Dynamical Counterparty Risk Valuation via Bessel Bridges

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AGENDA

• Combining pricing and counterparty default models
• Calibrating firm-value default models to CDS data
• Solving inverse hitting time distribution problems
• Conditional Brownian motion and Bessel bridges
• Applications to vanilla options and interest-rate swaps
Counterparty Risk

This is the exposure of a bank to a counterparty in some contract should the counterparty default at some specific time in the future.

*Note:* Exposure = \( \max(\text{value}, 0) \) [No exposure if we owe them!]

‘Exposure’ can be quantified in various ways:

- Quantile of loss distribution
- Expected shortfall
- \( \ldots \)

These are all functions of the post-default distribution of contract value.

*Example:* Long contract in (Black-Scholes) call option.

Value now: \( \text{BS}(S_0, K, r, \sigma, T) \)

Value at \( t \in (0, T) \): \( V_t = \text{BS}(S_t, K, r, \sigma, T - t) \)

so problem amounts to computing the distribution of \( S_t \).
Note this is in principle a ‘two-measure problem’:

- Distribution of $S_t$ in ‘real-world’ measure $\mathbb{P}$
- $V_t = \mathbb{E}_Q[e^{-r(T-t)}(S_t - K)^+|\mathcal{F}_t]$ where $Q$ is risk-neutral measure.

Nevertheless, computations are normally done using only $Q$-measure, because

- It is absolutely impossible to predict the $\mathbb{P}$-measure drift.
- For short horizons $t$, volatility, not drift, is the dominant factor.
Right-way/Wrong-way risk

The simplest approach is just to compute the distribution of $S_t$ (under $\mathbb{Q}$, say). In the Black-Scholes model we have

$$S_t = S_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

where $W$ is BM, i.e. $W_t \sim \mathcal{N}(0, t)$. However, this ignores any connection between exposure and the assumed counterparty default at $t$. We want the conditional distribution given that the default event occurs at $t$.

Right-way risk: Negative correlation between default and exposure.

Wrong-way risk: Positive correlation.

We first need a counterparty default model. We take the approach of John Hull: default time $\tau$ is the first hitting time of some barrier by BM.
Barrier calibration (John Hull)

CDS (credit default swap) spreads determine the $\mathbb{Q}$-measure distribution function $F$ of the counterparty default time $\tau$.

Calibrate a piecewise-constant barrier (levels $b_0, b_1, \ldots$) or a piecewise-linear barrier (level $b_0$ and slopes $d_0, d_1, \ldots$) so that $F$ is perfectly matched at discrete time points. Procedure for pw-constant barrier: choose $b_0$ so that hitting prob on $[0, t_1]$ is $F(t_1)$. Now get a discrete representation of the distribution of $B_{t_1}$ on $(-\infty, b_0)$ given that the barrier is not hit, and bootstrap for $b_1, b_2, \ldots$.

Practical point: Much better to use $X_t = B_t + \nu t$ with $\nu > 0$ (rotates barrier).
Barrier for exponential hitting time
Application to counterparty risk in Black-Scholes

As above, price process is

\[ S_t = S_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t) \]

Assume default is controlled by BM \( B_t \) as above, and

\[ W_t = \rho B_t + \sqrt{1 - \rho^2} B'_t \]

where \( B, B' \) are independent. Suppose default takes place at \( t = 0.25 \). Then \( B_t = b(t) \) so \( W_t = \rho b(t) + \sqrt{1 - \rho^2} B'_t \), i.e.

\[ W_t \sim N(\rho b(t), (1 - \rho^2)t). \]

Graph shows corresponding distribution of option value. (In this case \( b(t) = 0.5 \).)
BS value distribution with CP default at $t = 0.25$

$\rho = 0$ (blue), $\rho = .5$ (yellow), $\rho = -.5$ (red).
An alternative approach

The idea: instead of calibrating with a time-varying barrier, keep a flat barrier and vary the Brownian motion in some way. Always start at $a > 0$ and take 0 as the barrier.

Define $\tau^X_0 = \inf\{t : X_t \leq 0\}$ with Brownian motion $B_t$ and $X_t = a + \nu t + B_t$.

Then $\tau^X_0$ has distribution function $K$ with density

$$k(t) = \frac{d}{dt} K(t) = \frac{a}{\sqrt{2\pi t^3}} \exp \left(-\frac{(a + \nu t)^2}{2t}\right).$$
\[ K(\infty) = \mathbb{P}[\tau_0^X < \infty] = \begin{cases} 1, & \nu \leq 0 \\ e^{-2\nu a}, & \nu > 0 \end{cases}. \]

The moment generating function is given by
\[ \mathbb{E}_a[e^{-q\tau_0^X}] = e^{-a\Phi(q)}, \quad q > 0, \tag{1} \]
where \( \Phi(q) = \nu + \sqrt{\nu^2 + 2q} \). This is shown as follows. For \( \theta \in \mathbb{R} \), define
\[
Z_t = \theta B_t - \frac{1}{2} \theta^2 t
= \theta(X_t - a) - (\nu \theta + \frac{1}{2} \theta^2)t.
\]
For \( q > 0 \) the equation \( \theta^2/2 + \theta \nu = q \) has negative root \( \theta = -\Phi(q) \) with \( \Phi \) as above. Then \( \exp(Z_{t \wedge \tau_0^X}) \) is a bounded martingale and we conclude that
\[ 1 = \mathbb{E}_a[\exp(Z_{\tau_0^X})] = e^{a\Phi(q)} \mathbb{E}_a[e^{-q\tau_0^X}]. \]
1. Fixed starting point

Recall

\[ X_t = a + \nu t + B_t. \]  \hspace{1cm} (2)

Let \( \sigma(t) \) be a (deterministic) non-negative function and define

\[ I_t = \int_0^t \sigma^2(s)ds. \]

Then \( X(I_t) \) is equal in law to the process \( Y_t \) defined for some BM \( \tilde{B}_t \) by

\[ dY_t = \nu \sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \hspace{1cm} Y_0 = a. \]  \hspace{1cm} (3)

\( \tau^Y_0 \equiv \inf\{t : Y_t = 0\} \) is our model for the default time.
Theorem 1. Let $H$ be a distribution function on $\mathbb{R}^+ \cup \{+\infty\}$ with $H(0) = 0$, having a density function $h$. In (2), fix $a > 0$ and

$$\nu < -\frac{1}{2a} \log H(\infty),$$

and define $Y_t$ by (11) with

$$\sigma^2(s) = \begin{cases} 0, & 0 \leq s \leq \inf \{u : H(u) > 0\} \\ \frac{h(s)}{k(K^{-1}(H(s)))}, & s > s_0. \end{cases}$$

Then $\mathbb{P}[\tau^Y \leq t] = H(t)$, $t \in \mathbb{R}^+$.

Proof: With $\sigma^2(s)$ defined by (5) we find that

$$I_t = K^{-1}(H(t)).$$

This is well-defined because condition (4) ensures that $K(\infty) \geq H(\infty)$. Thus

$$\mathbb{P}[\tau^Y \leq t] = \mathbb{P}[\tau^X_0 \leq I_t] = K(I_t) = H(t).$$

Conclusion: we can realize any distribution on $\mathbb{R}^+$ having a density as the hitting time of zero by a drifting BM with deterministic time change.
2. Randomized starting point

We now introduce an extra degree of flexibility by taking $X_0 = A$, where $A$ is a random variable having distribution function $F$, independent of $B(\cdot)$.

We ask: given a distribution function $H$ on $\mathbb{R}^+ \cup \{\infty\}$ with density function $h$, can we choose $F$ on such that $\tau_0^X$ has distribution $H$ when

$$X_t = A + \nu t + B_t.$$ 

Such an $F$ must satisfy

$$H(t) = \mathbb{P}[\tau_0 \leq t] = \int_0^\infty \mathbb{P}_a[\tau_0^X \leq t]F(da).$$

Taking the Laplace-Stieltjes transform in $t$, $\mathcal{L}H(q) = \int_{\mathbb{R}^+} e^{-qt}H(dt)$, gives

$$\mathcal{L}H(q) = \int_0^\infty \mathbb{E}_a[e^{-q\tau_0^X}]F(da) = \int_0^\infty e^{-a \Phi(q)}F(da).$$

We know $\Phi$ has inverse $q = \psi(\theta) = \frac{1}{2}\theta^2 - \nu \theta$ so if $F$ exists it must satisfy

$$\mathcal{L}H(\psi(\theta)) = \mathcal{L}F(\theta)$$

(6)
Important example: Exponential distribution \( H(t) = 1 - e^{-\lambda t} \)

Here the left-hand side of (6) is

\[
\frac{\lambda}{\psi(\theta) + \lambda} = \frac{2\lambda}{\theta_+ - \theta_-} \left( \frac{1}{\theta - \theta_+} - \frac{1}{\theta - \theta_-} \right).
\]

This is the LT of a distribution on \( \mathbb{R}^+ \) if and only if \( \psi(\theta) > \lambda \) for all \( \theta \geq 0 \) and \( \nu^2 - \lambda \geq 0 \). We obtain the following solutions for the density \( f_\lambda(x) = (d/dx)F(x) \).

- \( \nu < -\sqrt{2\lambda} \) : \( f_\lambda(x) = \frac{2\lambda}{\theta_+ - \theta_-}(e^{\theta_+ x} - e^{\theta_- x}) \).

- \( \nu = -\sqrt{2\lambda} \) : \( f_\lambda(x) = 2\lambda x e^{-x\sqrt{2\lambda}} \).

Can we ‘realize’ an arbitrary density function \( h \) in this way? Answer: no. If a solution exists then the Laplace transform of the corresponding \( F \) must be given by the LHS of (6), but this formula may not correspond to a distribution having support in \( \mathbb{R}^+ \).
To go further, introduce a deterministic time change $I_t$ as before, where

$$I_t = \int_0^t \sigma^2(s)ds$$

and $X(I_t) =_d Y_t$ with

$$dY_t = \nu \sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad Y_0 = A. \quad (7)$$

Essentially we realize an exponential distribution as above and then massage the time scale to get the distribution we want. Thus we choose $A \sim F$ where $(dF/dx)(x) = f_\lambda(x)$ as above, so that when $\sigma \equiv 1$ then $\tau_0^Y \sim \exp(\lambda)$. 
Theorem 2. Let $H$ be a distribution on $\mathbb{R}^+$ with density $h$ and hazard function

$$\gamma(t) = \frac{h(t)}{\int_t^\infty h(s)ds} = \frac{h(t)}{H(t)}.$$

Define $Y_t$ by (7) with

$$\sigma(t) = \sqrt{\frac{\gamma(t)}{\lambda}}.$$

Then $\tau_0^Y \sim H$ where

$$\tau_0^Y = \inf\{t : Y_t = 0\}.$$

Indeed, since $h/H = -(d/dt) \log H$ we have

$$I_t = -\frac{1}{\lambda} \log H(t).$$

Since $Y_t = X(I_t)$ and $\tau_0^X$ has exponential distribution, we see that

$$\mathbb{P}[\tau_0^Y > t] = \mathbb{P}[\tau_0^X > I_t] = e^{-\lambda I_t} = e^{\log H(t)} = H(t).$$
Calibrating Risk-neutral default-time distributions from CDS Rates

For CDS contracts written on an underlying name ABC, we assume that premium payments are made at times $t_i$ and the available maturities are $T_j = t_{k(j)}$, $j = 1, \ldots, n$. For contract $j$ there is an upfront premium $\pi_j^0$ and a running premium rate $\pi_j^1$ (with accrual factors $\delta_i$). The recovery rate is $R \in (0, 1)$. The ‘fair premium’ $(\pi_j^0, \pi_j^1)$ then satisfies

$$\pi_j^0 + \pi_j^1 \sum_{i=0}^{k(j)-1} \delta_i p(0, t_i) \overline{H}(t_i) = (1 - R) \sum_{i=1}^{k(j)} p(0, t_i)(\overline{H}(t_{i-1}) - \overline{H}(t_i)). \quad (8)$$
We take the default distribution to have piecewise-constant hazard rate, i.e.

$$\overline{H}(t) = \exp \left( - \int_0^t \gamma(s) ds \right)$$

where $\gamma(s) = \gamma_i$ for $T_i \leq s < T_{i+1}$ (with $T_0 = 0$.)

We then back out $\gamma_0, \gamma_1, \ldots$ from (8) given the market data $(\pi_1, \pi_1), (\pi_2, \pi_2), \ldots$

Note this fits perfectly with our default model, which is hitting of 0 by $Y_t$ satisfying

$$dY_t = \nu \sigma^2(t) dt + \sigma(t) dB_t, \quad Y_0 = A,$$

where $\sigma^2(t) = \gamma(t)/\lambda$, i.e. $Y_t$ has piecewise-constant coefficients.
**Example:** Altria Group Inc

We have 12 CDS quote for maturities ranging from 6m to 16y.
Models conditioned on default time

To evaluate counterparty risk, we condition on default at a specific time \( s > 0 \). For path-dependent contracts we need the conditional law of the default risk process \( Y_t \) conditioned on the event \( (\tau_0^Y = s) \). We consider

1. Brownian motion case.

2. Case of general calibrated model as above.
Bessel Bridges, \( h\)-transforms etc.

The law of Brownian motion \( B \) starting at \( a > 0 \) and conditioned to hit 0 for the first time at \( \tau_0 = s \) is equal to that of the 3-dimensional Bessel Bridge from \( a \to 0 \) on \([0, s]\). We apply the Doob \( h\)-transform with \( h \) given by

\[
h(t, x) = \mathbb{P}[\tau_0 = s | B_t = x] = \mathbb{P}_x[\tau_0 \in [s - dt, s]]/dt = \frac{1}{\sqrt{2\pi(s-t)^3}} x e^{-x^2/2(s-t)}.\]

\( h(t, B_t) \) is a positive martingale, so (with \( h' = \partial h/\partial x \))

\[
dh = h'dB = h(h'/h)dB
\]

and

\[
\frac{h(t, B_t)}{h(0, a)} = \mathcal{E} \left( \frac{h'}{h} \cdot B \right).
\]

We apply a change of measure \( d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = h(t, B_t)/h(0, a) \). By Girsanov, the new drift is

\[
\frac{h'}{h} = (\log h)' = \left( \log x - \frac{1}{2(s-t)x^2} \right)' = \frac{1}{x} - \frac{x}{s-t}.
\]
Thus under $\mathbb{Q}$, $X_t$ satisfies the SDE

$$dX_t = \left(\frac{1}{X_t} - \frac{X_t}{1-t}\right) dt + dB, \quad t \in [0, s)$$ \quad (9)$$

Recall that the Brownian bridge $Z_t$ from $Z_0 = a$ to $Z_1 = 0$ satisfies

$$dZ_t = -\frac{Z_t}{1-t} dt + dW_t, \quad Z_0 = a.$$ 

It can also be represented as

$$Z_t = \frac{s-t}{s}a + B_t - \frac{t}{s}B_s$$

where $B_t$ is ordinary Brownian motion.

Taking a 3-vector $Z_t$ of independent Brownian bridges and applying the Ito formula, we find that $|Z|$ satisfies (9), so $X = \mathcal{L} |Z|$.

Finally, a result of Bertoin and Pitman states that

$$X = \mathcal{L} \sqrt{(a(s-t)/s + X_{1,t})^2 + X_{2,t}^2 + X_{3,t}^2}$$

where $X_i, \ i = 1, 2, 3$ are independent $0 \to 0$ Brownian Bridges.

This provides us with an efficient simulation method.
General case

Our main result is as follows. Recall that our general default time model is $\tau_0^Y$, the first hitting time of 0 by the process

$$Y_t = \nu\sigma^2(t) + \sigma(t)dB_t, \quad Y_0 = A.$$

**Proposition.** Conditioned on $\tau_0^Y = s > 0$, the process $Y_t$ satisfies

$$dY_t = \left(\frac{1}{Y_t} - \frac{Y_t}{\int_s^t \sigma^2(u)du}\right)\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad t \in (0, s)$$

$$Y_0 = A,$$

where $\tilde{B}$ is Brownian motion and $A \sim F$ is independent of $\tilde{B}$.

The (⋯) term can alternatively be expressed as

$$\left(\frac{1}{Y_t} - \frac{\lambda Y_t}{\log(H(t)/H(s))}\right).$$
Interest rate swaps

The zero-coupon (ZC) bond $p(t, T)$ gives the value at $t \leq T$ of £1 delivered at $T$ (so $p(T, T) = 1$). A short rate model specifies the ZC bond price as

$$p(t, T) = \mathbb{E}_Q \left[ e^{-\int_t^T r(s)ds} \bigg| \mathcal{F}_t \right],$$

where $r(t)$ is the short-rate process. Here we consider the Hull-White model

$$dr(t) = (\theta(t) - \mu r(t))dt + \beta dW_t.$$

This is an ‘affine process’ in that the ZC bond value takes the form

$$p(t, T) = \exp(A(t, T) - B(t, T)r(t)) \equiv p_{tT}(r(t)).$$

In an interest rate swap the parties exchange a fixed-rate payment $\delta_i K$ for a floating (Libor) payment $\delta_i L_i$ at times $T_1, \ldots, T_n$ where $\delta_i$ is the accrual factor and $L_i$ is the Libor rate set at $T_{i-1}$. The (payer’s) swap value at any time is

$$V_t = p(t, T_{k(t)}) - p(t, T_n) - K \sum_{i \geq k(t)} \delta_i p(t, T_i)$$

where $k(t)$ is the next coupon date after $t$. 

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The *swap rate* $S_t$ is the value of $K$ such that $V_t = 0$, i.e.

$$S_t = \frac{p(t, T_{k(t)}) - p(t, T_n)}{\sum_{i \geq k(t)} \delta_i p(t, T_i)}.$$ 

*Counterparty risk problem:* Calculate swap value distribution at $t > 0$ conditional on counterparty default at $t$. Note $V_t$ is a function of $r(t)$, a *path-dependent functional* so we can’t use the simple Black-Scholes method.
Counterparty risk

The essential problem is to get the distribution of $r(s)$ given that default happens (i.e. flat barrier with random starting point) is hit at $s$. Recall

$$dr(t) = (\theta(t) - \mu r(t))dt + \beta dW_t.$$  
(10)

We take the counterparty risk model developed above, i.e. the default time is $\tau^Y_0$ where $Y$ is the process

$$dY_t = \nu \sigma^2(t)dt + \sigma(t)dB_t, \quad Y_0 = A.$$  
(11)

Here $W, B$ are Brownian motions with correlation $\rho$, i.e. $E[dW dB] = \rho dt$, so we can represent $W$ as

$$W_t = \rho B_t + \tilde{\rho} \tilde{B}_t,$$

where $B, \tilde{B}$ are independent BMs and $\tilde{\rho} = \sqrt{1 - \rho^2}$.

*Calibration:* $\theta(\cdot)$ is calibrated from the swap market, $(\mu, \beta)$ from the swaption vol matrix and $\sigma(\cdot)$ from the counterparty CDS quotes.
The solution of (10) is

\[ r(s) = \alpha(s) + \int_0^s e^{-\mu(s-u)} \rho \beta dB_s + \int_0^s e^{-\lambda(s-u)} \bar{\rho} \beta d\tilde{B}_s \]  

(12)

where \( \alpha(s) \) is the deterministic function

\[ \alpha(s) = e^{-\mu s} r(0) + \int_0^s e^{-\mu(s-u)} \theta(u) du. \]

The third term in (12) is \( \Xi(s) \sim N(0, \Sigma^2(s)) \) where

\[ \Sigma^2(s) = \frac{\bar{\rho}^2 \sigma^2}{2\lambda} (1 - e^{-2\lambda s}). \]

From (11) we have

\[ dB = \frac{1}{\sigma(t)} dY - \nu \sigma(t) dt, \]

so the second term in (10) is

\[ \int_0^s e^{-\mu(s-u)} \frac{\rho \beta}{\sigma(u)} dY(u) - \int_0^s e^{-\mu(s-u)} \rho \beta \nu \sigma(u) du. \]
In summary we have

$$r(s) = \tilde{\alpha}(s) + \Xi(s) + \int_0^s e^{-\mu(s-u)} \frac{\rho \beta}{\sigma(u)} dY(u),$$

(13)

where

$$\tilde{\alpha}(s) = \alpha(s) - \int_0^s e^{-\mu(s-u)} \rho \beta \nu \sigma(u) du.$$  

If we condition on default at time $s$ then $Y_t$ satisfies the SDE

$$dY_t = \left( \frac{1}{Y_t} - \frac{Y_t}{\int_t^s \sigma^2(u) du} \right) \sigma^2(t) dt + \sigma(t) d\tilde{B}_t, \quad t \in (0, s)$$

(14)

By Monte Carlo simulation of (13),(14) we can obtain the empirical distribution of $r(s)$ at the assumed default time $s$.

Recall that in this model all zero-coupon bond values at $s$ are functions of $r(s)$. For $T_i > s$ we write

$$p_i(s, r(s)) = p(s, T_i)(r(s)).$$
Example: $s = 2.5$ years in a 5-year swap.

$\rho = 0$:

$\rho = -0.6, +0.6$
Risk measures

To a close approximation (exact at coupon dates) the swap value at time $s$ is

$$V_s = V_s(r(s)) = 1 - p_n(s, r(s)) - K \sum_{i \geq k(s)} \delta_i p_i(s, r(s))$$

The Expected Positive Exposure is

$$\text{EPE}_s = \mathbb{E}[V_s^+ | \tau_0 = s].$$

The Credit Value Adjustment is the value of compensation for losses on default, i.e.

$$\text{CVA} = \mathbb{E} \left[ e^{-\int_0^{\tau_0 \wedge T_n} r(u) du} V_{\tau_0 \wedge T_n}^+ \right]$$

We can evaluate these by Monte Carlo simulation. CVA as function of $\rho$: 

![Graph showing CVA as function of rho](image.png)
Concluding Remarks

We have developed a joint model for asset values and counterparty default risk, which enables us to estimate counterparty risk.

However there is lots more to do:

- More efficient computational methods.
- Multi-asset problems.
- Inclusion of credit assets (CDOs, ...)
- And the big one: how to get a consistent procedure for calibrating $\rho$. 
Appendix: A general (but maybe useless) representation

Suppose we have a general short rate model \( r(t) = r(t, X_t) \) where \( X_t \) is a multidimensional diffusion process driven by BM \( W \in \mathbb{R}^{n+1} \). As before we use barrier crossing by one component of \( W \), or some linear combination, as default indicator. Then we can represent \( X_t \) as the solution of an SDE in the form

\[
df(X_t) = L_0 f(X_t)dt + Z f(X_t) \circ dB^0_t + L_j f(X_t) \circ dB^j_t
\]

where \( L_0, \ldots, L_n, Z \) are vector fields, \( B^0, \ldots, B^n \) are independent BM and ‘\( \circ \)’ denotes the Stratonovich integral.

We want to replace \( B^0 \) by a Bessel bridge, say \( Y \).

We can use ideas of Doss, Sussman, Kunita to obtain a conditional representation of \( X_t \) given a sample path of \( B^0 \).
Let $\zeta_t(x) = \zeta(t, x)$ denote the flow of the vector field $Z$, i.e. the unique solution of the equation

$$
\frac{d}{dt} f(\zeta_t(x)) = Zf(\zeta_t(x)), \quad f \in C^\infty(S)
$$

$$
\zeta_0(x) = x.
$$

This is a diffeomorphism for each $t \geq 0$. Define

$$
\xi_t(x) = \zeta_{Y(t)}(x).
$$

As is easily checked, $\xi = \xi_t(x)$ is the solution of

$$
d\xi_t = Z(\xi_t) \circ dY_t
$$

and $\xi_t(\cdot)$ is almost surely a diffeomorphism for each $t > 0$. 
Now consider the equation

\[ df(\eta_t) = \xi_t^{-1} L_0 f(\eta_t) dt + \xi_t^{-1} L_j f(\eta_t) \circ dB_t^j. \quad (16) \]

This equation has a unique solution and it follows by applying the Ito formula that

\[ X_t(x) = \xi_t \circ \eta_t(x) = \zeta(Y(t), \eta_t(x)). \quad (17) \]

The representation (16), (17) describes the behaviour of \( X_t \) conditioned on \( Y \). Recall that the map \( \xi_t^{-1} \) is parametrized by \( Y \) and that \( B^0, B^j, j = 1, 2, \ldots \) are independent BM. Thus, conditional on \( B^0 \), \( \eta_t \) is a diffusion process whose differential generator is

\[ A_t^* = \xi_t^{-1} L_0 + \sum_j (\xi_t^{-1} L_j)^2 \]

and, for each \( t > 0 \), \( X_t \) is diffeomorphically related to \( \eta_t \) by equation (17).