Pareto Optimal Allocations for Law Invariant Robust Utilities

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Agenda

• Motivation

• Pareto Optimal Allocation (POA)

• Main result: Existence POA without and with constraints for Law Invariant Robust Utilities

• Example
Motivation

$X$ random variable representing future contingent claim, financial pay-off, endowment, etc.

Pareto Optimal Allocation of $X$: no agent can be better off without making any other agent worse off

No waste in society!

Pareto optimality: condition that any allocation should satisfy
Pareto Optimal Allocations (POA) with constraints

Risk/Endowment Sharing: POA + Individual Constraints

Individual Constraints define feasible allocations, e.g.

- no agent suffers a utility loss passing from initial to POA (optimal risk sharing)
- some agents suffer a (bounded) utility loss
- some agents require a utility improvements

- insurance and re-insurance contracts
- distributing liabilities among business units
- social welfare allocations
- OTC transactions
- ...
Existence of Pareto Optimal Allocations

State of art

- **Expected utility:** Borch (1962), Gerber (1978), Bühlmann (1984)

- **Law invariant monetary utility functions:**
  - on $L^\infty$; Jouini, Schachermayer, Touzi (2008)
  - on $L^p, p \in [1, \infty]$; Filipović and Svindland (2008)

- **Law invariant utilities with at least 1 monetary utility on $L^\infty$; Dana (2011)**

Our main result

- **Law invariant robust utilities on $L^1 (L^p, p \in [1, \infty])$**
Law Invariant Robust Utilities on $L^1$: What are they?

- Law invariant: utility functional depends only on law of $X$
- Robust utilities: very broad class of utility functionals
- $L^1$: large class of stochastic payoffs with at least first moment finite
Definition of Law Invariant Robust Utilities on $L^1(\Omega, \mathcal{F}, \mathbb{P})$

$$L^1 \ni X \mapsto U(X) := \inf_{Q \in \mathcal{Q}} \left( E_Q[u(X)] + \alpha(Q) \right) \in [-\infty, \infty)$$

$u : \mathbb{R} \to [-\infty, \infty)$ is a utility function on $\mathbb{R}$
i.e. concave, right-continuous, increasing and $\text{dom}(u) = \{u > -\infty\} \neq \emptyset$

$\mathcal{Q}$ is law invariant set i.e. if $Q \in \mathcal{Q}$, and $\hat{Q} \ll \mathbb{P}$ such that $\frac{d\hat{Q}}{d\mathbb{P}} \equiv \frac{dQ}{d\mathbb{P}}$,
then $\hat{Q} \in \mathcal{Q}$

$\alpha : \mathcal{Q} \to \mathbb{R}\cup\{\infty\}$ law invariant penalty function: $\frac{d\hat{Q}}{d\mathbb{P}} \equiv \frac{dQ}{d\mathbb{P}} \Rightarrow \alpha(Q) = \alpha(\hat{Q})$

**Proposition**: $U$ is a monotone, law invariant, closed, concave functional such that $\text{dom}U \subseteq \{X \in L^1 : u(X) \in L^1\}$
Law Invariant Robust Utilities on $L^1$: Examples

- **Expected utility**; von Neumann and Morgenstern (1944)


- **Law Invariant Robust Utility with Penalty Function**; Hansen and Sargent (2001), Maccheroni, Marinacci and Rustichini (2006)

- **Law Invariant Concave Utility**, which includes
  - Cash additive risk measure; Artzner et al. (1999), Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2005)
  - Cash sub-additive risk measure; El Karoui and Ravanelli (2009)

- ...
Main result: Existence of POA for Law Invariant Robust Utilities

2 agents with $\mathcal{U}_1$ and $\mathcal{U}_2$ law invariant robust utilities, $W_1, W_2 \in L^1$ initial endowments: $\mathcal{U}_1(W_1) > -\infty$, $\mathcal{U}_2(W_2) > -\infty$

Set of all allocations of $W := W_1 + W_2 \in L^1$: $\mathcal{A}(W) := \{(X_1, X_2) \in L^1 \times L^1 \mid X_1 + X_2 = W\}$

Individual Constraints:
$\mathcal{U}_i(X_i) \geq \mathcal{U}_i(W_i) - c_i$ for given $c_i \in \mathbb{R} \cup \{\infty\}$, $i = 1, 2$

Set of feasible allocations of $W$:
$\mathcal{A}^f(W) := \{(X_1, X_2) \in \mathcal{A}(W) \mid \mathcal{U}_i(X_i) > -\infty, \mathcal{U}_i(X_i) \geq \mathcal{U}_i(W_i) - c_i, i = 1, 2\}$

Pareto Optimal Allocation of $W$:
$(X_1, X_2) \in \mathcal{A}^f(W)$ s.t. $\forall (Y_1, Y_2) \in \mathcal{A}^f(W)$, $\mathcal{U}_i(Y_i) \geq \mathcal{U}_i(X_i)$ for $i = 1, 2$, then $\mathcal{U}_i(Y_i) = \mathcal{U}_i(X_i)$ for $i = 1, 2$

Theorem For any $W \in L^1$ which admits a $(X_1, X_2) \in \mathcal{A}^f(W)$ there exists a comonotone Pareto Optimal Allocation
Comonotonicity of POA

Pareto Optimal Allocation \((X_1, X_2) \in A^f(W)\) is comonotone when

\[ X_1 = f(W) \text{ and } X_2 = g(W), \]

\((f, g) \in \text{CF} := \{(f, g) \text{ s.t. } f, g : \mathbb{R} \to \mathbb{R}, f \text{ and } g \text{ increasing, } f + g = \text{Id}_\mathbb{R}\}\)

Comonotonicity of POA \((X_1, X_2)\), with \(X_1 + X_2 = W\), implies

- Payoffs \(X_1\) and \(X_2\) are increasing 1-Lipschitz-continuous functions of the risky payoff \(W\)

- if \(W \in L^p \subset L^1, p \in [1, \infty]\), nothing changes if optimal allocation problem is solved considering \(W\) on \(L^1\) or considering \(W\) on \(L^p\):

\[ W \in L^p \Rightarrow (f(W), g(W)) \in L^p \times L^p \text{ for all } (f, g) \in \text{CF} \]
Where the existence of POA is coming from?

(i) Characterization of POA as solution of $\lambda$-supconvolution

(ii) Existence of $\lambda$-supconvolution solution (main technical result)

using

- law invariance of robust utilities

- $U(\cdot) := \inf_{Q \in \mathcal{Q}} \left( E_Q[u(X)] + \alpha(Q) \right) = U(u(\cdot)),$
  i.e. any robust utility $U(\cdot)$ is a composition of
  - concave utility $u(\cdot)$ and
  - monetary utility function $U(\cdot) := \inf_{Q \in \mathcal{Q}} \left( E_Q[\cdot] + \alpha(Q) \right)$
Characterization of POA

Let $\lambda > 0$. The $\lambda$-supconvolution of $U_1$ and $U_2$ is

$$L^1 \ni W \mapsto U_1 \Box \lambda U_2(W) := \sup_{(X_1, X_2) \in A^f(W)} (U_1(X_1) + \lambda U_2(X_2)) \in (-\infty, \infty]$$

$(X_1, X_2) \in A^f(W)$ is $\lambda$-optimal if $U_1 \Box \lambda U_2(W) = U_1(X_1) + \lambda U_2(X_2)$

**Proposition:**

$\exists \lambda \geq 0 : (X_1, X_2) \in A^f(W)$ is $\lambda$-optimal $\iff (X_1, X_2) \in A^f(W)$ is POA

**Remark:**

If $(X_1, X_2) \in A^f(W)$ is $\lambda = 0$ optimal $\Rightarrow (X_1, X_2) \in A^f(W)$ “extremal” POA satisfying:

$$U_1(X_1) = \max\{U_1(Y) \mid (Y, X - Y) \in A^f(W)\}$$
Existence of $\lambda$-supconvolution solution (main technical result)

Definition: $d^i_L := \lim_{x \to \infty} u'_i(x)$, $d^i_H := \lim_{x \to a_i} u'_i(x)$ (possibly $\infty$), where $a_i := \inf \dom u_i \in \mathbb{R} \cup \{-\infty\}$, $i = 1, 2$

**Theorem 1**
If $a_i > -\infty$, $i = 1, 2$, then $\forall W \in L^1$ and $\forall \lambda > 0$ $\exists$ comonotone $\lambda$-optimal allocation

**Theorem 2**
- If $d^1_H = d^1_L = d^1 > 0$, $d^2_H = d^2_L = d^2 > 0$, $a_1 = a_2 = -\infty$ and $\lambda = \frac{d^1}{d^2}$, then $\forall W \in L^1$ $\exists$ comonotone $\lambda$-optimal allocation
- If $d^i_L < d^i_H$ for at least one $i \in \{1, 2\}$, and

$$\lambda \in \left(\frac{d^1_L}{d^2_H}, \frac{d^1_H}{d^2_L}\right),$$

then $\forall W \in L^1$ $\exists$ comonotone $\lambda$-optimal allocation
Idea of the proof

Definition: \( X \succeq_c Y \iff E[u(X)] \geq E[u(Y)] \) \( \forall \) concave functions \( u : \mathbb{R} \rightarrow \mathbb{R} \)

\( \text{CF} = \{(f+a, g-a) \mid (f, g) \in \text{CFN}, a \in \mathbb{R}\} \) where \( \text{CFN} := \{(f, g) \in \text{CF} \mid f(0) = g(0) = 0\} \)

i) \( U_1 \Box \lambda U_2(W) = \sup_{(f,g) \in \text{CFN}, a \in \mathbb{R}} U_1(f(W) - a) + \lambda U_2(g(W) + a) \)

\( \forall (Y, Z) \in \mathbb{A}^f(W) \exists (f, g) \in \text{CF} \) s.t. \( f(W) \succeq_c Y, g(W) \succeq_c Z \);

\( U \) law-invariant \( \iff X \succeq_c Y \implies U(X) \geq U(Y) \);
Dana (2005), Svindland (2008)

ii) \( U_1 \Box \lambda U_2(W) < \infty \) and if \( \exists (Y_1, Y_2) \in \mathbb{A}^f(X) \)

\[ U_1 \Box \lambda U_2(W) = \sup_{(f,g) \in \text{CFN}, a \in [-K,K]} U_1(f(W) - a) + \lambda U_2(g(W) + a) \in (-\infty, \infty) \]

\( \Box \) concavity of \( u_i \) and choice of \( \lambda \)

\[ \implies \exists \tilde{d}_L^1 \geq 0, \tilde{d}_H^i \geq 0 \text{ and } c \text{ s.t. } u_i(x) \leq \tilde{d}_L^1 x + c, \quad u_i(x) \leq \tilde{d}_H^i x + c, \]
\[ \tilde{d}_H^1 - \lambda \tilde{d}_L^2 > 0, \quad \lambda \tilde{d}_H^2 - \tilde{d}_L^1 > 0 \]
Example

Agent 1: $U_1(X) = -\text{AVaR}_\alpha(X), \ \alpha \in (0, 1)$

Agent 2: $U_2(X) = U(u(X))$

- $U$ strictly monotone and strictly risk averse conditionally on lower-tail events
- $u$ strictly increasing and $\text{dom}(u) = \mathbb{R}$

- Examples:
  
  $U(X) = -\frac{1}{\beta} \log E[e^{-\beta X}]$  

  $U(X) = E[X] - \delta E[|\max(0, E[X] - X)|^p]^{1/p}, \ \delta \in (0, 1), p \in (1, \infty)$
Proposition (POA without constraints):
For any given $X \in L^1$ there exists a unique (up to constant) comonotone POA which takes the following form:

$$(X_1, X_2) = (\min(0, X - l), \max(X, l)), \quad \text{for some } l \in \mathbb{R}$$
Conclusion

• Law Invariant Robust Utilities: broad class of utility functionals

• Economically relevant allocations:
  Pareto efficiency + Individual Constraints

• Main result: Existence of POA without and with Individual Constraints for law invariant robust utilities on $L^1$