Libor Market Models with Stochastic Basis

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- Extension of LMM model accounting for spread-widening between discount rates and LIBOR rates

- Related papers
  - Schoenbucher (1999)
  - Mercurio (2009)
  - Mercurio (2010)

- Formulae for IRS, caps, swaptions and guideline modeling different tenors simultaneously

- Very tractable application using SABR and shifted Lognormal models

- Fit to data is extraordinarily good
The framework models OIS forward rates $F_{k}^{x}(t)$ and spreads $S_{k}^{x}(t) := L_{k}^{x}(t) - F_{k}^{x}(t)$ as independent processes.

But $F$ and $S$ are dependent by construction.

Also conditional variance of the two is likely related.

![Graph showing 6mx12m FRA - 6mx12m EONIA](image_url)
Linear dependence regression \( \Delta F = \theta_0 + \theta_1 \Delta S + \varepsilon \)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Est.</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\theta}_0 )</td>
<td>-0.001</td>
<td>0.1672</td>
</tr>
<tr>
<td>( \hat{\theta}_1 )</td>
<td>-0.971</td>
<td>0.0000</td>
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</tbody>
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Variance dependence regression \( \Delta F^2 = \eta_0 + \eta_1 \Delta S^2 + \varepsilon \)

<table>
<thead>
<tr>
<th>Coefficient</th>
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<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\eta}_0 )</td>
<td>10.4962</td>
<td>0.0000</td>
</tr>
<tr>
<td>( \hat{\eta}_1 )</td>
<td>-1.6587</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Suppose that $F$ and $L$ are both driven by the same variance factor

\[
d \ln F^x_k(t) = -\frac{1}{2} V_k(t) dt + \sqrt{V_k(t)} dZ^F_k(t)
\]

\[
d \ln S^x_k(t) = -\frac{1}{2} V_k(t) dt + \sqrt{V_k(t)} dZ^S_k(t)
\]

\[
dV_k(t) = (b + \beta V_k(t)) dt + \sqrt{\alpha V_k(t)} dZ^V_k(t).
\]

They are instantaneously uncorrelated. For fixed time $T > 0$ correlation proportional to $V$.

Note that conditional on

\[
IV_k(t, T) := \int_t^T V_k(s) ds
\]

\[
\log F^x_k(T^x_{k-1}) | IV_k(t, T^x_{k-1}) \text{ and } \log S^x_k(T^x_{k-1}) | IV_k(t, T^x_{k-1})
\]

are independently normally distributed.
■ Condition on $IV_k(t, T^x_{k-1})$ instead of $S^x_k(T^x_{k-1})$

$$C_{\text{plt}}(t, K; T^x_{k-1}, T^x_k) = \tau^x_k P_D(t, T^x_k)$$

$$\cdot \mathbb{E}^{T^x_k} \mathbb{E}^{T^x_k} \left[ \mathbb{E}^{D_k} \left[ \left( L^x_k(T^x_{k-1}) - K \right)^+ | \mathcal{F}_t \lor IV_k(t, T^x_{k-1}) \right] | \mathcal{F}_t \right]$$

■ Inner expectation is an integration against Lognormal convolution

$$f(z) := \mathbb{E}^{T^x_k} \mathbb{E}^{D_k} \left[ \left( L^x_k(T^x_{k-1}) - K \right)^+ | \mathcal{F}_t \lor IV_k(t, T^x_{k-1}) = z \right]$$

■ Denote with $g(z)$ the conditional density of $IV_k(t, T^x_{k-1}) | v_k(t)$. We need to solve

$$\int_{0}^{\infty} f(z) g(z) dz$$
Approximate the conditional density of $IV_k(T) \mid V_k(t)$ using Filipović, Mayerhofer, and Schneider (2010) likelihood expansions.

Generate affine Markov process through embedding

$$V_k(t) \rightarrow (V_k(t), \int_0^t V_k(s)ds) =: IV_k(t)$$

$$dV_k(t) = (b + \beta V_k(t))dt + \sqrt{\alpha V_k(t)}dZ^V_k(t)$$

$$dIV_k(t) = V_k(t)dt$$

This process is polynomial and polynomial moments can be computed in closed-form.

Approximate the marginal distribution of $IV_k(T) \mid V_k(t)$ through polynomial expansion in a weighted $L^2$ space.

Expansion performs very accurately.
The Picture shows the percentage deviation of the conditional expectation obtained from true density (Fourier inversion) and Filipović, Mayerhofer, and Schneider (2010) expansion.
Praise

- Easy to use LMM adapted to current economic environment
- Formulae for IRS, caps, swaptions
- Guideline for modeling different tenors simultaneously

Future Topics

- Where does the spread come from?
- How can we make the model more realistic while maintaining tractability?

Mercurio, F. (2009), Interest Rates and The Credit Crunch: New Formulas and Market Models. working paper


Schoenbucher, F. (2000), A Libor Market Model with Default Risk. working paper
Consider

\[
\begin{align*}
    d \ln F_k^x(t) &= -\frac{1}{2} V_k(t) dt + \sqrt{V_k(t)} \left( \rho dZ_k^V(t) + \sqrt{1 - \rho^2} dZ_k^F(t) \right) \\
    d \ln S_k^x(t) &= -\frac{1}{2} V_k(t) dt + \sqrt{V_k(t)} \left( \eta dZ_k^V(t) + \sqrt{1 - \eta^2} dZ_k^S(t) \right) \\
    dV_k(t) &= (b + \beta V_k(t)) dt + \sqrt{\alpha} V_k(t) dZ_k^V(t).
\end{align*}
\]

Since

\[
\int_t^T \sqrt{V_k(s)} dZ_k^V(s) = -\frac{b(T - t) + \beta IV_k(T) + V_k(t) - V_k(T)}{\sqrt{\alpha}},
\]

\[
\ln F(T) - \ln F(t) \mid V(T), IV(T) \sim N \left( -\frac{1}{2} IV(T) + \rho \int_t^T \sqrt{V(s)} dZ_k^V(s), (1 - \rho^2) IV(T) \right)
\]

By approximating $V_k(T)$, $IV_k(T) \mid V_k(t)$ we could also induce instantaneous correlation.