RISK MANAGEMENT IN THE ENERGY MARKETS

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Talk Based on Three Papers

1. **R.C., and Y. Sun**: Implied and Local Correlations from Spread Options (July 2011)
2. **R.C., M. Coulon, and D. Schwarz**: A Structural Model for Electricity Prices (Oct. 2011)
3. **R.C., and J. Hinz**: Least Squares Monte Carlo for Control Problems with Convex Value Functions (Oct. 2011)
Commodities Forward Markets

- Forward curve is the **Basic Data**
- **Backwardation** / **Contango** ⇒ Theory of **Convenience Yield**

**In the Case of Power** several obstructions

- Cannot store the physical commodity
- Delivery **over** a period \([T_1, T_2]\) (**Benth**)
- Does the forward price converge as the time to maturity goes to 0?

**Mathematical spot?**

\[
S_t = \lim_{T \downarrow t} F(t, T)
\]

**Sparse Forward Data**

- Lack of **transparency** (manipulated indexes)
- Poor (or lack of) **reporting** by fear of law suits (**CCRO** white paper)
Mean Reversion toward the cost of production

Reduced Form Models
  - Nonlinear effects (exponential $OU^2$)
  - Jumps (Geman-Roncoroni, Benth, Cartea, Meyer-Brandis, ...)

Structural Models
  - Inelastic Demand
  - The Supply Stack

Barlow (based on merit order graph)
  - $s_t(x)$ supply at time $t$ when power price is $x$
  - $d_t(x)$ demand at time $t$ when power price is $x$

Power price at time $t$ is number $P_t$ such that

$$s_t(P_t) = d_t(P_t)$$
Example of a merit graph (Alberta Power Pool, courtesy M. Barlow)
Monte Carlo Sample from Barlow’s Spot Model (courtesy M. Barlow)
Consider the case of **PJM** (Pennsylvania - New Jersey - Maryland)

- Over 3,000 nodes in the transmission network
- Each day, and for each node
  - Real time prices
  - Day-ahead prices
  - Hour by hour load prediction for the following day

- **Historical prices**
  - In 2003 over 100,000 instances of **NEGATIVE PRICES**
    - Geographic clusters
    - Time of the year (**shoulder months**)  
    - Time of the day (**night**)  

- **Possible Explanations**
  - Load miss-predicted
  - High temperature volatility
MODELING THE DEMAND: LOAD / TEMPERATURE

Daily Load versus Daily Temperature (PJM)
MORE STRUCTURAL MODELS FOR POWER

Alternatives to reduced-form and equilibrium models

▶ Choice of Factors: Demand, Fuel Prices, Outages, etc.
▶ Choice of Function: $P_t = B(t, D_t, G_t, \ldots)$ to map to spot power.
▶ Calibration

Examples:

▶ Barlow (2002): $P_t = B(D_t) = (1 + \alpha_c D_t)^{1/\alpha_c}$
▶ Burger et al. (2004), Cartea et al. (2007): $P_t = B(t, D_t, \xi_t)$
▶ Pirrong- Jermakyan (2005): $P_t = B(D_t, G_t) = G_t f(D_t)$

Others include: Eydeland & Wolyniec (2003), Davison et. al. (2002), Cartea-Figueroa-Geman (2009), Aid et. al. (2009, 2011)

**Challenge**: Overlapping fuels in many markets!
The bid stack function

- Day-ahead generator bids arranged by price to form the bid stack
- Spot price $P_t$ is highest bid needed to match demand $D_t$
AN ALTERNATIVE PERSPECTIVE

- Can look at bid stack as a histogram of bids
- Merit order is often visible through clusters of bids

PJM sample bid histogram

- Natural gas
- Coal
- Nuclear (+ a few higher bids in tail)
Distribution-based Bid Stack Model

Coulon-Howison (2009)
- Fuel types $i = 1, \ldots, N$
- $F_1(x), \ldots, F_N(x)$ proportions of bids below $x$
- Weights $w_1, \ldots, w_N$ (observable percentage of total capacity $\bar{\xi}$ in the market).
- Assume $0 < D_t < \bar{\xi}$. (demand cannot exceed max capacity)
- Then the spot power price $P_t$ solves:

$$\sum_{i=1}^{N} w_i F_i(P_t) = D_t / \bar{\xi}$$

- The bid stack function is the quantile function of the distribution of bids.
- Extensions
  - $\bar{\xi}$ replaced by a process $\xi_t$ for capacity available, or alternatively $\xi_t = D_t + M_t$ where $M_t$ is reserve margin available.
  - Two-parameter distributions for bids (location $m_i$, scale $s_i$) such as Gaussian, Logistic, Cauchy, Weibull.
R.C - M.Coulon - D. Schwarz (2011)

- **Fact**: Multi-fuel case: no explicit expressions even for spot or forward.
- **Alternative**: Exchange *flexibility* in the stack for *closed-form* expressions for forwards, options, etc.
- **Key assumption**: within each fuel type, heat rate differences lead to *exponential* bid stacks. (multiplicative in fuel price)
  - Example: Two Fuel Case (coal and natural gas)
    - Capacities $\bar{\xi}_c$ and $\bar{\xi}_g$.
    - Aggregation of bids to get ‘sub bid stacks’:
      \[
      b_c(D) = C_t e^{k_c + m_c D}, \text{ for } 0 \leq D \leq \bar{\xi}_c
      \]
      \[
      b_g(D) = G_t e^{k_g + m_g D}, \text{ for } 0 \leq D \leq \bar{\xi}_g
      \]
A schematic of individual fuel bid curves and the resulting market bid stack for $l := \{1, 2, 3\}$, $q := \bar{q}$. Fuel bid curves $b_i$ (left), Market bid stack $b$ (right)
The total market bid stack (as a function of demand) is given by:

$$B(D) = (b_c^{-1} + b_g^{-1})^{-1}(D), \quad \text{for } 0 \leq D \leq \bar{\xi} = \bar{\xi}^c + \bar{\xi}^g$$

Hence, the result is **piecewise exponential**, although the precise form depends on ordering of start and endpoints of coal and gas stacks.

For example, if $C_t e^{k_c} < G_t e^{k_g} < C_t e^{k_c + m_c \bar{\xi}^c} < G_t e^{k_g + m_g \bar{\xi}^g}$ (coal below gas but some overlap), then spot price $P_t$ has three regions:

$$P_t(D, C_t, G_t) = \begin{cases} 
    b_c(D) = C_t e^{k_c + m_c D} & \text{for } 0 \leq D \leq D_1 \\
    C_t^{\alpha_c} G_t^{\alpha_g} e^{\beta + \gamma D} & \text{for } D_1 \leq D \leq D_2 \\
    b_g(D - \bar{\xi}^c) = G_t e^{k_g + m_g (D - \bar{\xi}^c)} & \text{for } D_2 \leq D \leq \bar{\xi} 
\end{cases}$$

$$\alpha_c = \frac{m_g}{m_c + m_g}, \quad \alpha_g = 1 - \alpha_c, \quad \beta = \frac{k_c m_g + k_g m_c}{m_c + m_g}, \quad \gamma = \frac{m_c m_g}{m_c + m_g},$$

and with $D_1 = \frac{1}{m_c} (\log(G_t/C_t) + k_g - k_c)$, and $D_2$ similar.
Power spot price $P_t$ must be given by one of the following five expressions:

<table>
<thead>
<tr>
<th>$P_t$</th>
<th>Criteria</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_c(D) = C_t e^{k_c + m_c D}$</td>
<td>$b_c(D) &lt; b_g(0)$</td>
<td>Coal sets price; No gas used</td>
</tr>
<tr>
<td>$b_g(D) = G_t e^{k_g + m_g D}$</td>
<td>$b_g(D) &lt; b_c(0)$</td>
<td>Gas sets price; No coal used</td>
</tr>
<tr>
<td>$b_c(D - \tilde{\xi}_g) = C_t e^{k_c + m_c (D - \tilde{\xi}_g)}$</td>
<td>$b_g(\tilde{\xi}_g) &lt; b_c(D - \tilde{\xi}_g)$</td>
<td>Coal sets price; All gas used</td>
</tr>
<tr>
<td>$b_g(D - \tilde{\xi}_c) = G_t e^{k_g + m_g (D - \tilde{\xi}_c)}$</td>
<td>$b_c(\tilde{\xi}_c) &lt; b_g(D - \tilde{\xi}_c)$</td>
<td>Gas sets price; All coal used</td>
</tr>
<tr>
<td>$b_{cg}(D) = C_t^{\alpha_C} G_t^{\alpha_G} e^{\beta + \gamma D}$</td>
<td>otherwise</td>
<td>Both set price (overlap region)</td>
</tr>
</tbody>
</table>

Note that this can be extended easily to more than two fuels (and still piecewise exponential in $D$). However, for $n$ fuels, number of cases is

$$\sum_{i=1}^{n} \binom{n}{i} \left[ \sum_{j=0}^{n-i} \binom{n-i}{j} \right].$$

(For $n = 3$, we have 19 cases, for $n = 6$, we have 665 cases!)
Alternatively, depicting power price $P_t$ as a function of $G_t$ (or similarly $C_t$) leads to three different demand ‘regimes’ (Case of $\bar{\xi}_c > \bar{\xi}_g$ plotted below):

1. **High Demand:** $D > \bar{\xi}_c$ (i.e., $D > \max(\bar{\xi}_c, \bar{\xi}_g)$)
2. **Medium Demand:** $\bar{\xi}_g < D < \bar{\xi}_c$
3. **Low Demand:** $D < \bar{\xi}_g$ (i.e., $D < \min(\bar{\xi}_c, \bar{\xi}_g)$)

Quadrilateral in middle of plot represents region of coal and gas price overlap (i.e., both generators at margin, setting price).
Summary

- **three** regimes for demand (low, medium, high)
- **three** cases (fuel price dependent) for each regime.
- For each regime, spot prices have a convenient form, e.g. for low $D$,

\[
P_{t}^{low} = C_t e^{\lambda^c(D)} \mathbb{I} \{ G_t > C_t e^{\lambda^c(D)} - \lambda^g(0) \} + G_t e^{\lambda^g(D)} \mathbb{I} \{ G_t < C_t e^{\lambda^c(0)} - \lambda^g(D) \} +
\]

\[
(C_t)^{\alpha^c} (G_t)^{\alpha^g} e^{\beta + \gamma D} \mathbb{I} \{ C_t e^{\lambda^c(0)} - \lambda^g(D) < G_t < C_t e^{\lambda^c(D)} - \lambda^g(0) \},
\]

where $\lambda^i(x) = k_i + m_i x$ for $i \in \{c, g\}$ and $x \in [0, \bar{\xi}^i]$.

- **Log-normal** case (at fixed maturity) $T$,

\[
\left( \begin{array}{c} \log C_T \\ \log G_T \end{array} \right) \sim N \left( \left( \begin{array}{c} \mu_c \\ \mu_g \end{array} \right), \left( \begin{array}{cc} \sigma_c^2 & \rho \sigma_c \sigma_g \\ \rho \sigma_c \sigma_g & \sigma_g^2 \end{array} \right) \right)
\]
For **FIXED** demand $D$, need formula for:

$$E^Q \left[ \tilde{a}_0 C_t \tilde{a}_1 G_t \mathbb{I}_{\{\tilde{b}_0 C_t \tilde{b}_1 G_t < 1\}} \right] = E \left[ \left( e^{a_0 + a_1 X + a_2 Y} \right) \mathbb{I}_{\{b_0 + b_1 X + b_2 Y < 0\}} \right]$$

for (correlated) jointly Gaussians $X$ and $Y$.

- We use:

  $$\int_{-\infty}^{\infty} e^{cx} \Phi \left( \frac{a + bx}{d} \right) e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} \, dx = e^{\frac{1}{2}c^2} \Phi \left( \frac{a + bc}{\sqrt{b^2 + d^2}} \right)$$

- and

  $$\int_{-\infty}^{h} e^{cx} \Phi \left( \frac{a + bx}{d} \right) e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} \, dx = e^{\frac{1}{2}c^2} \Phi_2 \left( h - c, \frac{a + bc}{\sqrt{b^2 + d^2}} ; \frac{-b}{\sqrt{b^2 + d^2}} \right)$$

where $\Phi(z)$ and $\Phi_2(z_1, z_2, \rho_{12})$ are the univariate and bivariate standard Gaussian cdf.
Example: **low D regime**

\[
F_{t}^{\text{low}} = b_c(D, F_t^c) \Phi \left( \frac{R_c(D, 0)}{\sigma} \right) + b_g(D, F_t^g) \Phi \left( \frac{R_g(D, 0)}{\sigma} \right) +
\]

\[
b_{cg}(D, F_t^c, F_t^g) e^{-\frac{1}{2} \alpha_c \alpha_g \sigma^2} \left[ 1 - \sum_{i \in I} \Phi \left( \frac{R_i(D, 0) + \alpha_j \sigma^2}{\sigma} \right) \right].
\]

where \( I = \{ c, g \} \), \( j = I \setminus \{ i \} \) and

\[
\sigma^2 = \sigma_c^2 - 2 \rho \sigma_c \sigma_g + \sigma_g^2,
\]

\[
R_i(\xi_i, \xi_j) = k_j + m_j \xi_j - k_i - m_i \xi_i + \log(F_i^j) - \log(F_t^i) - \frac{1}{2} \sigma^2.
\]

Similar expressions exist for \( F_{t}^{\text{mid}} \) and \( F_{t}^{\text{high}} \), the other regions.

\( D \) enters **linearly** inside cdf’s \( \Phi(\cdot) \) and in **exponential** outside.
Exponential Stacks - Random Demand

- Demand $D$ is random but **independent** (of fuels). Then integrate (or sum) over demand distribution $f(x)$:

$$
F_t^T = \int_0^\xi F_{t,\text{low}}(x)f(x)dx + \int_{\xi}^{\xi_c} F_{t,\text{med}}(x)f(x)dx + \int_{\xi_c}^{\xi} F_{t,\text{high}}(x)f(x)dx
$$

- Example: **Capped Gaussian** demand:

$$
D_t = \max \left( 0, \min(\xi, \tilde{D}_t) \right) \quad \text{where} \quad \tilde{D} \sim N(\mu_z, \sigma_z^2)
$$

- Again, closed form formulae for $F_t^T$ (though rather involved)

- Using notation:

$$
\Phi_2^{2 \times 1} \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y; \rho \right) = \Phi_2(x_1, y; \rho) - \Phi_2(x_2, y; \rho)
$$
**RANDOM DEMAND - FORWARD PRICES**

**T-Forward power price at time** \( t < T \)

\[
F^p_t(T) = \phi\left(\frac{-\mu_Z}{\sigma_Z}\right) \sum_{i \in I} b_i(0, F^i_t) \phi\left(\frac{R_i(0, 0)}{\sigma}\right) + \phi\left(\frac{\mu_Z - \bar{\varepsilon}_i}{\sigma_Z}\right) \sum_{i \in I} b_i(\bar{\varepsilon}_i, F^i_t) \phi\left(\frac{-R_i(\bar{\varepsilon}_i, \bar{\varepsilon}_i)}{\sigma}\right) + \\
\sum_{i \in I} b_i(\mu_Z, F^i_t) e^\frac{1}{2} m_i^2 \sigma_i^2 \phi_2^{2 \times 1} \left[ \begin{array}{c}
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - m_i \sigma_Z \\
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - m_i \sigma_Z \\
\end{array} \right] \cdot \frac{R_i(\mu_Z, 0) - m_i^2 \sigma_i^2}{\sigma_i, z} ; m_i \sigma_Z + \\
\sum_{i \in I} b_i(\mu_Z - \bar{\varepsilon}_i, F^i_t) e^\frac{1}{2} m_i^2 \sigma_i^2 \phi_2^{2 \times 1} \left[ \begin{array}{c}
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - m_i \sigma_Z \\
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - m_i \sigma_Z \\
\end{array} \right] \cdot \frac{R_i(\mu_Z - \bar{\varepsilon}_i, \bar{\varepsilon}_i) + m_i^2 \sigma_i^2}{\sigma_i, z} ; -m_i \sigma_Z + \\
(F^C_t)^\alpha c (F^g_t)^\alpha g e^\eta \left\{ - \sum_{i \in I} \phi_2^{2 \times 1} \left[ \begin{array}{c}
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - \gamma \sigma_Z \\
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - \gamma \sigma_Z \\
\end{array} \right] \cdot \frac{R_i(\mu_Z, 0) + \alpha_j \sigma_i^2 - \gamma m_i^2 \sigma_i^2}{\sigma_i, z} ; m_i \sigma_Z \right\} + \\
\sum_{i \in I} \phi_2^{2 \times 1} \left[ \begin{array}{c}
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - \gamma \sigma_Z \\
\frac{\bar{\varepsilon}_i - \mu_Z}{\sigma_Z} - \gamma \sigma_Z \\
\end{array} \right] \cdot \frac{R_i(\mu_Z - \bar{\varepsilon}_i, \bar{\varepsilon}_i) + \alpha_j \sigma_i^2 + \gamma m_i \sigma_i^2}{\sigma_i, z} ; -m_i \sigma_Z \right\} - \xi + \xi_x + \xi_y \right\}.
\]

where \( x := \text{argmax}_{i \in I}(\bar{\varepsilon}_i) \), \( y := \text{argmin}_{i \in I}(\bar{\varepsilon}_i) \), \( \sigma_{i,z}^2 = m_i^2 \sigma_z^2 + \sigma^2 \) for \( i \in I \) and \( \eta = \beta + \gamma \mu_Z + \frac{1}{2} \gamma^2 \sigma_z^2 + \frac{1}{2} \alpha_c \alpha g \sigma Z^2. \)
THE IMPORTANCE OF SPREAD OPTIONS

European Call written on

- the **Difference** between **two** Underlying Interests
- a **Linear Combination** of **several** Underlying Interests
CALENDAR SPREAD OPTIONS

- Single Commodity at two different times
  \[ \mathbb{E}\{ (I(T_2) - I(T_1) - K)^+ \} \]
- Mathematically easier (only one underlier)

- **Amaranth** largest (and fatal) positions
  - Shoulder Natural Gas Spread (play on inventories)
  - Long March Gas / Short April Gas
    - Depletion stops in March / injection starts in April
    - Can be fatal: *widow maker spread*
There is a long injection season from the spring through the fall when natural gas is injected and stored in caverns for use during the long winter to meet the higher residential demand, as in Figure 2.1. The figure illustrates the U.S. Department of Energy’s total (lower 48 states) working underground storage for natural gas inventories over 2006. Inventories stop being drawn down in March and begin to rise in April. As we will see in Section 2.1.3.2, the summer and fall futures contracts, when storage is rising, trade at a discount to the winter contracts, when storage peaks and levels off. Thus, the markets provide a return for storing natural gas. A storage operator can purchase summer futures and sell winter futures, the difference being the return for storage. At maturity of the summer contract, the storage owner can move the delivered physical gas into storage and release it when the winter contract matures. Storage is worth more if such spread bets are steep between near and far months.

2.1.3 Risk Management Instruments

Futures and forward contracts, swaps, spreads and options are the most standard tools for speculation and risk management in the natural gas market. Commodities market

![U.S. Natural Gas Inventories 2005-6](image-url)

**Weekly Storage in Billion Cubic Feet**

**U.S. Natural Gas Inventories 2005-6**

- **12/24/05**
- **2/12/06**
- **4/3/06**
- **5/23/06**
- **7/12/06**
- **8/31/06**
- **10/20/06**
- **12/9/06**
November 2006 bets were particularly large compared to the rest, as Amaranth accumulated the largest ever long position in the November futures contract in the month preceding its downfall. Regarding the Fund's overall strategy, Burton and Strasburg (2006a) write that Amaranth was generally long winter contracts and short summer and fall ones, a winning bet since 2004. Other sources affirm that Amaranth was long the far-end of the curve and short the front-end, and their positions lost value when far-forward gas contracts fell more than near-term contracts did in September 2006.

From these bets, Amaranth believed a stormy and exceptionally cold winter in 2006 would result in excess usage of natural gas in the winter and a shortage in March of the following year. Higher demand would result in a possible stockout by the end of February and higher March prices. Yet April prices would fall as supply increases at the start of the injection season. In this scenario, there is theoretically no ceiling on how much the price of the March contract can rise relative to the rest of the curve. Fischer (2006), natural gas trader at Chicago-based hedge fund Citadel Investment Group, believes Amaranth bet on similar hurricane patterns in the previous two years. As a result, the extreme event that hurt Amaranth was that nothing happened—there was no Hurricane Katrina or similar.

FIGURE 3.1: Natural Gas March-April Contract Spread Evolution

![Shoulder Month Spread](image.png)
CROSS COMMODITY SPREAD OPTIONS

- **Crush Spread**
  - between Soybean and soybean products (meal & oil)

- **Crack Spread**
  - gasoline crack spread between Crude and Unleaded
  - heating oil crack spread between Crude and HO

- **(Dirty) Spark Spread**
  - between price of 1 MWhe of Electric Power and cost of Natural Gas needed to produce it
  \[
  S_t = F_E(t) - H_{eff} F_G(t)
  \]

- **(Dirty) Dark Spread**
  - with Coal instead of Natural Gas
  \[
  S_t = F_E(t) - H_{eff} F_C(t)
  \]

- **(Clean) Spark Spread**
  - including the cost of CO₂ Emissions
  \[
  S_t = F_E(t) - H_{eff} F_G(t) - e_G A_t
  \]

`H_{eff}` **Heat Rate** of the plant
Real Option Approach

- Lifetime of the plant \([T_1, T_2]\)
- \(C\) capacity of the plant (in MWh)
- \(H\) heat rate of the plant (in MMBtu/MWh)
- \(P_t\) price of power on day \(t\)
- \(G_t\) price of fuel (gas) on day \(t\)
- \(K\) fixed Operating Costs
- Value of the Plant (ORACLE)

\[
C \sum_{t=T_1}^{T_2} e^{-rt} \mathbb{E}\{(P_t - HG_t - K)^+\}
\]

String of Spark Spread Options
(Flash Back)

The Calpine - Morgan Stanley Deal

- Calpine needs to refinance USD 8 MM by November 2004
- **Jan. 2004**: Deutsche Bank: no traction on the offering
- **Feb. 2004**: *The Street* thinks Calpine is ”heading South”
- **March 2004**: Morgan Stanley offers a (complex) structured deal
  - A strip of spark spread options on 14 Calpine plants
  - A similar bond offering
- **How were the options priced?**
  - By Morgan Stanley?
  - By Calpine?
CALPINE DEBT

![Bar chart showing Calpine Debt from 2005 to 2027](chart.png)
CALPINE DEBT WITH DEUTSCHE BANK FINANCING

Debt Distribution for Calpine with Deutsche Bank Refinancing

---|---|---|---|---|---|---|---|---|---|---
Debt ($ Millions) | 503 | 2655 | 1825 | 3654 | 900 | 363 | 200 | 900 | |
CALPINE DEBT WITH MORGAN STANLEY FINANCING

Debt Distribution for Calpine with Morgan Stanley Refinancing

Year
Debt ($ Millions)
250 503 926 805 3180 3959 280 900 1618 363 200 900

4500
4000
3500
3000
2500
2000
1500
1000
500
0
A Possible Model

Assume that Calpine owns only one plant

MS guarantees its spark spread will be at least $\kappa$ for $M$ years

Approach à la Leland’s Theory of the Value of the Firm

$$V = v - p_0 + \sup_{\tau \leq T} \mathbb{E} \left\{ \int_0^\tau e^{-rt} \delta_t \, dt \right\}$$

where

$$\delta_t = \begin{cases} 
(P_t - H \ast G_t - K) \vee \kappa - c_t & \text{if } 0 \leq t \leq M \\
(P_t - H \ast G_t - K)^+ - c_t & \text{if } M \leq t \leq T 
\end{cases}$$

and

- $v$ current value of firm’s assets
- $p_0$ option premium
- $M$ length of the option life
- $\kappa$ strike of the option
- $c_t$ cost of servicing the existing debt
Default Time

Expected Bankruptcy Time as function of Coupon

- M=0.1
- M=0.2
- M=0.5

Y-axis: Tau
X-axis: COUPON

Graph showing the expected bankruptcy time as a function of the coupon for different values of M (0.1, 0.2, 0.5).
PLANT VALUE

Plant Value as function of Coupon

![Graph showing Plant Value as a function of Coupon with different lines for M=0.1, M=0.2, and M=0.5.](image)
Debt Value as function of Coupon
Spread Valuation Mathematical Challenge

\[ p = e^{-rT} \mathbb{E}\{(l_2(T) - l_1(T) - K)^+\} \]

- Underlying indexes are spot prices
  - Geometric Brownian Motions \((K = 0\) Margrabe\)
  - Geometric Ornstein-Uhlembeck (OK for Gas)
  - Geometric Ornstein-Uhlembeck with jumps (OK for Power)
- Underlying indexes are forward/futures prices
  - HJM-type models with deterministic coefficients

**Problem**

Finding closed form formula and/or fast/sharp approximation for

\[ \mathbb{E}\{(\alpha e^{\gamma X_1} - \beta e^{\delta X_2} - \kappa)^+\} \]

for a Gaussian vector \((X_1, X_2)\) of \(N(0, 1)\) random variables with correlation \(\rho\).

**Sensitivities?**
Spread Valuation

- $K = 0$ (Easy Case) Exchange Option Margrabe Formula
- $K \neq 0$ Approximations
  - Fourier Approximations (Madan, Carr, Dempster, Hurd et. al)
  - Bachelier approximation (Alexander, Borovkova)
  - Zero-strike approximation
  - Kirk approximation
  - CD Upper and Lower Bounds (R.C. - V. Durrleman)
  - Bjerksund - Stensland approximation
  - Alos-Eydeland-Laurence approximation for 3 log-normal interests

  Can we also approximate the Greeks?

- New Electricity pricing formula (R.C.-Coulon-Schwartz)
- Clean spread (including price of carbon) (R.C.-Coulon-Schwartz)
**Implied Correlation**

**R.C. - Y. Sun**

Given market prices of

- Options on individual underlying interests
- Spread options

**Infer / Imply** a (Pearson) correlation and

- Smiles
- Skews

in the spirit of **implied volatility**

**Major Difficulty:**

- Data NOT REALLY available!
- Need to rely on trader’s observations / speculations
**Multi-Scale Stochastic Volatility Model**

R.C. - Y. Sun à la Fouque-Sircar-Papanicolaou

\[
\begin{align*}
\text{d}X_t &= \mu_1 X_t \text{d}t + X_t f(Z_t) f_1(V_t) \text{d}W_t^{(X)}, \\
\text{d}Y_t &= \mu_2 Y_t \text{d}t + Y_t f(Z_t) f_2(V_t) \text{d}W_t^{(Y)}, \\
\text{d}Z_t &= \frac{1}{\epsilon} (m - Z_t) \text{d}t + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \sqrt{Z_t} \text{d}W_t^{(Z)}, \\
\text{d}V_t &= \delta c(V_t) \text{d}t + \sqrt{\delta} g(V_t) \text{d}W_t^{(V)}.
\end{align*}
\]

- **\( Z_t \)** *Fast scale* volatility factor
- **\( V_t \)** *Slow scale* volatility factor

\[W_t = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\rho & \sqrt{1 - \rho^2} & 0 & 0 \\
\rho_{11} & \tilde{\rho}_{21} & \sqrt{1 - \rho_{11}^2 - \tilde{\rho}_{21}^2} & 0 \\
\rho_{12} & \tilde{\rho}_{22} & \tilde{\rho}_0 & \sqrt{1 - \rho_{12}^2 - \tilde{\rho}_{22}^2 - \tilde{\rho}_0^2}
\end{pmatrix} W_t^0.\]
Under Pricing Measure

\begin{align*}
  \text{d}X_t & = rX_t \text{d}t + X_t f(Z_t) f_1(V_t) \text{d}W_t^{(X)}, \\
  \text{d}Y_t & = rY_t \text{d}t + Y_t f(Z_t) f_2(V_t) \text{d}W_t^{(Y)}, \\
  \text{d}Z_t & = \left[ \frac{1}{\epsilon} (m - Z_t) - \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} \sqrt{Z_t} \wedge (Z_t, V_t) \right] \text{d}t + \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} \sqrt{Z_t} \text{d}W_t^{(Z)}, \\
  \text{d}V_t & = \left[ \delta c(V_t) - \sqrt{\delta g(V_t)} \Gamma(Z_t, V_t) \right] \text{d}t + \sqrt{\delta g(V_t)} \text{d}W_t^{(V)},
\end{align*}

Option Prices

\[ C^{\epsilon, \delta}(x, y, z, v, t) = e^{-r(T-t)} \mathbb{E}^Q \left[ h(X_T, Y_T) | X_t = x, Y_t = y, Z_t = z, V_t = v \right] \]
Pricing PDE (Feynman-Kac)

\[\frac{1}{2} x^2 f^2(z)f_1^2(v)C_{xx} + \frac{1}{2} y^2 f^2(z)f_2^2(v)C_{yy} + \frac{\nu^2}{\varepsilon} z C_{zz} + \frac{1}{2} \delta g^2(v)C_{vv} + \rho xy f^2(z)f_1(v)f_2(v)C_{xy} + \rho_{11} \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} x \sqrt{z} f(z)f_1(v)C_{xz} + \rho_{21} \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} y \sqrt{z} f(z)f_2(v)C_{yz} + \rho_{12} x \sqrt{\delta} g(v)f(z)f_1(v)C_{xv} + \rho_{22} y \sqrt{\delta} g(v)f(z)f_2(v)C_{yv} + \rho_0 \sqrt{\frac{\delta}{\varepsilon}} \nu \sqrt{2} \sqrt{z} g(v)C_{zv} + \left[ \frac{1}{\varepsilon} (m - z) - \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} \sqrt{z} \Lambda(z, v) \right] C_z + \left[ \delta c(v) - \sqrt{\delta} g(v) \Gamma(z, v) \right] C_v + rxC_x + ryC_y \]

\[-rC + C_t = 0\]

with terminal condition

\[C^{\epsilon, \delta}(x, y, z, v, T) = h(x, y)\]
**Asymptotic Expansions**

**Singular Perturbation Theory**

**Option Price Approximation Formula**

\[ C^{\epsilon,\delta} \approx \tilde{C}^{\epsilon,\delta} := C_0 + \sqrt{\epsilon} C_1 + \sqrt{\delta} D_1 \]

where coefficients \( C_0, C_1 \) and \( D_1 \) can be **calibrated** without the full knowledge of the functions \( f, f_1, f_2, \Lambda \) and \( \Gamma \)!

*(Fouque-Sircar-Papanicolaou)*

**Control of the error**: for fixed \( (x, y, z, \nu, t) \), there exists \( c > 0 \) s.t.

\[ | C^{\epsilon,\delta} - \tilde{C}^{\epsilon,\delta} | \leq c(\epsilon + \delta + \sqrt{\epsilon \delta}) . \]
Implied volatility for options on electric power futures maturing in August 2011, and traded in April 2010, May 2010, until April 2011 (left) and for options on natural gas futures maturing in August 2011, and traded in April 2010, May 2010, until April 2011 (right)
**Implied Correlation Approximations**

\[ l^{\epsilon, \delta} \approx \rho + \sqrt{\epsilon} l_1 + \sqrt{\delta} l_2 \]

Implied correlation (IC) with fitted parameters and strike \( K = 0 \) (left) and time-to-maturity \( TTM = 1 \) year (right)
BACK TO THE EXPONENTIAL BID-STACKS

Spread price (for some $T$) on fuel $v \in I$ (with $w = I \setminus v$) is given by

$$V_t = \Phi(-\xi_8) \sum_{i \in I} b_i(\xi^i, F^i_t) \Phi\left(\frac{-R_i(\xi^i, \xi^i)}{\sigma}\right) - HF^v_t (1 - \Phi(\xi_7) + \Phi(\xi_6) - \Phi(\xi_5)) +$$

$$b_v(\mu_z, F^v_t)e^{t_2^v} \frac{m^2_v \sigma^2_z}{\Phi^2_z} \left(\begin{bmatrix} \xi^v & \xi_3 \xi_4 \xi_2 \end{bmatrix}, \frac{R_v(\mu_z, 0) - m^2_v \sigma^2_z}{\sigma_v, \sigma_z}; m_v \sigma_z \right) +$$

$$b_v(\mu_z - \xi^w, F^v_t)e^{t_2^v} \frac{m^2_v \sigma^2_z}{\Phi^2_z} \left(\begin{bmatrix} \xi_8 & \xi_6 \xi_7 \xi_5 \end{bmatrix}, \frac{-R_v(\mu_z - \xi^w, \xi^w) + m^2_v \sigma^2_z}{\sigma_v, \sigma_z}; -m_v \sigma_z \right) +$$

$$b_w(\mu_z - \xi^v, F^w_t)e^{t_2^w} \frac{m^2_w \sigma^2_z}{\Phi^2_z} \left(\begin{bmatrix} \xi_8 & \xi_6 \x_7 \xi_5 \end{bmatrix}, \frac{-R_w(\mu_z - \xi^v, \xi^v) + m^2_w \sigma^2_z}{\sigma_w, \sigma_z}; -m_w \sigma_z \right) -$$

$$HF^v_t \Phi^2_z \left(\begin{bmatrix} \xi_7 & \xi_5 & \xi_3 \xi_4 \xi_2 \end{bmatrix}, \frac{\tilde{R}_v ((\log H - \beta - \gamma \mu_z)/\alpha w)}{\sigma_{\gamma z}}; -\gamma \sigma_z \right) +$$

$$(F^c_t)^{\alpha_c} (F^g_t)^{\alpha_g} e^\eta \left\{ \Phi^2_x \left(\begin{bmatrix} \xi^v & \xi_3 \xi_4 \xi_2 \end{bmatrix}, \frac{-R_v(\mu_z, 0) - \alpha w \sigma^2 + \gamma m_v \sigma^2_z}{\sigma_v, \sigma_z}; -m_v \sigma_z \right) -$$

$$\Phi^2_z \left(\begin{bmatrix} \xi_8 & \xi_6 \x_7 \xi_5 \end{bmatrix}, \frac{-R_v(\mu_z - \xi^w, \xi^w) - \alpha w \sigma^2 + \gamma m_v \sigma^2_z}{\sigma_v, \sigma_z}; -m_v \sigma_z \right) +$$

$$\Phi^2_z \left(\begin{bmatrix} \xi_8 & \xi_6 \x_7 \xi_5 \end{bmatrix}, \frac{R_w(\mu_z - \xi^v, \xi^v) + \alpha w \sigma^2 - \gamma m_w \sigma^2_z}{\sigma_w, \sigma_z}; m_w \sigma_z \right) -$$

$$\Phi^2_z \left(\begin{bmatrix} \xi_7 & \xi_5 & \xi_3 \xi_4 \xi_2 \end{bmatrix}, \frac{-\tilde{R}_v ((\log H - \beta - \gamma \mu_z)/\alpha w) - \alpha w \sigma^2 - \gamma^2 \sigma^2_z/\alpha w}{\sigma_{\gamma z}}; \gamma \sigma_z \right) \right\}$$

with $\tilde{R}_i(z) = z + \log(F^i_t) - \log(F^i_t) - \frac{1}{2} \sigma^2$.
Applications

- Computation of higher moments and covariances:

\[
\left( \text{e.g., } \mathbb{E}_t[P_t^2], \mathbb{E}_t[P_T C_T], \mathbb{E}_t[P_T G_T] \right)
\]

- Sensitivities (Greeks).
- Calibration to power forwards (with fuel forwards as inputs).
- Spikes (and negative prices) is possible
- Extension to carbon markets possible: merit order affected by allowance price, and accumulated emissions also driven by merit order.
- Key advantage: Structural link between electricity and fuel prices.
Implied Correlation for high demand, - Spark (left) - Dark (right)
Implied Correlation - low demand - Spark (left), Dark (right)
Implied Correlation III

Implied Correlation Surface (vs h and T)

Implied Correlation (Dark Spread)

Implied correlation varying $\bar{\xi}^g$ (left) – Implied correlation varying $\mu_d$ (right), Dark (right)
Implied correlation surfaces, for symmetric case (left) and for spark spread in case of coal in contango, gas in backwardation (right)
The bid stack model (without spikes) typically prices spread options lower than the Margrabe formula due to strong structural link. (We first match means and variances in the two approaches.)
In addition, the stack model automatically adjusts to information about likely future merit order changes. (Here we choose gas forwards in contango, coal in backwardation.)
POWER PLANT VALUATION REVISITED

- Choose Bid-Stack
- Choose NG Forward Curve
- Choose Coal Forward Curve

Examples
- 3 or 6 yrs tolling agreement
- 10MW
- No O&M \((K = 0)\) Compare to Price from Margrabe formula
- Plot Value as function of mean demand \(\mu_d\)
Power Plant Valuation Ex#1

Coal or gas power plant value (10 MW)

\( \mu_d \) (avg demand level)
Coal or gas power plant value (10 MW)

Coal Power Plant Valuation – 6 yrs (gas contango, coal backwarded)

- Stack model ($\rho = -0.5$)
- Stack model ($\rho = 0$)
- Stack model ($\rho = 0.5$)
- Margrabe ($\rho = -0.5$)
- Margrabe ($\rho = 0$)
- Margrabe ($\rho = 0.5$)
Coal Power Plant Valuation – 3 yrs (gas bids above coal)

- Stack model ($\rho = -0.5$)
- Stack model ($\rho = 0$)
- Stack model ($\rho = 0.5$)
- Margrabe ($\rho = -0.5$)
- Margrabe ($\rho = 0$)
- Margrabe ($\rho = 0.5$)
Coal or gas power plant value (10 MW) vs. \( \mu_d \) (avg demand level)

- Stack model (\( \rho = -0.5 \))
- Stack model (\( \rho = 0 \))
- Stack model (\( \rho = 0.5 \))
- Margrabe (\( \rho = -0.5 \))
- Margrabe (\( \rho = 0 \))
- Margrabe (\( \rho = 0.5 \))

Gas Power Plant Valuation – 3 yrs (gas bids above coal)
Stochastic (Control) Optimization to Take Full Advantage of the Optionality

- **Physical Asset:** Fossil Fuel Power Plant, Oil Refinery, Pipeline, Gas Storage Facility, Hydro, . . .

- **Owner** (of the asset or a tolling contract)
  - Decides *when* and *how* to use the asset (e.g. run the power plant)
  - Has someone else do the leg work

- **Optimal Switching R.C - M. Ludkovski**

- **Extensions**
  - Accomodate *outages*, switch separation
  - Duality upper bounds (*Meinshausen-Hambly*)
  - More (rigorous) **Mathematical Analysis**
    - *Porchet-Touzi* (BSDEs)
    - *Forsythe-Ware* (Numeric scheme to solve HJB QVI)
    - *Bernhart-Pham* (reflected BSDEs)
    - *Bouchard-Warin* (numerics of reflected BSDEs)

- **Financial Hedging:** Extending the Analysis Adding Access to a Financial Market (indifference pricing)
Given

- $P(t)$ sale price of 1 MWhr of electricity
- $G(t)$ price of 1 MBtu natural gas
- $A(t)$ price of an allowance for 1 ton of CO$_2$ equivalent

compute

$$e^{-rT} \mathbb{E}\{(P(T) - H_{\text{eff}}G(T) - e_G A(T))^+\}$$

where $e_G$ is the emission coefficient of the technology.

Requires

- Joint model for $\{(P(t), G(t), A(t))\}_{0 \leq t \leq T}$