Robustness, Model Uncertainty and Pricing

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Motivation

- Pricing contracts in incomplete markets
- Examples:
  - Pricing very long-dated cash flows \( T \sim 30 - 100 \) years
  - Pricing long-dated equity options \( T > 5 \) years
  - Pricing pension & insurance liabilities
- Actuarial premium principles typically “ignore” financial markets
  - Actuarial pricing is “static”: price at \( t = 0 \) only
- Financial pricing considers “dynamic” pricing problem:
  - How does price evolve over time until time \( T \)?
- Financial pricing typically “ignores” unhedgeable risks
Main Ideas

- Pricing contracts in incomplete markets in a “market-consistent” way
- Use model uncertainty and ambiguity aversion as “umbrella”
  - Agent does not know the “true” drift rate of stochastic processes
  - Agent does know confidence interval for drift
  - Agent is worried about model mis-specification
  - Agent can trade in financial markets
  - Agent is “robust”; i.e. tries to maximise worst-case expected outcome
- Results:
  1. Robust agent perfectly hedges financial risks: leads to “risk-neutral” pricing
  2. Robust agent prices unhedgeable risks using a “worst case” drift
  3. Drift depends on type of liability: leads to non-linear pricing
Outline of This Talk

1. Literature Overview
2. Complete Market
3. Incomplete Market
4. Applications
Literature Overview

- **Martingale Pricing (Föllmer-Schweizer-Schied)**
  - Many possible martingale measures in incomplete market
  - Minimum variance measures
  - Quantile Hedging

- **Utility Based Pricing (Carmona-book)**
  - Specify utility function & find “utility indifference” price
  - Very hard problem to solve, except for special cases
  - “Horizon problem”: specify utility at $T$
  - “Short call” problem

- **Monetary Utility Functions (ADEH, Schachermayer, Filipović)**
  - Coherent & Convex risk measures with sign-change
  - Axiomatic approach
  - Characterise as: minimum over set of “test measures” of expectation plus penalty term
  - Construct “time-consistent” risk-measures via backward induction
Approaches not really different, only different “language”

Example:
- Minimum entropy martingale measure $\iff$
- Exponential utility indifference price $\iff$
- Convex risk measure with entropy penalty term

Model Uncertainty & Robustness (Hansen-Sargent book)
- Choose worst-case drift within “confidence interval” $\iff$
- Coherent risk measure with given set of “test measures”

Model Uncertainty gives economic meaning to “set of test measures”
- Econometric estimation of parameters gives confidence intervals
- Disagreement between panel of experts
Complete Market

Tree Setup

Suppose we have a stock price $S$ with return process $x = \ln S$:

$$dx = m\,dt + \sigma\,dW_x,$$

Discretisation in binomial tree:

$$x(t + \Delta t) = x(t) + \begin{cases} 
+\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 + \frac{m}{\sigma}\sqrt{\Delta t}) \\
-\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 - \frac{m}{\sigma}\sqrt{\Delta t})
\end{cases}$$

Model uncertainty as $m \in [m_L, m_H]$. This implies that prob. in

$$[p_L = \frac{1}{2}(1 + \frac{m_L}{\sigma}\sqrt{\Delta t}), p_H = \frac{1}{2}(1 + \frac{m_H}{\sigma}\sqrt{\Delta t})].$$
Derivative Contract

Suppose we have a derivative contract with value \( f(t + \Delta t, x(t + \Delta t)) \) at time \( t + \Delta t \).

Taylor expansion & binomial tree:

\[
f_0 = f_1 + \begin{cases} 
  +f_x \sigma \sqrt{\Delta t} + \frac{1}{2} f_{xx} \sigma^2 \Delta t & \text{with prob. } \frac{1}{2} \left(1 + \frac{m}{\sigma} \sqrt{\Delta t}\right) \\
  -f_x \sigma \sqrt{\Delta t} + \frac{1}{2} f_{xx} \sigma^2 \Delta t & \text{with prob. } \frac{1}{2} \left(1 - \frac{m}{\sigma} \sqrt{\Delta t}\right),
\end{cases}
\]

where \( f_0 := f(t, x(t)) \), \( f_1 := f(t + \Delta t, x(t)) \), \( f_x := \partial f(t, x(t))/\partial x \) and \( f_{xx} := \partial^2 f(t, x(t))/\partial x^2 \).
Discounted Expectation

Rational agent calculates discounted expectation with no model uncertainty:

\[ e^{-r\Delta t}E_t[f(t + \Delta t, x(t + \Delta t))] = e^{-r\Delta t}(f_1 + (f_xm + \frac{1}{2}f_{xx}\sigma^2)\Delta t) \]

Limit for \( \Delta t \downarrow 0 \) leads to pde (Feynman-Kač formula):

\[ f_t + f_xm + \frac{1}{2}f_{xx}\sigma^2 - rf = 0 \]

Note: no “risk-neutral valuation”, drift \( m \) is real-world drift.
Valuation with Model Uncertainty

Given uncertainty about drift $m$, “robust” rational agent will consider “worst case” discounted certainty equivalent:

$$\min_{m \in [m_L, m_H]} e^{-r\Delta t} \mathbb{E}_t^m [f(t + \Delta t, x(t + \Delta t))]$$

Explicit solution for binomial tree:

$$\begin{cases} 
  e^{-r\Delta t} \left( f_1 + (f_x m_L + \frac{1}{2} f_{xx} \sigma^2) \Delta t \right) & \text{if } f_x > 0 \\
  e^{-r\Delta t} \left( f_1 + \left( \frac{1}{2} f_{xx} \sigma^2 \right) \Delta t \right) & \text{if } f_x = 0 \\
  e^{-r\Delta t} \left( f_1 + (f_x m_H + \frac{1}{2} f_{xx} \sigma^2) \Delta t \right) & \text{if } f_x < 0.
\end{cases}$$

$\Delta t \downarrow 0$ leads to “semi-linear” pde: $f_t + f_x \bar{m} - |f_x| h + \frac{1}{2} f_{xx} \sigma^2 - rf = 0$ with $\bar{m} = \frac{1}{2} (m_H + m_L)$ and $h = \frac{1}{2} (m_H - m_L)$.

- Actuarial notion of prudence (not “risk-neutral”)
- Coherent time-consistent risk-measure with “$Q \in [p_L, p_H]$”
- Solution exists & unique: theory of BSDE’s
Suppose that rational agent can trade in the share price $S$.

Buy $\theta/S(t)$ shares at $t$, financed by borrowing an amount $\theta$ from the bank account $B$.

At time $t + \Delta t$, net position has value $(e^{x(t + \Delta t) - x(t)} - e^{r\Delta t})\theta$.

Find optimal amount $\theta$ that maximises worst-case expectation:

$$\max_{\theta} \min_{m \in [m_L, m_H]} e^{-r\Delta t} \left( f_1 + (f_x m + \frac{1}{2} f_{xx} \sigma^2 + (m + \frac{1}{2} \sigma^2 - r)\theta) \Delta t \right)$$

Two-player game: “mother nature” vs. agent.
Optimum \((m, \theta)\) depends on sign of partial deriv’s:

\[
\frac{\partial}{\partial \theta} : e^{-r\Delta t}(m + \frac{1}{2}\sigma^2 - r)\Delta t \quad \frac{\partial}{\partial m} : e^{-r\Delta t}(f_x + \theta)\sigma\Delta t
\]

Optimal choice for \(m\) depends on sign of \(\frac{\partial}{\partial m}\)

- Suppose agent chooses \(\theta\) such that \(f_x + \theta > 0\),
- then “mother nature” chooses \(m = m_L\).
- If \(m_L < r - \frac{1}{2}\sigma^2\), then agent can improve by lowering \(\theta\),
- until \(\theta = -f_x\).
- Similar argument for \(f_x + \theta < 0\), if \(m_H > r - \frac{1}{2}\sigma^2\)
Conclusion: optimal choice for agent is $\theta^* = -f_x$.

- But this is delta-hedge for derivative $f$
- Leads to risk-neutral valuation!

How severe is restriction $m_L < r - \frac{1}{2}\sigma^2$? (Equivalent to $\mu_L < r$)

Thought-experiment:

- Suppose 25 years of data
- $\hat{\mu} = 8\%$, $\sigma = 15\%$
- Then std.err. of estimate for $\hat{\mu}$ is $\sigma/\sqrt{25} = 15\%/5 = 3\%$
- So, 95%-conf.intv. for $\hat{\mu}$ is $8\% \pm 6\%$.
- Need about $(2 \times 15/(8 - 4))^2 \approx 50$ years of data to distinguish between 8\% and 4\% if $\sigma = 15\%$!
Tree Setup

Introduce additional non-traded process $y$:

$$dy = a\, dt + b\, dW_y,$$

with $dW_x\, dW_y = \rho\, dt$.

“Quadrinomial” discretisation:

<table>
<thead>
<tr>
<th>State:</th>
<th>$y + b\sqrt{\Delta t}$</th>
<th>$y - b\sqrt{\Delta t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + \sigma\sqrt{\Delta t}$</td>
<td>$p_{++} = \left(\frac{(1+\rho)+(\frac{m}{\sigma}+\frac{a}{b})\sqrt{\Delta t}}{4}\right)$</td>
<td>$p_{+-} = \left(\frac{(1-\rho)+(\frac{m}{\sigma}-\frac{a}{b})\sqrt{\Delta t}}{4}\right)$</td>
</tr>
<tr>
<td>$x - \sigma\sqrt{\Delta t}$</td>
<td>$p_{-+} = \left(\frac{(1-\rho)-(\frac{m}{\sigma}-\frac{a}{b})\sqrt{\Delta t}}{4}\right)$</td>
<td>$p_{--} = \left(\frac{(1+\rho)-(\frac{m}{\sigma}+\frac{a}{b})\sqrt{\Delta t}}{4}\right)$</td>
</tr>
</tbody>
</table>
Model Uncertainty

Model uncertainty in both $m$ and $a$.

Additional notation:

$$\mu := \begin{pmatrix} m \\ a \end{pmatrix}, \quad \Sigma := \begin{pmatrix} \sigma^2 & \rho \sigma b \\ \rho \sigma b & b^2 \end{pmatrix}.$$  

Describe uncertainty set as ellipsoid:

$$\mathcal{K} := \{ \mu_0 + \varepsilon \mid \varepsilon' \Sigma^{-1} \varepsilon \leq k^2 \}.$$  

Motivated by shape of confidence interval of estimator $\hat{\mu}$. 
Ellipsoid Uncertainty Set
Robust Optimisation Problem

Robust rational agent solves the following optimisation problem

$$\max_{\theta} \min_{\mu \in \mathcal{K}} e^{-r\Delta t} \left( f_1 + (f'_x \mu + \theta(e'_1 \mu - r + \frac{1}{2}\sigma^2)) + \frac{1}{2} \text{tr}(f_{xx} \Sigma) \right) \Delta t,$$

where $f_x$ denotes gradient $(f_x, f_y)'$ and $e_1$ denotes the vector $(1, 0)'$.

Reformulate & simplify problem

$$\max_{\theta} \min_{\varepsilon} \theta q + \varepsilon'(f_x + \theta e_1)$$

s.t. $\varepsilon' \Sigma^{-1} \varepsilon \leq k^2$.

with $q = (e'_1 \mu_0 - r + \frac{1}{2}\sigma^2)$. 

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Optimal Response for Mother Nature

Two-player game: agent vs. “mother nature”

Worst-case choice for “mother nature” given any \( \theta \) is “opposite direction” of vector \( (f_x + \theta e_1) \):

\[
\varepsilon^* := - \left( \frac{k}{\sqrt{(f_x + \theta e_1)'\Sigma(f_x + \theta e_1)}} \right) \Sigma(f_x + \theta e_1).
\]

If we use this value for \( \varepsilon^* \) we obtain the reduced optimisation problem for the agent:

\[
\max_{\theta} \quad \theta q - k \sqrt{(f_x + \theta e_1)'\Sigma(f_x + \theta e_1)}.
\]

Maximise expected excess return \( \theta q \) minus \( k \) times st.dev. of total portfolio.
Optimal Response for Agent

Solution to reduced optimisation problem for agent:

\[ \theta^* := - \left( f_x + \frac{b \rho}{\sigma} f_y \right) + \frac{q/\sigma}{\sqrt{k^2 - (q/\sigma)^2}} \frac{b \sqrt{1 - \rho^2}}{\sigma} |f_y|. \]

Note, switch of notation: back to scalar expressions \( f_x \) and \( f_y \)!

Nice economic interpretation:

- Left term is “best possible” hedge
- Right term is “speculative” position, which is product of:
  - “Market confidence factor”
  - Residual unhedgeable risk
Agent’s Valuation of Contract

If we substitute optimal $\varepsilon^*$ and $\theta^*$ into original expectation, we obtain “semi-linear” pde

$$f_t + f_x(r - \frac{1}{2}\sigma^2) + f_y a^* + \frac{1}{2}\sigma^2 f_{xx} + \rho \sigma b f_{xy} + \frac{1}{2} b^2 f_{yy} - rf = 0,$$

where the drift term $a^*$ for the insurance process is given by

$$a^* = \left( a_0 - q \frac{\rho b}{\sigma} \right) + b \sqrt{1 - \rho^2} \cdot \begin{cases} 
- \sqrt{k^2 - (q/\sigma)^2} & \text{for } f_y > 0, \\
+ \sqrt{k^2 - (q/\sigma)^2} & \text{for } f_y < 0.
\end{cases}$$

Again, nice economic interpretation for $a^*$. 
Agent’s Valuation of Contract – Graphical

“Inf-convolution” of probability measures (Barrieu & El Karoui)
Generalisation to $N$ Risk-Drivers

Suppose we have an $N$-dim vector $x$ of risk-processes with covar matrix $\Sigma$ and uncertainty in mean $\mu$ given by $\mathcal{K} := \{\mu_0 + \varepsilon | \varepsilon'\Sigma^{-1}\varepsilon \leq k^2\}$.

Suppose we can trade in $J < N$ (linear combinations of) assets. We can define a $(N \times J)$ hedge-matrix $H$.

Optimal hedge $\theta^*$ for agent is $\theta^* = (H'\Sigma H)^{-1}(H'\Sigma(-f_x) + \alpha H'q)$ with

$$\alpha = \sqrt{\frac{f'_x (\Sigma - \Sigma H(H'\Sigma H)^{-1}H'\Sigma) f_x}{k^2 - q'H(H'\Sigma H)^{-1}H'q}}$$

This leads to “semi-linear” pricing pde:

$$f_t + (r + q'(I - H(H'\Sigma H)^{-1}H'\Sigma))f_x + \frac{1}{2}\text{tr}(\Sigma f_{xx}) + (\sqrt{k^2 - q'H(H'\Sigma H)^{-1}H'q}) \sqrt{f'_x (\Sigma - \Sigma H(H'\Sigma H)^{-1}H'\Sigma) f_x} - rf = 0$$

Solution exists & unique: BSDE theory
Applications

- Pricing long-dated cash flows with interest rate risk.
  - $N$ cash flows and only $J$ bonds traded
- Pricing LT cash flow with equity & int.rate risk.
- Pricing cash flows with mortality risk.