Discussion on:

On the Theory of Continuous-Time Recursive Utility

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Studying the connection between

- **Recursive utility:**  \[ V(c_0, c_1, ..) = W(c_0, \mu[V(c_1, c_2, \ldots)]] \]

Epstein-Zin (1989)
Studying the connection between

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  cont.time: stochastic differential utility

  preferences formalized via a differential equation

  Duffie-Epstein (1992)

  Kraft-Seifried (2010)
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- **Time-global problem formulation**

  Issue of **Time-Consistency**

  \[ \rightarrow \text{optimize not among admissible, but only among consistent strategies!} \]
  - (e.g. mean-variance problem without precommitment)
  - Basak-Chabakauri (2010), Björk-Murgoci (2010)
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**Result:** The resulting optimal strategies *sometimes* coincide

(Example: Black-Scholes market)
Model Assumptions

\[ dB(t) = B(t) r \, dt \]

\[ dS(t) = S(t) \left[ (r + \lambda(t, Y(t))) \, dt + \sigma(t, Y(t)) \, dW(t) \right] \]

\[ dY(t) = \alpha(t, Y(t)) \, dt + \beta(t, Y(t)) \left( \rho \, dW(t) + \sqrt{1 - \rho^2} \, d\tilde{W}(t) \right) \]

Wealth process \( X(t) \): invest \( \pi X \) in \( S \) and \( (1 - \pi)X \) in \( B \), consume at some rate \( c \)
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Classical problem: value function

\[ V(t, x, y) = \sup_{c, \pi} E_{t,x,y} \left[ \int_t^T e^{-\delta(s-t)} \frac{1}{1 - \gamma} c^{1-\gamma(s)} \, ds \right] \]

Linearity: local \( \rightarrow \) global
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New problem: value function \( \theta \). elasticity of intertemporal substitution

\[ V(t, x, y) = \sup_{c, \pi} \left[ \int_t^n \delta e^{-\delta(s-t)} \left( E_{t,x,y} \left[ \frac{1}{1-\gamma} \, c^{1-\gamma(s)} \right] \right)^{1/\theta} \, ds \right]^\theta \]
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New problem: value function

\[ V(t, x, y) = \sup_{c, \pi} \left[ \int_t^\infty \delta e^{-\delta(s-t)} \left( E_{t,x,y} \left[ \frac{1}{1-\gamma} \, c^{1-\gamma(s)} \right] \right)^{1/\theta} ds \right]^\theta \]

certainty equivalent on \( c \)

Non-Linear: look only for optimal control among consistent ones
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More flexibility, conceptually: analytical advantages over recursive utility

Bellman-type equation with \textit{additional terms}, which cancel if

- \( \theta = 1 \): time-additive utility
- Separation condition and \( \beta = 0 \)
Questions

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- Restriction to time-consistent strategies? Is this an issue? (Markov assumptions)
  What is the 'price' of such a restriction?
- Could one replace power utility by a distance to prespecified consumption stream?
Final Remarks

Related problem in insurance:

Maximize (utility of) expected discounted dividend payments until ruin

\[
\max_c E \left( \int_0^\tau e^{-\delta s} U(c_s) \, ds \right), \quad \max_C E \left[ U \left( \int_0^\tau e^{-\delta s} dC_s \right) \right]
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- How to specify utility functions?
  Can the same utility function be used for all time horizons?

**Time Consistency of Valuations** (rather than Actions)

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**Time Consistency of Valuations** (rather than Actions)


- Risk measures:

  Coherence vs. Time Consistency