Discussion of “A preferred-Habitat Model of the Term structure of interest rates”
by Dimitri Vayanos and Jean-Luc Vila

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Discussion of “A preferred-Habitat Model of the Term structure of interest rates” by Dimitri Vayanos and Jean-Luc Vila
Summary of the paper

- Propose a theoretical model of the term structure based on
  - Agents with preferred-habitat (i.e., exogenous demand for specific maturity bonds)
  - Risk-averse arbitrageurs

- That can explain many stylized facts:
  - Predictability in long-term bonds
  - Forward rates are biased predictors of expected future short rates
  - Two first principal components explain most of the variation in the yield curve
  - Relation between forward rates and investor demand.

- Ultimate goals:
  - 'Considering this mechanism leads to a new set of intuitions and predictions.'
  - 'Because model is structural rather than reduced-form, can suggest which specifications [...] are more economically plausible.'
The stylized facts

- Predictability in bond returns
  - Returns on long-term bonds are larger when the slope of the yield curve is steep
  - Average term premia are small.
  - Term premia are time varying (and even may change sign).


\begin{table}[h]
\centering
\caption{Regressions of excess returns to Treasury bonds}
\begin{tabular}{cccc}
\text{Maturity (years)} & \text{Mean excess return [\%]} & \text{Slope volatility} & \text{Std. dev. of fitted excess returns} \\
0 < m \leq 1 & 0.011 & 0.027 & 0.116 & 0.030 \\
\text{(1.76)} & \text{(0.98)} & \text{(0.98)} & \text{(0.30)} \\
1 < m \leq 2 & 0.045 & 0.085 & 0.413 & 0.119 \\
\text{(1.85)} & \text{(1.27)} & \text{(1.27)} & \text{(1.27)} \\
2 < m \leq 3 & 0.064 & 0.232 & 0.582 & 0.170 \\
\text{(1.88)} & \text{(1.29)} & \text{(1.29)} & \text{(1.29)} \\
3 < m \leq 4 & 0.074 & 0.387 & 0.796 & 0.241 \\
\text{(2.38)} & \text{(1.36)} & \text{(1.36)} & \text{(1.36)} \\
4 < m \leq 5 & 0.083 & 0.214 & 0.692 & 0.268 \\
\text{(2.37)} & \text{(1.18)} & \text{(1.18)} & \text{(1.18)} \\
5 < m \leq 10 & 0.081 & 0.206 & 0.584 & 0.354 \\
\text{(2.06)} & \text{(1.08)} & \text{(1.08)} & \text{(1.08)} \\
\end{tabular}
\end{table}

- In contradiction with expectation’s hypothesis.
- But consistent with evidence from equity and currency markets about predictability: ‘high yields forecast high returns... presumably reflecting the fact that discount rates are high’
Absence of arbitrage in Short rate models

- Suppose the short rate model is exogenously given by:
  \[ dr_t = \kappa(\theta - r_t)dt + \sigma_r dz_t \]

- Assume zero-coupon bond \( P(t, T_i) = P^i(t, r_t) \). Construct a portfolio \( V(t) = n_1 P^1(t) + n_2 P^2(t) \), that is self financing:
  \[
  dV_t = n_1(t)dP^1(t) + n_2(t)dP^2(t) \\
  = \left[ n_1 \left( \frac{1}{2} P^1_{rr} \sigma_r^2 + P^1_r \mu_r + P^1_t \right) + n_2 \left( \frac{1}{2} P^2_{rr} \sigma_r^2 + P^2_r \mu_r + P^2_t \right) \right] dt \\
  + \left( n_1 P^1_r + n_2 P^2_r \right) \sigma_r dw_t
  \]

If we choose \( \{n_1, n_2\} \) such that the portfolio is locally risk-free

\[ n_1 P^1_r + n_2 P^2_r = 0 \quad (1) \]

it should earn the risk-free rate:

\[
 n_1 \left( \frac{1}{2} P^1_{rr} \sigma_r^2 + P^1_r \mu_r + P^1_t \right) + n_2 \left( \frac{1}{2} P^2_{rr} \sigma_r^2 + P^2_r \mu_r + P^2_t \right) = r(n_1 P^1 + n_2 P^2) \quad (2)
\]
Absence of arbitrage in Short rate models

- Combining equations (1) and (2), we find

\[
\frac{1}{P^1_r} \left[ \frac{1}{2} P^1_r \sigma^2_r + P^1_r \mu_r + P^1_t - rP^1 \right] = \frac{1}{P^2_r} \left[ \frac{1}{2} P^2_r \sigma^2_r + P^2_r \mu_r + P^2_t - rP^2 \right].
\]  

(3)

Therefore, there must exist a market price of risk process \( \gamma(t) \) independent of maturities such that:

\[
\gamma(t) = \frac{1}{2} P^1_r \sigma^2_r + P^1_r \mu_r + P^1_t - rP^1 \equiv \frac{\mu_P(t, T) - r_t}{\sigma_P(t, T)}.
\]  

(4)

⇒ Absence of arbitrage implies that the instantaneous Sharpe ratio across all bonds is equalized (in a one-factor model).
Specification of risk-premia and predictability in bond returns

- How do we choose the instantaneous Sharpe ratio?
  - First generation: 'completely affine' models.
    - Vasicek: $\gamma$ constant.
    - CIR: $\gamma(r) = \gamma \sqrt{r}$ proportional to short rate volatility.
  - Second Generation: 'essentially affine' (Duffee (2002), DS (2002)):
    - Vasicek: $\gamma(r) = \gamma_0 + \gamma r r$.
    - CIR: $\gamma(r) = \gamma_0 \sqrt{r} + \frac{\gamma r}{\sqrt{r}}$.

- Why are essentially affine models of the market price of risk useful?
  Can be seen from the definition of excess expected return:

  \[ \mu_P(t, T) - r_t = \gamma(r_t)\sigma_P(t, T) \]

  ⇒ First generation risk-premia impose that compensation for risk is a fixed multiple of short-rate volatility. In particular, risk-premia cannot switch sign over time.

  ⇒ More general structure of risk-premia breaks this link, which is necessary to capture evidence on predictability (Duffee (2002)).

N.B.: $\sigma_P(t, T) = -\sigma_r \frac{P_r(t, T)}{P(t, T)} < 0$ so need $\gamma_r > 0$ to generate relation between slope and expected return.
Intuition for the Habitat-model

- Risk-averse arbitrageurs can freely trade bonds of all maturities.

⇒ No-arbitrage holds and Sharpe ratios ($\gamma(t)$) across all bonds are equalized.

⇒ Arbitrageurs are indifferent between all bonds. Can achieve same payoff with one bond and short rate.

- Consider portfolio choice with one bond (maturity $T$) and risk-free rate.

\[
dW(t) = r(t)W(t) + X(t)\left(\frac{dP(t, T)}{P(t, T)} - r(t)\right)dt
\]

\[
= r(t)W(t) + X(t)\sigma_P(t, T)(\gamma(t)dt + dz(t))
\]

- Arbitrageurs are instantaneous mean-variance optimizers:

\[
\max_{X(t)} \mathbb{E}[dW(t)] - \frac{a}{2} \mathbb{V}[dW(t)]
\]

So first order condition is

\[
X(t) = \frac{\gamma(t)}{a\sigma_P(t, T)}
\]
Now, in equilibrium net demand is zero. So if $Y(t, T)$ is dollar demand for bond of maturity $T$ by Habitat-agent we get:

$$X(t) = -Y(t, T)$$

Using first order condition of arbitrageurs

$$\frac{\gamma(t)}{a \sigma_P(t, T)} = -Y(t, T)$$

So, in particular, if we assume that demand of Habitat-agents is linear in short rate

$$Y(t, T) = -\frac{\beta(t) + \beta_r r(t)}{\sigma_P(t, T)}$$

we get Duffee’s essentially affine model in equilibrium, with mpr linear in short rate:

$$\gamma(t) = a\beta(t) + a\beta_r r(t)$$

Note: Can have stochastic demand shocks $\beta(t)$ which leads to a two-factor model of the term structure where one factor is a pure risk-premium factor (it only affects bond prices and not the dynamics of the short rate - Duffee (2002)).
Intuition

- To solve the predictability puzzles we need Sharpe ratio to go up when rates go down (and slope increases), i.e.,

$$\beta_r > 0$$

⇒ Demand of Habitat-agents must be increasing in the short rate (remember $\sigma_P(t, T) < 0$!).

- Intuition:
  - For Risk-averse arbitrageurs the risk-free benchmark is the short rate (their 'habitat').
  - When rates decrease, Habitat-agent demand decreases ($Y \downarrow$) and since arbitrageurs must take the other side ($X \uparrow$).
  - In equilibrium, Sharpe ratio on long-term bond has to increase to incite arbitrageurs to hold larger amount of risky bonds. (Arbitrageurs are instantaneous mean-variance optimizers so they only care about Sharpe ratio).
Comments

- In effect, the paper provides one possible justification for the essentially affine model of Duffee (2002).

- There are others, e.g., Lucas (1978) exchange economy with log $C_t$ utility agent and aggregate consumption given by:

$$
\frac{dC_t}{C_t} = (r_t + \gamma(r_t)^2)dt + \gamma(r_t)dz_t
$$

- In particular, all the implications for prices are identical to those of Duffee:
  - predictability
  - forward vs. expected future spot
  - PCA

- How can we test this model’s specific implications?
  ⇒ Can we identify the Habitat-agents and tie variation in risk-premia to their demand for long-term bonds.

- Who are the Habitat agents?
  - Pension funds in the UK during and post pension reform?
  - Wouldn’t their demand for long term bonds go up when rates decrease (as the PV of liabilities increase)?
Comments about the model

- Arbitrageurs are also Habitat agents with a Habitat equal to instantaneous maturity.

- Would be interesting to model Habitat from Micro-economic foundations rather than ad-hoc demand function (e.g., Ingersoll (1987)).

- Problem with perfect market, continuous time set-up with finite factors is that portfolio problem is ill-defined. In a one factor model, only one bond (in addition of the risk-free rate) is needed to replicate any other bond perfectly. So notion of Habitat seems a bit awkward.

- Interesting to relax 'perfect-market' assumptions (restrictions to trading etc...) or finite factor assumption such as in a string or Brownian field setup (e.g., PCD and Goldstein (2004)).
Conclusion

- Interesting idea to propose microeconomic foundations for essentially affine term structure models.

- Can we go further in modeling microeconomic foundations of these Habitat agents?

- Can we identify their asset demand empirically to test the model’s new predictions?

- Is habitat the most likely explanation for predictability in long-term bond returns?

- Can it explain similar phenomenon (high yield predicting high future returns) we see in equity and currency markets?