Endogenous Liquidity and Defaultable Bonds

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Background: Fundamental vs Liquidity

- **Fundamental** and **liquidity** are interconnected as evident from recent financial crisis
  - Liquidity: funding liquidity, price impact, transaction costs, etc
  - Today’s paper: liquidity $\leftrightarrow$ fundamental, two-way feedback
    Liquidity solved jointly with fundamental (default decision)
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- **Pattern of bond illiquidity:**
  - OTC transactions have average transaction cost of around 100bps
  - Illiquidity higher for longer time-to-maturity, closer to default
  - Barclays Capital report (2009) shows high correlation between default and liquidity spreads, both time-series and cross-sectional
Motivation: Corporate Bonds

Average Bond Illiquidity (Transaction Cost) in 2008

Group 1, shortest maturity
Group 2
Group 3
Group 4
Group 5, longest maturity

Average Bond Illiquidity (Transaction Cost) in 2008

Group 1, lowest CDS
Group 2
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- **Empirical approach to bond liquidity:**
  - State-of-the-art empirical literature decomposes spreads into independent liquidity and default premium
Mechanism and Results

**Building blocks** for interaction between fundamental and liquidity:

- How does bond illiquidity arise, and how is it affected by maturity and state of the firm?
  - Over-the-counter market with search friction à la Duffie et al (2005)
- How do corporate decisions interact with secondary market liquidity?
  - Endogenous default à la Leland Toft (1996)

**Main results:**

- Closed-form solution for bond & equity values, default boundary
- Novel liquidity-default spiral, can be quantitatively important for understanding credit spreads
- Ability to target empirical pattern of bond illiquidity, match to credit spreads than can be decomposed into default and liquidity components
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Related Literature

**Search in asset markets:**
- Duffie, Garleanu, Pedersen ’05, ’07
  OTC search market with simplified ’derivative’

**Capital structure models:**
- Leland, Toft ’96 (LT96)
  Rollover increases exposure of equity holders to fundamental risk
- He, Xiong ’12 (HX12)
  Exogenously given secondary market liquidity affects default decision

**Empirical literature:**
- Bao, Pan, Wang ’11; Edwards, Harris, Piwowar ’07; Hong, Warga ’00; Hong, Warga, Schultz ’01; Harris, Piwowar ’06; Feldhütter ’11

**Feedback models:**
- Many many more papers...
The Model: Basics & Liquidity Shocks

Preferences: Everyone risk-neutral with common discount rate $r$

Firm:
- Assets produce per-period cash-flow $\delta_t$, $d\delta_t = \mu \delta_t dt + \sigma \delta_t dZ_t^Q$
- Debt in place with aggregate (constant) face value $p$ and coupon $c$
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Idiosyncratic liquidity shock for bond investors:
- With intensity \( \xi \), jump in individual discount rate to \( \bar{r} > r \)
- Let \( H \) be high-value (\( r \)) type, \( L \) low-value/liquidity (\( \bar{r} \)) type
- Idiosyncratic liquidity shock not insurable (incomplete market)
- Holding restriction: \( \{0, 1\} \) (as in DGP '05)
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- Idiosyncratic liquidity shock *not* insurable (incomplete market)
- Holding restriction: $\{0, 1\}$ (as in DGP '05)

Trade:
- Efficient for L types to sell to H types with higher valuation
- $D_H$ and $D_L$ are the values of debt for H/L types taking into account future liquidity shocks/re-trading opportunities/default/maturity
The Model: Illiquid Secondary Bond Market

Search friction in secondary bond market:

- $L$ meets dealers with intensity $\lambda$ & bargains over sale
- $L$’s outside option ($D_L$) is waiting for other dealers/default/maturity
- Dealer immediately sells bond on for $D_H$ to $H$ type outside investors
  - For simplicity, frictionless contact with $H$ investors
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Bargaining:

- Nash-Bargaining over surplus from intermediation, $S \equiv D_H - D_L$
- Endogenous price $X$ implements $\beta$ ($L$ type) and $(1 - \beta)$ (dealer) surplus split:

\[
\begin{align*}
D_H - X &= (1 - \beta) (D_H - D_L) \\
X - D_L &= \beta (D_H - D_L)
\end{align*}
\]
**Bonds mature at** $\tau = 0$:

- At maturity bonds equal to face value, $D_H(\delta, 0) = D_L(\delta, 0) = p$ for $\delta > \delta_B$
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Bonds default at $\delta = \delta_B$:
- Bonds have *equal seniority* in default
- Cash recovery value constant at $\alpha V_B = \alpha \frac{\delta_B}{r-\mu}$ with $\alpha \leq 1$
- Legal delay: Cash-payout $\alpha V_B$ only after an exponential delay with intensity $\theta$
- Post-default trading possible with intermediation intensity $\lambda_B$
The Model: Boundary Conditions - Maturity and Default

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⇒ **Result:** \( D_H(\delta_B, \tau) = \alpha_H V_B, \ D_L(\delta_B, \tau) = \alpha_L V_B \). Wedge in *effective* bankruptcy discounts

\[ \alpha_L < \alpha_H < \alpha \]
The Model: Bargaining

(A) AAA at issuance

Outside Option:

$D_L$
The Model: Bargaining and Maturity

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Outside Option:
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(B) Close to maturity

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The Model: Bargaining and Default

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Debt structure:

- **Stationary principal & staggered maturity** (as in LT96):
  - Maturity structure evenly staggered (i.e., uniform) on $[0, T]$
  - Maturing bonds reissued with same $(c, p, T)$
  - Mass $1/T \cdot dt$ of bonds matures every instant
The Model: Debt Structure, Rollover & Default

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Rollover:

- Primary market competitive & liquid, so issue at $D_H$ to $H$ types.
- Rollover further exposes equity to movement in $\delta$ via repricing.

\[
Net\text{CashFlow}_t = \underbrace{\delta_t}_{CF} - \underbrace{(1 - \pi)c}_{Coupon} + \underbrace{\frac{1}{T}[D_H(\delta_t, T) - p]}_{Rollover \ gain/loss}
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\[
NetCashFlow_t = \begin{cases} 
\delta_t - (1 - \pi) c + \frac{1}{T} \left[ D_H (\delta_t, T) - p \right] 
\end{cases}
\]

Optimal default:

- Equity defaults at \(\delta_B\) when absorbing further losses unprofitable
Above analysis outside default
Schematic Representation: The Primary Market

Above analysis outside default
Above analysis outside default
Above analysis outside default
Closed form solutions for all important objects:

**Debt** $D_H, D_L$: mixture of distorted LT96 solutions

**Equity** $E$: solved directly as no 'adding up' as in LT96

**Optimal default boundary** $\delta_B$
Bond Liquidity: Relative Bid-Ask Spread

Consistent with empirical pattern:
BA spread lower for shorter-term bonds and higher quality bonds
Liquidity and Default: Feedback Loop

**Counterfactual: Fixed illiquidity / transaction cost**

- Fixed transaction cost $k$ (bid-ask spread of $\frac{k}{1-k/2}$) with immediate sale after shock (as in Amihud Mendelson ’86, He Xiong ’12)
- Our model: pro-cyclical liquidity, i.e., liquidity dries up as fundamental $\delta$ worsens
- Thought experiment to get feedback: Investors erroneously believe current liquidity will stay constant
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Parameters: normalize $\delta_0 = 1$

- Calibrate so at $\delta_0$ bid-ask is 100bps
- Benchmark of HX12: $k = 99.5\text{bps}$ (so 100bps bid-ask spread)
- Benchmark of LT96: $k = 0$ (no illiquidity)
- Effective bankruptcy discounts $\alpha_H = 67\%$ and $\alpha_L = 55\%$
Liquidity and Default: Pro-cyclical Liquidity

Pro-cyclical liquidity:

- Illiquidity increases as distance to default shrinks
- Illiquidity non-zero for large $\delta$ / AAA-rated bonds
Liquidity and Default: Rollover Losses & Default

Rollover loss amplified:

- Possible future illiquidity depresses primary market price $D_H(\delta, T)$
- Higher rollover losses for every $\delta$ lead to earlier default

Endogenous Liquidity

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>-0.04</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.03</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.02</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.01</td>
</tr>
<tr>
<td>1.4</td>
<td>0.01</td>
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<tr>
<td>1.6</td>
<td>0.02</td>
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Liquidity and Default: Full Feedback Loop

Equilibrium feedback loop:

- Compare to counterfactual *constant* transaction costs

![Diagram showing the feedback loop between cash-flow, liquidity, debt values, and debt rollover costs.]

- Cash-flow $\delta$ declines
- Liquidity decreases
- Debt values decline
- Debt rollover more expensive
- Equity holders default earlier
Liquidity and Default: Full Feedback Loop

Equilibrium feedback loop:

- Compare to counterfactual *constant* transaction costs

> Default is just *one* channel to affect fundamental

  - Simple extension: endogenous investment by equity to improve asset-in-place creates feedback of illiquidity on cash-flows
Maturity: Rollover Risk vs Liquidity Provision

Negative: Short-term debt leads to earlier default
- Higher rollover frequency increases equity’s exposure to $\delta$

$$Rollover\ gain/\ loss_t = \frac{1}{T} \times [D_H(\delta_t, T) - p]$$

- Higher exposure to $\delta$ leads to higher default boundary $\delta_B$
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- Short maturity improves bargaining outcome between seller & dealer
- Issuing to H types more frequently improves allocative efficiency as it ‘recycles’ L types to H types quicker (lower SS mass of L holdings)
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$\Rightarrow$ Finite maturity $T^* < \infty$ optimal if moderate initial leverage; $T^*$ lower the less liquid secondary market (i.e. the lower $\lambda$)
Current Work: Aggregate Shocks & Serious Calibration

Advantage of structural model:

- Added discipline of **jointly** matching credit spreads and liquidity

Changes to model:

- Sacrifice deterministic maturity, use random maturity to handle shifts in aggregate state while maintaining tractability:
  - **Good** period with normal cash-flows and well intermediated OTC markets
  - **Bad / Crisis** period with shock to intermediation intensity (financial crisis), riskier cash-flows, and higher price of risk (Chen 2010)

Implementation:

- Extract $\alpha_H, \alpha_L$ from bond ultimate recovery and trading prices at default (Moody’s Default & Recovery Database)
- Target bid-ask spread to one observed in data, match total credit spreads of bonds of different ratings
- **Decompose** credit-spreads into default-, liquidity- and interaction terms, and see how they vary cross-sectionally and across states
Model-Based Decomposition: Methodology

- Model allows to decompose *total credit spread* in more refined way:
  - **“Pure Default”**: Yield of a defaultable bond free from liquidity frictions with adjusted default boundary reflecting improved secondary market liquidity (both before and after default)
  - **“Liquidity Driven Default”**: Yield of a defaultable bond free from liquidity frictions with original default boundary minus “Pure Default”
  - **“Pure Liquidity”**: Yield of a default free bond subject to the same liquidity frictions
  - **“Default Driven Liquidity”**: The residual

- None of the above parts are directly observable from data: We need a structural model to construct this decomposition
- The decomposition scheme is designed to quantify the interaction between liquidity and default
**Model Based Decomposition: Superior Grade Bonds**

<table>
<thead>
<tr>
<th></th>
<th>State G</th>
<th>State B</th>
<th>Change (in bps)</th>
<th>Change (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Credit Spread</td>
<td>84.73</td>
<td>124.13</td>
<td>39.39</td>
<td>100.00</td>
</tr>
<tr>
<td>Pure Default</td>
<td>22.46</td>
<td>40.16</td>
<td>17.70</td>
<td>44.92</td>
</tr>
<tr>
<td>Liquidity Driven Default</td>
<td>9.04</td>
<td>14.87</td>
<td>5.83</td>
<td>14.80</td>
</tr>
<tr>
<td>Pure Liquidity</td>
<td>45.59</td>
<td>53.68</td>
<td>8.27</td>
<td>20.98</td>
</tr>
<tr>
<td>Default Driven Liquidity</td>
<td>7.64</td>
<td>15.25</td>
<td>7.60</td>
<td>19.30</td>
</tr>
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</table>

*Table: Model Based Decomposition: Superior Grade Bonds*
<table>
<thead>
<tr>
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<th>Change (in bps)</th>
<th>Change (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Credit Spread</td>
<td>196.82</td>
<td>288.77</td>
<td>91.95</td>
<td>100.00</td>
</tr>
<tr>
<td>Pure Default</td>
<td>86.20</td>
<td>139.63</td>
<td>53.43</td>
<td>58.11</td>
</tr>
<tr>
<td>Liquidity Driven Default</td>
<td>24.63</td>
<td>33.14</td>
<td>8.51</td>
<td>9.26</td>
</tr>
<tr>
<td>Pure Liquidity</td>
<td>56.69</td>
<td>67.03</td>
<td>10.34</td>
<td>11.24</td>
</tr>
<tr>
<td>Default Driven Liquidity</td>
<td>29.29</td>
<td>48.97</td>
<td>19.67</td>
<td>21.39</td>
</tr>
</tbody>
</table>

**Table:** Model Based Decomposition: Investment Grade Bonds
### Model Based Decomposition: Junk Grade Bonds

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</thead>
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<tr>
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<td>574.54</td>
<td>178.45</td>
<td>100.00</td>
</tr>
<tr>
<td>Pure Default</td>
<td>210.46</td>
<td>319.81</td>
<td>109.35</td>
<td>61.28</td>
</tr>
<tr>
<td>Liquidity Driven Default</td>
<td>48.08</td>
<td>63.47</td>
<td>15.39</td>
<td>8.62</td>
</tr>
<tr>
<td>Pure Liquidity</td>
<td>74.74</td>
<td>88.49</td>
<td>13.76</td>
<td>7.71</td>
</tr>
<tr>
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<td>62.81</td>
<td>102.76</td>
<td>39.96</td>
<td>22.39</td>
</tr>
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</table>

**Table:** Model Based Decomposition: Junk Grade Bonds
What did we learn from this decomposition?

- Liquidity driven default is quantitatively important, especially in bad times and for risky bonds.

- Default driven (endogenous) liquidity is as important as pure liquidity (search frictions) for risky bonds.

- Increase in default driven illiquidity responsible for most of the contribution of liquidity to credit spread when the economy switches to bad state.
Conclusion

**Fully solved non-stationary dynamic search model:**
- Closed form solution for debt, equity, default boundary

**Liquidity-default spiral:**
- Lower liquidity in secondary market lowers the distance to default, which further lowers liquidity in secondary market,…

**What about adverse selection?**
- Definitely reasonable but challenging. Probably generates similar empirical illiquidity pattern
- For understanding the role of liquidity in credit spreads, search framework (simple, easy to be integrated) delivers first-order effects

**Empirical implementation:**
- Targeting liquidity, we match bond credit spreads and are then able to decompose into liquidity and default components
Future work: Aggregate Shocks & Serious Calibration

TRACE implied bid-ask spread (in %, Bao et al 2011) by year and by rating class.
Solution: Derivation of Closed-Forms

**Debt** $D_H, D_L$:
- Mix of two distorted LT96 solutions

\[
rd_H(\delta, \tau) = A^\delta D_H(\delta, \tau) - \frac{\partial D_H}{\partial \tau}(\delta, \tau) + c + \xi [D_L(\delta, \tau) - D_H(\delta, \tau)] ^{CF \text{ dynamics} \atop \text{Maturity} \atop \text{Liquidity shock}}
\]

\[
\bar{r}D_L(\delta, \tau) = A^\delta D_L(\delta, \tau) - \frac{\partial D_L}{\partial \tau}(\delta, \tau) + c + \lambda [X(\delta, \tau) - D_L(\delta, \tau)] ^{CF \text{ dynamics} \atop \text{Maturity} \atop \text{Secondary market}}
\]

**Equity** $E$:
- No 'adding up' as in LT96, solve for equity via ODE directly

\[
r \cdot E(\delta) = A^\delta E(\delta) + \delta - (1 - \pi) c + 1/T [D_H(\delta, T) - p] ^{CF \atop \text{Coupon} \atop \text{Rollover gain/loss}}
\]

**Optimal default boundary** $\delta_B$:
- Unique fixed-point $\delta_B$ from smooth pasting
Solution: Derivation of Closed-Forms

Debt $D_H, D_L$:

- Mix of two distorted LT96 solutions

\[
rD_H (\delta, \tau) = \frac{A^\delta D_H (\delta, \tau)}{CF \text{ dynamics}} - \frac{\partial D_H (\delta, \tau)}{\partial \tau} + c + \xi \left[ D_L (\delta, \tau) - D_H (\delta, \tau) \right]
\]

\[
\bar{r}D_L (\delta, \tau) = \frac{A^\delta D_L (\delta, \tau)}{CF \text{ dynamics}} - \frac{\partial D_L (\delta, \tau)}{\partial \tau} + c + \lambda \beta \left[ D_H (\delta, \tau) - D_L (\delta, \tau) \right]
\]

Equity $E$:

- No 'adding up' as in LT96, solve for equity via ODE directly

\[
r \cdot E (\delta) = A^\delta E (\delta) + \delta - (1 - \pi) c + \frac{1}{T} \left[ D_H (\delta, T) - p \right]
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