Discussion of
“Stable Diffusions Interacting through Their Ranks, as Models of Large Equity Markets” by Ioannis Karatzas

Discussant: Semyon Malamud, EPF Lausanne and SFI
Standard Model

\[ d \log B(t) = r(t) \, dt \]
\[ d \log S_i(t) = \gamma_i(t) \, dt + \sum_{\nu} \sigma_{i\nu}(t) \, dW_{\nu} \]  

(1)

\( (a_{ij}) \) is the instantaneous variance-covariance matrix of the stocks
Log Wealth Process of a Portfolio Strategy $\pi$

$$
    d \log(V^{w,\pi}) = \left( \gamma^{\pi}(t) \, dt + \sum_{\nu} \sigma^{\pi}_{\nu}(t) \, dW_{\nu} \right)
$$

with volatilities

$$
    \sigma^{\pi}_{\nu}(t) = \sum_{i} \pi_{i}(t) \sigma_{i\nu}(t),
$$

the growth rate

$$
    \gamma^{\pi}(t) = \sum_{i} \pi_{i}(t) \gamma_{i}(t) + \gamma^{\pi}_{\pi}
$$

and the excess growth rate

$$
    \gamma^{\pi}_{\pi} = \frac{1}{2} \left( \sum_{i} a_{i\pi}(t) \pi_{i}(t) - \frac{1}{2} \sum_{i,j} \pi_{i}(t) \pi_{j}(t) a_{ij}(t) \right)
$$

Pure diversification gain Theorem: excess growth rate is always positive for a long-only portfolio
The Market Portfolio $\mu$

$$\mu_i(t) = \frac{S_i(t)}{\sum_j S_j(t)}$$

Order Statistics

$$\mu(1) \geq \mu(2) \geq \cdots \geq \mu(n)$$

Weath process

$$V^{w,\mu} = \sum_i S_i(t)$$
Questions

- Can we outperform the market with a simple long only portfolio?
- In a model-independent way?
- Without any statistical parameter estimation?
Answers:

- Yes, we can
- Yes, we can
- Yes, we can
- Yes, we can

But we need ...
Answers:

- Yes, we can
- Yes, we can
- Yes, we can
- Yes, we can

But we need ...
Diversity over $[0, T]$:

**Strong diversity:** $\mu_{(1)} \leq 1 - \delta$ a.s. for some $\delta > 0$

**Weak diversity:** $\frac{1}{T} \int_0^T \mu_{(1)}(t) dt \leq 1 - \delta$ a.s.

Definitely True in the Data!
Can we find a simple model with diversity?

- No!
- Because Diversity leads to arbitrage!

Example: a diversity-weighted portfolio

\[ \pi_i(p) = \frac{\mu_i(t)^p}{\sum_j \mu_j(t)^p}, \quad p \in (0, 1) \]

outperforms the market with probability 1

- Strict local martingales
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- Strict local martingales
Pure Diversity Based Portfolios

- Can be constructed
- Are based only on the perfectly observable capital distribution curve

\[ \log k \rightarrow \log \mu(k) \]
Local “Rank Occupation” Times

The dynamics of $\mu(k)$ is to a large extent driven by the directly observable local times $\Lambda^{k,k+1}(t)$ accumulated at the origin by $\log(\mu(k)/\mu(k+1))$.
Empirical Regularities

- Capital distribution curve $\log k \rightarrow \log \mu(k)$ is extremely stable over time, flat for small $k$, concave for large $k$
- Local times $\Lambda^{k,k+1}(t)$ are almost linear in $t$ and decrease very fast with $k$
- Variance is almost linear in the rank
Questions:

- Amazing ?!
- What is the economics behind this?
- Can we build a model reproducing these regularities?
Answers:

- Amazing ?! Yes!
- What is the economics behind this? No idea!
- Can we build a model reproducing these regularities? This is what Ioannis’ talk was about.
Hybrid Atlas Models

- Postulate that
  \[ d(\log S_i(t)) = (\gamma + \gamma_i + g_k) + \sigma_k dW_i(t) + \sum_j \rho_{ij} dW_j(t) \]

  if \( \mu_i = \mu(k) \)

- stability conditions: **global cancellation:**
  \[ \sum_k g_k + \sum_i \gamma_i = 0 \]

- no \( k \leq n - 1 \) stocks dominate:
  \[ \sum_{l=1}^{k} (g_l + \gamma_{i_l}) < 0 \]

  for all \( k \leq n - 1 \) and all permutations \( \{i_1, \ldots, i_n\} \).

- These conditions imply stochastic stability: there is unique invariant distribution and SLLN holds \( \Rightarrow \) local times grow linearly with \( t \)

- closed form invariant measure when \( \rho_{ij} = 0 \)
Long-Run Slopes of the Local Times

\[
\lim_{T \to \infty} \frac{1}{T} \Lambda_{k,k+1}^k(T) = -2 \sum_{l=1}^{k} \left( g_l + \sum_{i=1}^{l} \gamma_i \theta_{l,i} \right)
\]

where \( \theta_{l,i} \) is the long-run occupation time.
Empirically, the slopes decrease fast with \( k \). What does this mean?
Concavity of the Capital Distribution Curve

For any permutation $p$ define

$$\lambda_{p,k} = -\frac{4 \sum_{l=1}^{k} (g_l + \gamma_{p(l)})}{\sigma_k^2 + \sigma_{k+1}^2}$$

(the exponents for the invariant distribution).

Then, the curve is concave if the sequence

$$\lambda_{p,k} \log \frac{k + 1}{k}$$

is decreasing in $k$ for any $p$.

What does this condition mean?
Empirical Diversity

Define diversity as the entropy (other definitions also work):

\[ \sum_{i} \mu_i(t) \log(\mu_i(t)) \]
An additional Risk Factor?

Alpha as diversity beta?
The shape of capital distribution fluctuates over time?