Capital Conservation and Risk Management
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Basic background [Cherny and Madan ’09, ’10]

- **Needed:** Framework to study capital conservation, risk management and hedging in illiquid derivative markets.
  - Illiquid derivative markets as competitive counterparties creating new financial products and efficiently using liquid hedging instruments.
  - Ask and bid prices reflect the cost of holding unhedgeable risk, rather than processing, inventory or transaction costs.

- **Approach:** Convex cone $\mathcal{A}$ of acceptable cash-flows:
  \[ X \in \mathcal{A} \iff E^Q(X) \geq 0 \text{ for all } Q \in \mathcal{M} \]  
  for some convex set $\mathcal{M}$ of measures equivalent to $P$ [Artzner et. al. ’99].

- **Liquid hedging instruments:** Modeled as a vector space $\mathcal{H}$, given a set $\mathcal{R}$ of risk-neutral measures equivalent to $P$:
  \[ H \in \mathcal{H} \iff E^Q(H) = 0 \text{ for all } Q \in \mathcal{R} \]  

- **Competitive bid-ask spread:** Modeled through $\mathcal{M}$ and $\mathcal{R}$:
  \[ a(X) = \inf \{ a : a + H - X \in \mathcal{A} \text{ for some } H \in \mathcal{H} \} = \sup_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q(X) \]  
  \[ b(X) = \sup \{ b : -b - H + X \in \mathcal{A} \text{ for some } H \in \mathcal{H} \} = \inf_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q(X) \]
  Distinct, e.g., from superhedging-type approaches.
Convex cone $A$ of market-acceptable cash flows

This is a family of convex cones of random variables containing the positive orthant $L_\infty^+$ and decreasing in $x$. The value $\alpha(X)$ is then the largest number $x$ such that $X$ belongs to the level-$x$ acceptability set:

$$\alpha(X) = \sup_{x \in \mathbb{R}_+} f_x(X)$$

Thus, for a risk measure, all the positions are split in two classes: acceptable and not acceptable. In contrast, for an acceptability index we have a whole continuum of degrees of acceptability defined by the system $(A_x)_{x \in \mathbb{R}_+}$, and the index measures the degree of acceptability of a trade.

To provide a visual illustration of the relation between coherent risks, acceptability indices, acceptability sets, and acceptability systems, we present in Figure 1 an example for the case when $\Omega$ consists of two points $\omega_1, \omega_2$. Then any random variable $X$ is represented as a point $(X(\omega_1), X(\omega_2))$ on the plane. The left-hand graph illustrates the acceptability cone $A$ of a coherent risk measure $\rho$. The right-hand graph illustrates the flow of acceptability cones $A_x$ of an acceptability index $\alpha$.

To conclude this section, let us remark that Theorem 1 provides a description of indices satisfying the first four properties of the previous section. The law invariance and the second order monotonicity for acceptability indices are studied in Subsection 3.6, where it is shown that they are equivalent one to the other, and the description of corresponding indices is provided. As for the remaining two properties of the previous section, it can be shown that an acceptability index is arbitrage consistent if and only if the closure of $\bigcup_{x \in \mathbb{R}_+} \mathcal{D}_x$ coincides with $\mathcal{P}$, where $(\mathcal{D}_x)_{x \in \mathbb{R}_+}$ is the system of supporting kernels; the expectation consistency for an acceptability index is equivalent to the property $\mathcal{D}_0 = f \mathcal{P} g$. 

\[14\]
Concave distortions [Cherny and Madan ’09, ’10]

Model of market acceptable cash flows: Given distribution function $F_X(x)$,

$$X \in \mathcal{A} \iff E^Q(X) \geq 0 \text{ for all } Q \in \mathcal{M} \iff \int xd(\Psi \circ F_X)(x) \geq 0$$

where $\Psi(u)$ is a concave distribution on $[0, 1]$.

⇒ Convex set $\mathcal{M}$ is fully characterized in terms of $\Psi$ [Cherny ’06].

Density $\psi(x) := (\Psi' \circ F)(x)$ with respect to original measure $P$:

⇒ $\Psi' \circ F_X$ defines market-preferences by a "stressed" distribution that shifts probability mass towards negative cash flows.

⇒ Like utility kernels, $\Psi' \circ F_X$ can be taken to put arbitrarily large (small) mass on large negative (positive) cash flows [e.g., for MINMAXVAR $\Psi$’s]

Parametric bid and ask:

$$a(X) = \inf \{ a : a + \int xd(\Psi \circ F_{H - X})(x) \geq 0 \text{ for some } H \in \mathcal{H} \}$$

$$= \inf_{H \in \mathcal{H}} - \int xd(\Psi \circ F_{H - X})(x)$$

(3)

$$b(X) = \sup \{ b : -b + \int xd(\Psi \circ F_{X - H})(x) \geq 0 \text{ for some } H \in \mathcal{H} \}$$

$$= \sup_{H \in \mathcal{H}} \int xd(\Psi \circ F_{X - H})(x)$$

(4)
Example: Stressed densities $\Psi' \circ F_X$

![Graphs showing stressed densities](image)

**Figure 2.** (a) Extreme measure densities for $\Psi(x) = 1 - (1 - x)^3$.

(b) Extreme measure densities for $\Psi(x) = x^{1/3}$.

- **MINVAR** $[\Psi_\gamma(u) = 1 - (1 - u)^{1+\gamma}]$: implies an infinity (zero) mass at large negative (positive) cash flows values.

- **MAXVAR** $[\Psi_\gamma(u) = u^{1/(1+\gamma)}]$: implies a bounded (zero) mass at large negative (positive) cash flows values.

- **MINMAXVAR** $[\Psi_\gamma(u) = 1 - (1 - u^{1/(1+\gamma)})^{1+\gamma}]$: implies an infinity (zero) mass at large negative (positive) cash flows values.
Quantile exposures and risk charges [Carr et al. ’10]

- **Idea:** Split the price of a contingent payoff into (i) a quantile exposure and (ii) a charge for quantile risk.

- **Bid and ask prices:** Given in terms of the inverse distribution function $G_H(u)$ of a hedged cash flow $X - H$ with median $m = G_H(1/2)$:

  $$a(X) = m + \inf_{H \in \mathcal{H}} \int_0^1 [\Psi(1 - u) - \mathbb{I}(u \leq 1/2)] dG_H(u)$$

  $$b(X) = m + \sup_{H \in \mathcal{H}} \int_0^1 [\mathbb{I}(u \geq 1/2) - \Psi(u)] dG_H(u)$$

- $dG_H(u)$ is the sensitivity of the cash flow to a change in the quantile: $\Rightarrow$ It gives the risk exposure of that particular quantile under distribution $F_H(x)$.

- Over interval $dG_H(u)$, the charge for ask and bid prices is:

  $$\Psi(1 - u) - \mathbb{I}(u \leq 1/2) ; \quad \mathbb{I}(u \geq 1/2) - \Psi(u)$$

  $\Rightarrow$ Equation (5) defines the $\Psi$–dependent risk charge per unit of quantile risk exposure.

- Similar interpretations for bid-ask related quantities, like capital, profit, etc., see below.
Profit, capital and leverage [Carr et al. ’10]

- **Capital**: Cost of unwinding a position, i.e., the bis-ask spread:
  \[ k(X) = a(X) - b(X) = \int_0^1 K(u)dG(u) \]
  where \( K(u) \) is symmetric about 1/2.

- **Profit** [given fixed risk neutral probability \( P \)]:
  - Market distributes half of bid-ask spread to market participants.
  - Cash flow production cost is its risk neutral expectation.
  \[ \pi(X) := m(X) - c(X) \]
  \[ := \frac{a(X) + b(X)}{2} - E^P(X) = \int_0^1 H(u)dG(u) \]
  where \( H(u) \) is antisymmetric about 1/2.

- **Rate of return**:
  \[ \rho(X) := \pi(X)/k(X) \]

- **Scale**: Translation-invariant measure of scale of operations (associated with leverage to be granted for given capital \( k(X) \)):
  \[ \text{scale}(X) := E^P(\{|X - m(X)|\}) = \int_0^1 S(u)dG(u) \]
Profit and capital charges \([H(u), K(u)]\)

Figure 1: The profit charge on quantiles for MINMAXVAR at three stress levels of 0.1, 0.25 and 0.5.

Figure 2: Capital charges for different quantile levels for MINMAXVAR at three stress levels of 0.1, 0.25 and 0.5.
Figure 3: Graph of Capital Charges against Scale for various settings of the stress parameter in minmaxvar.
Applications

▶ Variance-swap hedging: [Illiquid markets with (skewed) VG underlying]
⇒ Standard hedge reduces bid-ask spreads and raises returns on earlier maturities.
⇒ Standard hedge produces losses on longer maturities, due to a larger unhedged cash flow risk.
⇒ A hedge minimizing first the ask and then the capital committed can avoid the lossed of the standard hedge.

▶ Call option hedging: [left skewed VG underlying]
⇒ Capital minimization is not well achieved by expected utility optimization.

▶ Delta hedging: [left skewed (VG) returns]
⇒ Under concave distortion $\Psi(u)$ downside risk is more heavily priced than upside risk.
⇒ To minimize capital, the optimal delta should be revised downwards in presence of $\Gamma$ exposure.

▶ Dynamic extensions via dynamically consistent non-linear expectations [Thm 6.1, Cohen and Elliott, ’10]:
⇒ Solution of backward stochastic difference equation with corresponding driver:

$$Y_t^j = E_t[Y_{t+1}^j] + \int_{-\infty}^{\infty} xd(\Psi \circ \Theta_t^j)(x)$$  \hspace{1cm} (6)

where $\Theta_t^j$ is the distribution function of $Y_{t+1}^j - E_t[Y_{t+1}^j]$, $j = \text{bid, ask}$. 
Comments (I)

Model of financial market as competitive capital optimizer: Aspects...

▶ General:

▶ Largely based on univariate hedging problems (because of law invariance), thus abstracting from potential portfolio dependencies (centralized vs. decentralized markets; exchanges vs. over-the-counter)?

▶ Can the approach be reconciled with demand pressure effects documented in, e.g., index and individual option markets [Garleanu et al. '09]?

▶ Concrete specifications implicitly linked to parametric assumptions on "market-preferences" via chosen distortion $\Psi(u)$ (i.e., cone $\mathcal{A}$).

⇒ How to identify $F(x)$ and $\Psi(u)$ only from cross-sectional information without parametric assumptions?

⇒ Not always clear in the draft whether this is with respect to risk-neutral or physical probabilities...

⇒ Time-series information might help to separate probabilistic cash flow features from market-driven price distortions?

▶ Definition of profits related to cash flow "replication costs" in incomplete markets; uniquely defined?

▶ Deeper interpretation of (virtual) assumption that profits are evenly redistributed in competitive markets? How could this effectively function?
Model of financial market as competitive capital optimizer: Aspects...

- Some (among many) potential applications:
  - Joint explanations of bid and ask prices of, e.g., put and call option smiles? Comparison to fit of standard approaches?
  - Time variation of bid ask spreads in terms of time variation in implied distortions:
    ⇒ Joint cross-sectional and time series study!?
    ⇒ Proxies of time-varying market fear, e.g., linked to time-varying uncertainty or uncertainty aversion!?
    ⇒ Deeper implied (possibly multivariate) liquidity-market depth proxies in terms of estimated cone of acceptable cash flows?

- Overall, very interesting framework to study a variety of questions in illiquid financial markets!
Appendix I: MINMAXVAR features [Cherny ’06]

- **MINMAXVAR as weighted Tail VAR (WVAR):**

\[
WVAR_\mu(X) = \int_{(0,1]} TVAR_\lambda \mu(d\lambda) \tag{7}
\]

given measure \(\mu\) on \((0,1]\) and tail Value at Risk \(TVAR_\lambda = -E[X|X \leq q_\lambda(X)]\).

- **Föllmer and Schied ’04:** One-to-one relation between concave distortions and measures on \((0,1]\):

\[
WVAR_\mu(X) = -\int_{(0,1]} \left( \lambda^{-1} \int_{(-\infty,q_\lambda(X)]} ydF_X(y) \right) \mu(d\lambda) \\
= -\int_{\mathbb{R}} y \left( \int_{(\mathbb{R},1]} \lambda^{-1} \mu(d\lambda) \right) dF_X(y) \\
= -\int_{\mathbb{R}} yd(\Psi_\mu \circ F_X)(y) \tag{8}
\]

where \(\Psi_\mu(u) := \int_{0}^{u} \int_{(z,1]} \lambda^{-1} \mu(d\lambda)dz\).