Duffie and Epstein (1992) made it possible to separate the effects of EIS and risk aversion. Since then, the Constant Elasticity of

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\(^1\) See, e.g., Blundell and Preston (1998) and Carroll, Slakalek and Tokuoka (2013).

\(^2\) See also Caballero (1991) and Aiyagari (1994).
Intertemporal Substitution - Constant Relative Risk Aversion (CEIS-CRRA) utility specification (often referred to as the isoelastic recursive utility, or the Epstein-Zin preferences) has become a major workhorse model in the macro-finance literature. See, for example, Weil (1989, 1993), Epstein and Zin (1991), Bansal and Yaron (2004), Hansen, Heaton, Roussanov, and Lee (2007), Hansen, Heaton and Li (2008), and Hansen and Scheinkman (2012). Given the role the CEIS-CRRA preferences specification plays in economics, it is important to understand the optimal saving and consumption decisions they imply. This is the goal of the present paper.

My main finding is that the concavity of the consumption function depends crucially on the preferences for timing the resolution of uncertainty. Let us denote by $\psi$ and $\gamma$ the EIS and risk aversion of the CEIS-CRRA agent, respectively. First, considering a two-period model, I show that when the agent prefers late resolution of uncertainty (i.e., when $\psi^{-1} > \gamma$), concavity of the consumption function is preserved. By contrast, if the agent prefers early resolution of uncertainty, there always exists a distribution of uninsurable income shocks for which the consumption function is convex. Then, I extend the concavity result to dynamic settings and show that concavity is preserved if $\psi^{-1} > \gamma$ and $\psi^{-1} \geq 2$ and $\gamma > \frac{\psi^{-1} - 2}{3}$.

The intuition for my result is as follows. An agent prefers early resolution of uncertainty when the risk aversion effect dominates the inter-temporal substitution effect. If the agent anticipates large uncertainty about income tomorrow, he may find it optimal to reduce today’s consumption more than proportionally with respect to his current wealth and to increase the buffer stock of savings against future shocks. As his wealth becomes sufficiently large, the MPC converges to one. As a result, MPC becomes an increasing function of wealth, which is below one for low wealth levels. By contrast, when the substitution effect dominates the risk aversion effect, the agent will worry more about today’s consumption than about tomorrow’s consumption as long as he is sufficiently constrained, implying that MPC is decreasing. The proof of the result is non-trivial and is based on subtle properties of the derivatives of the value function.

2 Setup

Time is discrete, $t = 0, \cdots, T$. Uncertainty is described by a filtration $\mathcal{F}_t$, $t = 0, \cdots, T$ satisfying standard conditions. An economic agent faces uninsurable income shocks given by a bounded stochastic process $w_t$, adapted to $\mathcal{F}_t$. In or-

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3Weil (1993) studies an interesting class of recursive preferences, the constant EIS-constant absolute risk aversion utility. He shows that, under this specification, the consumption function is linear in wealth.

4While inequality $\psi^{-1} > \gamma$ is necessary for the validity of the result, inequality $\gamma > \frac{\psi^{-1} - 2}{3}$ is a technical condition needed for the proof. It can probably be relaxed, but I have not been able to prove this.
der to reallocate wealth across time periods, the agent can use risk-free bonds that pay a risk-free interest rate \( r_t \) at time \( t \). The process \( r_t \) is predictable with respect to the filtration \( \mathcal{F}_t \). The agent maximizes a recursive functional \( J_t \) defined on the set of positive and bounded consumption stream \( c_t, t = 0, \ldots, T \), subject to the inter-temporal budget constraint

\[
c_t = x_{t-1}e^{r_t} + w_t - x_t
\]

where \( x_t \) is the (adapted) process of savings, which can take arbitrary values, \( x_t \in \mathbb{R} \). We assume that the utility functional \( J_t \) is the CEIS-CRRA recursive utility defined via

\[
J_t = \left( c_t^{1-\lambda} + e^{-\rho \left( E_t[J_{t+1}^{1-\gamma}] \right)^{1-\lambda}} \right)^{1/(1-\lambda)}, \quad J_{T+1} = 0.
\]

Here, \( \gamma \) is the coefficient of relative risk aversion, \( \psi = \frac{1}{\lambda} \) is the elasticity of intertemporal substitution, and \( \rho \) is the time discount rate. The case \( \gamma > \psi^{-1} \) (respectively, \( \gamma < \psi^{-1} \)) corresponds to preferences for early (respectively, late) resolution of uncertainty. See Kreps-Porteus (1978) and Epstein-Zin (1989).

We can then define the value function as

\[
V_\tau(y) \equiv \max_{c_t,x_t,t \geq \tau} J_\tau((c_t)_{t \geq \tau})
\]

assuming that the income process of the agent is given by \( \tilde{w}_t, t > \tau \), where we have defined the process \( \tilde{w}_t \) via \( \tilde{w}_t = w_t, t > \tau \) and \( \tilde{w}_\tau = y \).

### 3 Main results

The next result shows that, for the case \( T = 1 \), preferences for late resolution of uncertainty are both necessary and sufficient for the concavity of the consumption function.

**Theorem 3.1** Let \( T = 1 \). Then the consumption function \( c_0 \) is a concave function of wealth if

\[
\lambda \geq \gamma.
\]

Conversely, if \( \lambda < \gamma \), there exists an income shock \( w_1 \) and an open set \( \Omega \subset \mathbb{R} \) such that \( c_0(w_0) \) is strictly convex for \( w_0 \in \Omega \).

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5This functional is unique, up to an ordinarily equivalent transformation. See, Kreps-Porteus (1978) and Epstein-Zin (1989). The particular form of normalization chosen in my paper is used purely for technical convenience.

6While the result only claims that the consumption function is convex on some set, extensive numerical experiments indicate that, in many cases, the consumption function is actually strictly convex everywhere.
I now discuss the proof of Theorem 3.1. To illustrate the new effects due to time-nonseparability, note that we can rewrite the maximization problem as
\[
\max_{x_0} \left( (w_0 - x_0)^{1-\lambda} + e^{-\rho} \left( E[(w_1 + e^r x_0)^{1-\gamma}] \right)^{1/(1-\gamma)} \right)
\]
where \( r \equiv r_1 \). Differentiating, we get the first order condition
\[
g(x_0) \equiv E[(w_1 + e^r x_0)^{1-\gamma}]^{\beta} E[(w_1 + e^r x_0)^{-\gamma}]^{-\frac{1}{\beta}} = e^{(r-\rho)/\lambda} c_0, \tag{3.1}
\]
where we have defined
\[
\beta = \frac{\lambda - \gamma}{1 - \gamma}.
\]
As I show in the Appendix, \( g(x) \) is monotone increasing; hence, \( (3.1) \) is equivalent to \( x_0 = g^{-1}(e^{(r-\rho)/\lambda} c_0) \). The budget constraint
\[
c_0 + g^{-1}(e^{(r-\rho)/\lambda} c_0) = w_0
\]
uniquely determines the consumption function \( c_0 = c_0(w_0) \). Thus, \( c_0 \) is concave if and only if \( g^{-1} \) is convex, which is in turn equivalent to \( g \) being concave. Therefore, Theorem 3.1 is a direct consequence of the following lemma.

**Lemma 3.2** The function \( g(x) \) is monotone increasing. It is concave if \( \lambda \geq \gamma \). However, if \( \gamma > \lambda \), there exists a random variable \( w_1 \) such that \( g(x) \) is not concave for \( x \) in some open set of \( \mathbb{R} \).

It is important to note that the convexity result of Theorem 3.1 immediately extends to the case \( T > 1 \). Indeed, assuming that the income process is zero for \( T > 1 \) and using homogeneity of isoelastic preferences, we get that the time-zero consumption function \( c_0 \) coincides with that for the car \( T = 1 \) but with a different discount factor \( e^{-\rho} \).

We now turn to the dynamic version of the problem. In this case, we can write down the dynamic programming equation as
\[
V_t(y) = \max_x \left( (y - x)^{1-\lambda} + e^{-\rho} \left( E_t[(V_{t+1}(e^{r_{t+1}} x + w_{t+1})^{1-\gamma}] \right)^{1/(1-\lambda)} \right) \tag{3.2}
\]
Standard results about envelopes of concave functions, combined with the fact that the function \( (x_1^{1-\lambda} + x_2^{1-\lambda})^{1/(1-\lambda)} \) is jointly concave in \( (x_1, x_2) \) imply that the following is true.

**Lemma 3.3** The function \( V_t(y) \) is increasing and concave for any \( t \geq 0 \).

Using the envelope theorem, we and differentiating (3.2), we get
\[
\left( (1 - \lambda)^{-1} V_t^{1-\lambda}(y) \right)' = c_t(y)^{-\lambda}. \tag{3.3}
\]
Differentiating (3.3), we arrive at the following result.
Lemma 3.4 The consumption function $c_t(y)$ is concave if and only if

\[ (V_t(y)^{1-\lambda})'' (V_t(y)^{1-\lambda})' \geq \frac{\lambda + 1}{\lambda} \left( (V_t(y)^{1-\lambda})'' \right)^2. \]  

Lemma 3.4 is a direct analog of Lemma 3 in Carroll and Kimball (1996). Thus, proving concavity reduces to proving that the value function $V_t$ satisfies (3.4). The proof of this result in Carroll and Kimball (1996) is based on an important inter-temporal aggregation property. Namely, they show that if the value function $V_{t+1}$ satisfies condition (3.4) and the von Neumann-Morgenstern utility function satisfies the same condition, then so does $V_t$.\(^7\) In the time-separable case, Carroll and Kimball present a short and elegant argument to prove this important result. Namely, their key observation is that if a function $V(x)$ satisfies $V''V' \geq k(V'')^2$ then so does the function $E[V(x + w)]$ for any random variable $w$. However, to apply the argument of Carroll and Kimball in the time-nonseparable case, one would need to show that the function $E[V(x + w)]^{\frac{1}{1-\lambda}}$ inherits the property $V''V' \geq k(V'')^2$. I have not been able to prove this result and, in fact, I conjecture that it is not true in general. While it may still be true that the aggregation result does hold in some more general form, I have only been able to prove it under additional conditions. The following is true.

Lemma 3.5 Suppose that the agent has preferences for late resolution of uncertainty, i.e. $\lambda > \gamma$. If $V_{t+1}(y)$ satisfies (3.4) and we have that

- either $\lambda > 2$ and $\gamma \geq \frac{\lambda - 2}{3}$,
- or $\lambda \leq 2$ and the income shock $w_t$ is independent of $F_{t-1}$ for each $t \geq 1$.

Then, $V_t(y)$ also satisfies (3.4).

Lemmas 3.4 and 3.5 immediately yield the following result.

Proposition 3.6 Suppose that the agent has preferences for late resolution of uncertainty and that

1. either $\lambda > 2$ and $\gamma \geq \frac{\lambda - 2}{3}$,
2. or $\lambda \leq 2$ and the income shock $w_t$ is independent of $F_{t-1}$ for each $t \geq 1$.

Then the consumption function is concave.

In conjecture that, in general, the assumption of uncorrelated shocks in item (2) of Proposition 3.6 cannot be relaxed. The reason is that with correlated shocks the value function depends on $w_t$ not only through the current wealth, but also through the impact of $w_t$ on the distribution of future shocks. This introduces additional non-linear dependence into the value function and

\(^7\)See Lemma 2 in Carroll and Kimball (1996).
distorts the consumption/saving behaviour. Note however that item (2) al-

ows the distribution of $w_t$ to change over time in an arbitrary fashion. In

particular, it covers the standard life-cycle income specification with mean $w_t$

exhibiting a hump-shaped dependence on the “age” $t$.

It is also interesting to know whether Proposition 3.6 covers situations that

are considered to be empirically relevant. In the finance literature, the most

commonly used specification is $\psi \approx 1.5$ and $\gamma \geq 5$, used in Bansal and Yaron

(2004). In the macroeconomics literature, there is less consensus on what the

“true” value of EIS is. For example, Hall (1988) finds an estimate of $\psi$ that

is very low (less than 0.2), while Ogaki and Reinhart (1998) find an estimate

of EIS that is slightly above one. Given that the risk aversion parameter $\gamma$

is commonly assumed to be above one, only Hall’s estimate would imply

preferences for early resolution of uncertainty.

4 References


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A Proofs

Proof of Lemma 3.2. We have

\[
g'(x) = \frac{\beta}{\lambda} E[(w_1 + x)^{1-\gamma}]^{\frac{\gamma}{\lambda} - 1} (1 - \gamma) E[(w_1 + x)^{-\gamma}]^{1 - \frac{1}{\lambda}} \\
+ E[(w_1 + x)^{1-\gamma}]^{\frac{\gamma}{\lambda} - 1} E[(w_1 + x)^{-\gamma}]^{1 - \frac{1}{\lambda}} E[(w_1 + x)^{-\gamma - 1}] \\
+ \frac{\gamma}{\lambda} E[(w_1 + x)^{1-\gamma}]^{\frac{\gamma}{\lambda} - 1} E[(w_1 + x)^{-\gamma}]^{1 - \frac{1}{\lambda}} \\
\times \left( E[(w_1 + x)^{-\gamma - 1}] E[(w_1 + x)^{1-\gamma}] - (E[(w_1 + x)^{-\gamma}])^2 \right) \]  

(A.1)

By the Cauchy-Schwarz inequality,

\[ E[(w_1 + x)^{-\gamma - 1}] E[(w_1 + x)^{1-\gamma}] \geq (E[(w_1 + x)^{-\gamma}])^2, \]

and hence \( g'(x) \geq 0 \). Denote

\[ f(\alpha) \equiv E[(w_1 + x)^{-\alpha}]. \]
Then, differentiating (A.1), we get
\[ g''(x) = \frac{\gamma \beta}{\lambda^2} f(\gamma - 1) \frac{1}{x^{\frac{1}{x^2}}} (1 - \gamma) f(\gamma) f(\gamma)^{-\frac{1}{x^2}} f(\gamma + 1) \]
\[ - \frac{\gamma}{\lambda} (\gamma - 1) f(\gamma - 1) \frac{2}{x^{\frac{1}{x^2}}} f(\gamma)^{-\frac{1}{x^2}} (-\gamma) f(\gamma + 1)^2 \]
\[ - \frac{\gamma}{\lambda} (\gamma + 1) f(\gamma - 1) \frac{2}{x^{\frac{1}{x^2}}} f(\gamma)^{-\frac{1}{x^2}} (-\gamma) f(\gamma + 2) \]
\[ + \left( 1 - \frac{\gamma}{\lambda} \right) \left( \beta - 1 \right) f(\gamma - 1) \frac{2}{x^{\frac{1}{x^2}}} (1 - \gamma) f(\gamma) f(\gamma)^{1 - \frac{1}{x^2}} \]
\[ + \left( 1 - \frac{\gamma}{\lambda} \right) \left( 1 - \frac{1}{\lambda} \right) f(\gamma - 1) \frac{2}{x^{\frac{1}{x^2}}} f(\gamma)^{-\frac{1}{x^2}} (-\gamma) f(\gamma + 1) \]
\[ = \frac{1}{\lambda^2} f(\gamma - 1) \frac{2}{x^{\frac{1}{x^2}}} f(\gamma)^{-\frac{1}{x^2}} \]
\[ \times \left[ \gamma(\lambda - \gamma) f(\gamma - 1) f(\gamma)^2 f(\gamma + 1) + \gamma^2(\lambda + 1) f(\gamma - 1)^2 f(\gamma + 1)^2 \right. \]
\[ - \gamma \lambda (\gamma + 1) f(\gamma - 1)^2 f(\gamma) f(\gamma + 2) + (\lambda - \gamma)(\beta - \lambda)(1 - \gamma) f(\gamma)^4 \]
\[ - \gamma(\lambda - \gamma)(\lambda - 1) f(\gamma - 1) f(\gamma)^2 f(\gamma + 1). \]
\[ = \frac{1}{\lambda^2} f(\gamma - 1) \frac{2}{x^{\frac{1}{x^2}}} f(\gamma)^{-\frac{1}{x^2}} \]
\[ \times \left[ \gamma^2 (\lambda + 1) f(\gamma - 1) f(\gamma)^2 (f(\gamma + 2) - f(\gamma + 1)^2) + \gamma (\lambda - \gamma) \times \right. \]
\[ \left[ (f(\gamma - 1)^2 f(\gamma) f(\gamma + 2) + f(\gamma)^4 - 2 f(\gamma - 1) f(\gamma)^2 f(\gamma + 1)) \right. \]
\[ + \lambda f(\gamma)^2 (f(\gamma - 1) f(\gamma + 1) - f(\gamma)^2) \]
By the Cauchy-Schwarz inequality,
\[ f(\gamma) f(\gamma + 2) \geq f(\gamma + 1)^2, \quad f(\gamma - 1) f(\gamma + 1) \geq f(\gamma)^2 \]
and therefore
\[ f(\gamma - 1)^2 f(\gamma) f(\gamma + 2) + f(\gamma)^4 - 2 f(\gamma - 1) f(\gamma)^2 f(\gamma + 1) \]
\[ \geq f(\gamma - 1)^2 f(\gamma + 1)^2 + f(\gamma)^4 - 2 f(\gamma - 1) f(\gamma)^2 f(\gamma + 1) \quad (A.3) \]
\[ = (f(\gamma - 1) f(\gamma + 1) - f(\gamma)^2)^2 \geq 0, \]
and the required concavity follows.

To prove the last statement, suppose that \( w_1 \) is such that \((w_1 + x)^{-1/N}\) is uniformly distributed on \([0, 2]\). Then, \( f(\alpha) = 0.5 \int_0^2 z^N \alpha dz = \frac{1}{N\alpha} 2^{N\alpha-1} \) When \( N \) is sufficiently large, \( f(\gamma + 2)/f(\gamma + 1) \) and \( f(\gamma + 1)/f(\gamma) \) can also be made arbitrarily large. Thus, choosing a large \( N \), we can make the terms in (A.2), containing \( f(\gamma + 2) \) dominate all other terms. If \( \gamma > \lambda \), this will imply that \( g(x) \) is locally convex. \( \square \)
Proof of Proposition 3.4. Let $\tilde{V} = V^{1-\lambda}$. Then,

$$\tilde{V}_t'' = -\lambda c_t^{-\lambda-1} c_t'$$

and

$$\tilde{V}_t''' = \lambda (\lambda + 1) c_t^{-\lambda-2} (c_t')^2 - \lambda c_t^{-\lambda-1} c_t''.$$ 

Hence,

$$\lambda \tilde{V}_t''' \tilde{V}_t'' - (\lambda + 1) (\tilde{V}_t'')^2 = -\lambda^2 c_t^{-\lambda-1} c_t''$$

and the claim follows. □

Proof of Lemma 3.5. For simplicity, I omit the index $t+1$ and denote $V = V(x, w_{t+1}) \equiv V_{t+1}(x)$. I will also assume for simplicity that $\rho = r = 0$. All the calculations can be easily modified for the case of non-zero rates. I will also assume without loss of generality that $t = 0$.

The agent is facing the problem.

$$\max_x \left\{ (y - x)^{1-\lambda} + (E[V(x + w_1, w_1)^{1-\gamma}])^{\frac{1-\lambda}{1-\gamma}} \right\}$$

and the first order condition is

$$E[V^{1-\gamma}]^{-\beta} E[V^{-\gamma} V'(x)] = c_t^{\lambda},$$

or, equivalently,

$$g(x) = c_t$$

with

$$g(x) \equiv E[V^{1-\gamma}]^\frac{\beta}{2} E[V^{-\gamma} V']^{-\frac{1}{2}},$$

where $V'$ denote the derivative with respect to wealth. The same argument as in the proof of Theorem 3.1 implies that we need to show that $g(x)$ is concave. We have

$$g'(x) = \frac{\gamma}{\lambda} E[V^{1-\gamma}]^\frac{\beta}{2} E[V^{-\gamma} V']^{-1-\frac{1}{2}} E[V^{-\gamma-1} (V')^2]$$

$$+ \frac{\lambda - \gamma}{\lambda} E[V^{1-\gamma}]^\frac{\beta}{2-1} E[V^{-\gamma} V']^{-\frac{1}{2}+1}$$

$$- \frac{1}{\lambda} E[V^{1-\gamma}]^\frac{\beta}{2} E[V^{-\gamma} V']^{-\frac{1}{2}-1} E[V^{-\gamma} V''],$$

(A.4)
and therefore

\[ g''(x) = \frac{\gamma \beta (1 - \gamma)}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\gamma - 1} E[V^{-\gamma} V']^{-\frac{1}{2}} E[V^{-\gamma - 1} (V')^2] \\
+ \frac{\gamma}{\lambda} \left( -\frac{1}{\lambda} - 1 \right) E[V^{1-\gamma}]^\frac{2}{\gamma} E[V^{-\gamma} V']^{-\frac{1}{2}-2} E[-\gamma V^{-\gamma - 1} (V')^2 + V^{-\gamma} V''] E[V^{-\gamma - 1} (V')^2] \\
+ \frac{\gamma}{\lambda} E[V^{1-\gamma}]^\frac{2}{\gamma} E[V^{-\gamma} V']^{-\frac{1}{2}-1} E[-(\gamma + 1) V^{-\gamma - 2} (V')^2 + 2 V^{-\gamma - 1} V' V''] \\
+ \frac{\lambda - \gamma}{\lambda} \frac{\beta - \lambda}{\lambda} E[V^{1-\gamma}]^\frac{2}{\gamma - 2} (1 - \gamma) E[V^{-\gamma} V']^{-\frac{1}{2}+2} \\
+ \frac{\lambda - \gamma}{\lambda} \left( 1 - \frac{1}{\lambda} \right) E[V^{1-\gamma}]^\frac{2}{\gamma - 1} E[V^{-\gamma} V']^{-\frac{1}{2}} E[-\gamma V^{-\gamma - 1} (V')^2 + V^{-\gamma} V''] \\
- \frac{\beta}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\gamma - 2} (1 - \gamma) E[V^{-\gamma} V']^{-\frac{1}{2}} E[V^{-\gamma} V''] \\
+ \frac{1}{\lambda} \left( \frac{1}{\lambda} + 1 \right) E[V^{1-\gamma}]^\frac{2}{\gamma} E[V^{-\gamma} V']^{-\frac{1}{2}-2} E[-\gamma V^{-\gamma - 1} (V')^2 + V^{-\gamma} V''] E[V^{-\gamma} V''] \\
- \frac{1}{\lambda} E[V^{1-\gamma}]^\frac{2}{\gamma} E[V^{-\gamma} V']^{-\frac{1}{2}-1} E[-\gamma V^{-\gamma - 1} V' V'' + V^{-\gamma} V''] \\
= -\frac{1}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\gamma - 2} E[V^{-\gamma} V']^{-\frac{1}{2}-2} \\
\times \left[ \gamma^2 (\lambda + 1) E[V^{1-\gamma}] (E[V^{-\gamma} V'] E[V^{-\gamma - 2} (V')^3] - E[V^{-\gamma - 1} (V')^2]^2) + \gamma (\lambda - \gamma) \times \\
\left[ (E[V^{1-\gamma}]^2 E[V^{-\gamma} V'] E[V^{-\gamma - 2} (V')^3] + E[V^{-\gamma} V']^4 - 2 E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma - 1} (V')^2]) \\
+ \lambda E[V^{-\gamma} V']^2 (E[V^{1-\gamma}] E[V^{-\gamma - 1} (V')^2] - E[V^{-\gamma} V']^2) \right] \right] \\
- \frac{\gamma (1 + \lambda)}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\gamma} E[V^{-\gamma} V']^{-\frac{1}{2}-2} E[V^{-\gamma V}'] E[V^{-\gamma - 1} (V')^2] \\
+ 2 \frac{\gamma}{\lambda} E[V^{1-\gamma}]^\frac{2}{\gamma} E[V^{-\gamma} V']^{-\frac{1}{2}-1} E[V^{-\gamma - 1} V' V''] \\
+ \frac{(\lambda - \gamma) (\lambda - 1)}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\gamma - 1} E[V^{-\gamma V'}]^{-\frac{1}{2}} E[V^{-\gamma V}'] \\
- \frac{\beta (1 - \gamma)}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\gamma - 1} E[V^{-\gamma V'}]^{-\frac{1}{2}} E[V^{-\gamma V}'] \\
- \frac{(1 + \lambda) \gamma}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\gamma} E[V^{-\gamma V'}]^{-\frac{1}{2}-2} E[V^{-\gamma - 1} (V')^2] E[V^{-\gamma V}'] \right] \\
(A.5) \]
of Theorem 3.1. By direct calculation, (3.4) is equivalent to

\[ \lambda V'' \geq (\lambda + 1) \left( (V'')^2 (V')^{-1} + \lambda (\lambda - 2) V' V'' V^{-1} \right) \]  

where

\[ K = -\frac{1}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\lambda^2} E[V^{-\gamma} V']^{-\frac{1}{\lambda^2}} E[V^{-\gamma} V'']^2 \]

\[ -\frac{\gamma^2 (\lambda + 1)}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\lambda^2} E[V^{-\gamma} V']^{-\frac{1}{\lambda^2}} E[V^{-\gamma - 1} (V')^3] \]

\[ - E[V^{-\gamma - 1} (V')^2] + \frac{1}{\lambda^2} E[V^{1-\gamma}]^\frac{2}{\lambda^2} E[V^{-\gamma} V']^{-\frac{1}{\lambda^2}} \]

\[ \times \left( -2\gamma (1 + \lambda) E[V^{1-\gamma}] E[V^{-\gamma - 1} (V')^2] E[V^{-\gamma} V''] \right) \]

\[ + 3\gamma \lambda E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma - 1} V' V''] \]

\[ + (\lambda - \gamma)(\lambda - 2) E[V^{-\gamma} V']^2 E[V^{-\gamma} V''] \]

\[ + (1 + \lambda) E[V^{1-\gamma}] E[V^{-\gamma} V'']^2 \]

\[ - \lambda E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma} V'''] \right), \quad (A.6) \]
Using this inequality, we get
\[
\left( E[V^{1-\gamma}]^\frac{2}{\lambda^2} E[V^{-\gamma} V']^{-\frac{1}{\lambda^2}} \right)^{-1} (g''(x) - K) \\
\leq -\frac{\gamma^2}{\lambda^2} \left( E[V^{-\gamma} V''] E[V^{-\gamma-2} (V')^3] - E[V^{-\gamma-1} (V')^2] \right) \\
+ \left( - 2 \frac{\gamma}{\lambda^2} (1 + \lambda) E[V^{1-\gamma}] E[V^{-\gamma-1} (V')^2] E[V^{-\gamma} V''] \\
+ 3 \frac{\lambda - \gamma}{\lambda^2} E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma-1} V' V''] \\
+ \frac{1 + \lambda}{\lambda^2} E[V^{1-\gamma}] E[V^{-\gamma} V'']^2 \\
- E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma} (\lambda + 1) ((V'')^2 (V')^{-1} + \lambda (\lambda - 2) V' V' V'')] \right) \\
\text{(A.9)}
\]

Define the quadratic polynomial
\[
P(\alpha) \equiv - (\lambda + 1) \frac{\gamma^2}{\lambda^2} E[V^{1-\gamma}] \left( E[V^{-\gamma} V'] E[V^{-\gamma-2} V' (V' - \alpha V (V')^{-1} V'')] \right) \\
- E[V^{-\gamma-1} V' (V' - \alpha V (V')^{-1} V'')]^2 \\
= - (\lambda + 1) \frac{\gamma^2}{\lambda^2} E[V^{1-\gamma}] \left( E[V^{-\gamma} V'] E[V^{-\gamma-2} (V')^3] \\
- 2\alpha E[V^{-\gamma} V'] E[V^{-\gamma-1} V' V''] + \alpha^2 E[V^{-\gamma} V'] E[V^{-\gamma} (V')^{-1} (V'')^2] \\
- E[V^{-\gamma-1} (V')^2]^2 - \alpha^2 E[V^{-\gamma} V'']^2 + 2\alpha E[V^{-\gamma-1} (V')^2] E[V^{-\gamma} V''] \right) \\
\text{(A.10)}
\]

By the Cauchy-Schwarz inequality, \( P(\alpha) \leq 0 \). Adding and subtracting the terms on the last two lines of (A.10) to the right-hand side of (A.9), we get that the following inequality holds true for all \( \alpha \in \mathbb{R} \):
\[
g(x) \leq K + E[V^{1-\gamma}]^\frac{2}{\lambda^2} E[V^{-\gamma} V']^{-\frac{1}{\lambda^2}} P(\alpha) \\
+ E[V^{1-\gamma}]^\frac{2}{\lambda^2} E[V^{-\gamma} V']^{-\frac{1}{\lambda^2}} Q(\alpha), \\
\text{(A.11)}
\]
where we have defined

\[
Q(\alpha) \equiv (1 + \lambda) \left( 2\alpha \frac{\gamma^2}{\lambda^2} - 2 \frac{\gamma}{\lambda^2} \right) E[V^{1-\gamma}] E[V^{-\gamma-1}(V')^2] E[V^{-\gamma} V'']
+ \left( 3 \frac{\gamma}{\lambda} - 1 + \frac{2}{\lambda} - 2(\lambda + 1) \frac{\gamma}{\lambda^2} \right) E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma-1} V' V'']
+ \frac{(\lambda - \gamma)(\lambda - 2)}{\lambda^2} E[V^{-\gamma} V'']^2 E[V^{-\gamma} V''']
+ (\lambda + 1) \left( \frac{1}{\lambda^2} - \frac{\gamma^2}{\lambda^2} \alpha^2 \right) E[V^{1-\gamma}] E[V^{-\gamma} V''']^2
+ (\lambda + 1) \left( 2\alpha \frac{\gamma^2}{\lambda^2} - 2 \frac{\gamma}{\lambda^2} \right) E[V^{1-\gamma}] E[V^{-\gamma} (V')^2] E[V^{-\gamma} V']
+ (\lambda + 1) \left( \frac{\gamma^2}{\lambda^2} \alpha^2 - \frac{1}{\lambda^2} \right) E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma} (V')^{-1} (V'')^2]
\]

(A.12)

Since \( K \) and \( P(\alpha) \) are non-positive, and we have full freedom to choose the parameter \( \alpha \), it remains to show that the minimum of \( Q(\alpha) \) over \( \alpha \) is negative. By the Cauchy-Schwarz inequality,

\[
E[V^{-\gamma} V'] E[V^{-\gamma} (V')^{-1} (V'')^2] - E[V^{-\gamma} V'']^2 \geq 0,
\]

and hence \( Q(\alpha) \) is convex in \( \alpha \) and its minimum is attained at

\[
\alpha^* = \frac{E[V^{-\gamma} V'] E[V^{-\gamma-1} V' V''] - E[V^{-\gamma-1}(V')^2] E[V^{-\gamma} V'']}{E[V^{-\gamma} V'] E[V^{-\gamma} (V')^{-1} (V'')^2] - E[V^{-\gamma} V'']^2}.
\]

The corresponding value is given by

\[
Q(\alpha^*) = \left( E[V^{-\gamma} V'] E[V^{-\gamma} (V')^{-1} (V'')^2] - E[V^{-\gamma} V'']^2 \right)^{-1}
\times \left( - (E[V^{-\gamma} V'] E[V^{-\gamma-1} V' V''] - E[V^{-\gamma-1}(V')^2] E[V^{-\gamma} V''])^2 \right)
\times 2(\lambda + 1) \frac{\gamma^2}{\lambda^2} E[V^{1-\gamma}]
+ \left( \left( 3 \frac{\gamma}{\lambda} - 1 + \frac{2}{\lambda} \right) E[V^{1-\gamma}] E[V^{-\gamma} V'] E[V^{-\gamma-1} V' V''] \right.
\left. + \frac{(\lambda - \gamma)(\lambda - 2)}{\lambda^2} E[V^{-\gamma} V'']^2 E[V^{-\gamma} V'''] \right)
\times \left( E[V^{-\gamma} V'] E[V^{-\gamma} (V')^{-1} (V'')^2] - E[V^{-\gamma} V'']^2 \right)
\]

(A.13)

By Lemma \( V' \geq 0 \) and \( V'' \leq 0 \), and therefore \( Q(\alpha^*) < 0 \) if

\[
\lambda \geq 2
\]

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and
\[ \frac{3\gamma}{\lambda} - 1 + \frac{2}{\lambda} \geq 0 \iff \gamma \geq \frac{\lambda - 2}{3}. \]
Suppose now that \( \lambda \leq 2 \) and \( w_t \) is independent of \( \mathcal{F}_{t-1} \). Then, \( V(x + w_1, w_1) = V(x + w_1) \), and (A.8) implies that \( V''' \geq 0 \). Let us show that, under this condition,
\[ E[V^{1-\gamma}] E[V^{-\gamma}V'] E[V^{-\gamma-1}V'(-V'')] \geq E[V^{-\gamma}V']^2 E[V^{-\gamma}(-V'')] \]  
(A.14)
To this end, define a new measure \( M \) with the density \( \frac{V^{1-\gamma}}{E[V^{1-\gamma}]} \). Then, denoting the expectation under this measure by \( E^M[\cdot] \), we can rewrite the desired inequality (A.14) as
\[ E^M[V^{-2}V'(-V'')] \geq E^M[V^{-1}V'] E^M[V^{-1}(-V'')] \]  
(A.15)
Since \( V'' \leq 0 \leq V' \) and \( V''' \geq 0 \), we get that the functions \( V'/V \) and \( (-V'')/V \) are both monotone decreasing. Inequality (A.15) follows therefore from the Chebyshev sum inequality stating that
\[ E[f(X)g(X)] \geq E[f(X)]E[g(X)] \]
for any random variable \( X \) and any decreasing functions \( f, g \). Thus,
\[ \left( \frac{3\gamma}{\lambda} - 1 + \frac{2}{\lambda} \right) E[V^{1-\gamma}] E[V^{-\gamma}V'] E[V^{-\gamma-1}V'(-V'')] \]
\[ + \frac{(\lambda - \gamma)(\lambda - 2)}{\lambda^2} E[V^{-\gamma}V']^2 E[V^{-\gamma}(-V'')] \]
\[ \geq \left( \frac{3\gamma}{\lambda} - 1 + \frac{2}{\lambda} + \frac{(\lambda - \gamma)(\lambda - 2)}{\lambda^2} \right) E[V^{-\gamma}V']^2 E[V^{-\gamma}(-V'')] \]  
(A.16)
\[ = \frac{2\gamma(\lambda + 1)}{\lambda^2} E[V^{-\gamma}V']^2 E[V^{-\gamma}(-V'')] \geq 0, \]
and therefore \( Q(\alpha^*) \) is non-positive. The proof is complete. \( \square \)

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\(^8\)See Hardy, Littlewood, and Pólya (1988).