A Bayesian Adaptive Singular Control Problem arising from Corporate Finance

Jean-Paul Décamps  
(Toulouse School of Economics, IDEI)  
Stéphane Villeneuve  
(Toulouse School of Economics, IDEI)  

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In Corporate Finance, Liquidity Risk refers to the difficulty to meet its operational needs.

Liquidity risk models take cash reserve as state variable and dividend as control variable $\rightarrow$ Singular control problem.

Solvency risk refers to the inability to honour its debt commitment. This means that the insolvent company owes more than it owns.

Solvency risk models take firm cash-flows as state variable. This leads to determine the optimal time to liquidate $\rightarrow$ Optimal Stopping problem

How to merge these two risks in a tractable model?
Solvency Risk

- Solvency risk models focus on firm profitability.
- But they often assume costless external financing $\rightarrow$ Cash reserves do not matter.
- As long as the firm asset is higher than the firm liability, shareholders inject cash at no costs to meet the operational needs.
- The liquidation of the firm asset is endogeneous.
The dominant paradigm: deep pocket shareholders (Leland).

- Cash-flows process (EBIT) = regular diffusion or Lévy process \( X_t \)
- The shareholders’ decision is to select the default policy that maximizes the value of their share.

\[
E(x) = \sup_{\tau \in \mathbb{T}_0, \infty} \mathbb{E}_x \left[ \int_0^\tau e^{-rt} X_t \, dt \right]
\]

- The entire after tax cash flow is distributed as dividends.
- Optimal default policy

\[
\tau_{x^*} = \inf\{ t \geq 0 : X_t \leq x^* \} \quad x^* < 0
\]

- Equity-holders are ready to inject cash to maintain the firm’s activity when \( x^* < X_t < 0 \).
Liquidity risk models focus on optimal cash management (dividend and issuance policy) when external financing is costly.

Financial frictions generate risk aversion even for well-diversified firm. Too little cash leads to liquidation of productive assets but holding too much cash reduces profitability.

Optimal policy: firms have target cash levels (cash in excess of certain threshold is returned to shareholders).
The dominant paradigm: No external Funding (Jeanblanc-Shiryaev)

- **Cumulative cash flow process** $dR_t = \mu \, dt + \sigma \, dB_t$
- **Strong assumption** External financing is too costly $\rightarrow$ liquidation when the cash reserves hit zero.

- **Cash reserves process** $dM_t = (r - \lambda)M_t \, dt + dR_t - dZ_t$
- **Liquidation time** $\tau_0 \equiv \inf\{t \geq 0 : M_t = 0\}$

- **Firm value** $V^*(m) = \sup_{Z} \mathbb{E}_m \left[ \int_{0}^{\tau_0} e^{-rs} \, dZ_s \right]$
- **First best value** $V_{FB}(m) = m + \frac{\mu}{r}$. 
The dominant paradigm: No external funding (Jeanblanc-Shiryaev)

- The value of the firm is an increasing and concave function of the level $m$ of its cash reserves.
- Any excess above $m^*$ is paid to equity-holders.
- The marginal value of cash is strictly greater than 1 for $m \in (0, m^*)$, where $m^* = \inf\{m > 0 \mid V^*(m) = 1\}$, and equal to 1 for $m \in [m^*, \infty)$.
- The volatility of the firm value decreases with the level of cash.
Optimal Dividend policy.

The model

No dividends  $x^*$  Dividends

$V_{FB}$

$V_{JS}$

Literature on optimal liquidity management policies: Jeanblanc and Shiryaev (1995); Asmussen, Højgaard and Taksar (1999); Sethi and Taksar (2002); Choulli, Taksar and Zhou (2003); Lokka and Zervos, (2005); Cadenillas, Choulli, Taksar and Zhang (2006), Décamps, Mariotti, Rochet and Villeneuve (2007).
Merging Liquidity and Solvency Risks

- Liquidity and solvency concerns are recognized to be driving forces behind management decisions.
- Finance literature has mainly focused on each type of financial distress independently.
- Standard trade-off models initiated by Leland (94) put aside the role of cash balances and focus only on endogeneous default triggered by shareholders. There is no financial constraints on external financing.
- Literature on liquidity management initiated by Radner and Shepp (95) considers costly external financing but neglects the optimal choice of leverage.
Davydenko (2007) summarized ”Neither solvency nor liquidity concerns alone can fully explain the observed corporate decisions”.

Following Gryglewicz (2011, JFE), the contribution of this study is the integration of liquidity and solvency concerns in a dynamic model.

Understand liquidity management as a means to avoid inefficient default as documented by Lins, Servaes and Tufano (2008).
Dynamic model for a cash-constrained firm with uncertainty about the profitability of its project

- Cumulative cash-flow process: \( R = (R_t)_{t \geq 0} \) follows an arithmetic Brownian motion with unknown drift \( Y \)

\[
dR_t = Y \, dt + \sigma \, dB_t
\]

Brownian motion \( B \) is independent of \( Y \).

- Firm’s profitability \( Y \) takes either of the two values \( y < 0 < \bar{y} \).
Shareholders’ belief.

- Let us define $Y_t = \mathbb{E}[Y \mid \mathcal{F}^R_t]$ the shareholders’ belief, Filtering theory (Lipster-Shiryaev)

$$dY_t = \frac{1}{\sigma}(Y_t - \bar{y})(\bar{y} - Y_t)dW_t$$

where $W = (W_t)_{t \geq 0}$ is a $\mathcal{F}^R$-Brownian motion called innovation

$$dW_t = \frac{1}{\sigma}(dR_t - Y_t dt).$$

- Itô’s formula gives

$$dR_t = d\phi(Y_t) + \frac{1}{2} (\bar{y} + y) dt$$

where

$$\phi(y) = \frac{\sigma^2}{\bar{y} - y} \ln \left( \frac{y - y}{\bar{y} - y} \right).$$
The control problem.

- The cash reserves $X = (X_t)_{t \geq 0}$ of the firm evolve according to
  \[ dX_t = dR_t - dL_t \]  
  \( (1) \)

- The process $L = (L_t)_{t \geq 0}$ is $\mathcal{F}^R$ adapted and right-continuous and nondecreasing processes with $L_{0-} = 0$.

- The firm ceases its activity for two possible reasons:
  (i) it cannot meet its short-term operating costs by drawing cash from its reserves *liquidity problem*
  (ii) the firm’s management decides to default for profitability reasons because the belief that the drift parameter $Y$ is $y < 0$ is strong *solvency problem*.

- Equation (1) represents the dynamics of the cash reserve up to the time $\tau_0$ where $\tau_0 = \inf\{t \geq 0 \mid X_t = 0\}$

Stéphane Villeneuve 14
The control problem.

- A control policy $\pi = (L_t; t \geq 0)$ is admissible if
  
  $$X^{\pi}_t \geq 0, e^{-rt}X^{\pi}_t \text{ integrable and } \lim_{t \to \infty} e^{-rt}X^{\pi}_t = 0 \text{ a.s and in } L^1.$$ 

- For a given control $\pi$ we define the firm value for all $(x, y) \in [0, \infty) \times (y, \bar{y})$.

  $$V_\pi(x, y) = \mathbb{E}_{(x,y)} \left[ \int_0^{\tau_0} e^{-rt} dL_t \right],$$

  where $\Delta L_{\tau_0} = \max(X_{\tau_0}^-, 0)$. The case $\Delta L_{\tau_0} = X_{\tau_0}^-$ corresponds to a strategic default.

- The objective is to find the optimal value function

  $$V^*(x, y) = \sup_\pi V(x, y).$$
Introduction

Benchmarks

The model

Benchmark: deep pocket shareholders.

- Itô’s formula gives

\[ V^*(x, y) = x + \sup_{\pi} \mathbb{E}_{(x,y)} \left[ \int_0^{\tau_0} e^{-rs} (-rX_\pi^s + Y_s) \, ds \right]. \]

- For all \((x, y) \in [0, \infty) \times (\underline{y}, \bar{y}), V^*(x, y) \leq \bar{V}(x, y)\) where

\[ \bar{V}(x, y) \equiv x + \sup_{\tau \in \mathcal{T}_R} \mathbb{E}_{y} \left[ \int_0^{\tau} e^{-rs} Y_s \, ds \right]. \]  \hspace{1cm} (2)

- Optimal stopping for (2)

\[ \tau_{y^*} = \inf \{ t \geq 0 : Y_t \leq y^* \} \quad y^* < 0. \]
Proposition

- The mapping \((x, y) \rightarrow V^*(x, y)\) is continuous on 
  \([0, \infty) \times (-1, 1)\).

- For any \(x \in [0, \infty)\), the mapping \(y \rightarrow V^*(x, y)\) is increasing
  and convex on \((\underline{y}, \overline{y})\). Positive value of information

- For any \(y \in (\underline{y}, \overline{y})\), the mapping \(x \rightarrow V^*(x, y)\) is increasing
  and concave on \([0, \infty)\). Precautionary role of cash reserves
Properties of the value function

\( V^*(x,.) \) can be continuously extended at \( y = \bar{y} \):

Let us consider the standard Jeanblanc-Shiryaev problem with known drift \( \bar{y} \):

- Controlled cash reserves: \( dX_t = \bar{y} dt + \sigma dB_t - dL_t \)
- \( V_{JS}(x) \equiv \sup_L \mathbb{E}_x \left[ \int_0^{\tau_0} e^{-rt} dL_t \right] \)

**Proposition**

*The following holds*

For every \( x > 0 \) \( \lim_{y \rightarrow \bar{y}} V^*(x,y) = V_{JS}(x) \).
HJB Equation and Verification Theorem

Proposition

Assume there exists a twice continuously differentiable function with bounded first derivatives $V$ defined on $[0, \infty) \times (y, \bar{y})$ that satisfies $V(0, .) = 0$, $V(., y)$ concave and

$$\max(\mathcal{A} V - rV, 1 - V_x) \leq 0,$$

where

$$\mathcal{A} V(x, y) = \frac{1}{2\sigma^2} (y - \bar{y})^2 (\bar{y} - y)^2 V_{yy} + \frac{1}{2} \sigma^2 V_{xx} + (1 - y^2) V_{xy} + y V_x.$$

then $V \geq V^*$. 
Heuristics about the target cash level

Let $g(y) = \inf\{x, \ V_x(x, y) = 1\}$ the target cash level.

Concavity of $x \mapsto V(x, y)$: $V_x(x, y) > 1$ for $x \in (0, g(y))$, and $V_x(x, y) = 1$ for $x \geq g(y)$.

For all $y \in (\underline{y}, \bar{y})$, $g(y) \leq \bar{x}$.

where $\bar{x}$ is the dividend boundary associated to the JS problem with cash reserves $dX_t = \bar{y}dt + \sigma dB_t - dL_t$. 
Goal: Find a concave function $V$ and a boundary $g$ that solve the free-boundary problem

\[
\begin{align*}
g(y) &= \bar{x} \\
V(0, y) &= 0 \quad \forall y \in [0, 1), \\
\mathcal{A}V(x, y) - rV(x, y) &= 0 \text{ on } \{(x, y), \, 0 < x < g(y)\}, \\
V_x(g(y), y) &= 1, \\
V_{xy}(g(y), y) &= 0.
\end{align*}
\]
Solving the free-boundary problem

- \( \bar{y} = -y = 1 \).
- Deterministic time-independent relationship between the cumulative cash-flow process and the belief process.

\[ dR_t = d\phi(Y_t). \]

- Cash reserve process

\[ X_t = \phi(Y_t) - \phi(y) + x - L_t \]

where \( X_0 = x \) and \( Y_0 = y \).
Solving the free-boundary problem... and the control problem

Idea: to consider the change of variable $Z_t = X_t - \phi(Y_t)$, to define $U(z, y) \equiv V(\phi(y) + z, y)$, to re-state the free-boundary problem in the $(z,y)$-space.

Equation $\mathcal{A} V(x, y) - rV(x, y) = 0$ takes the form

$$\frac{1}{2\sigma^2}(y + 1)^2(1 - y)^2 U_{yy}(z, y) - rU(z, y) = 0.$$
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The boundary $g$ is an increasing function defined as the unique solution to $g'(y) = f(y, g(y))$ with terminal condition $g(1) = x_1$. 

Attainability: $(V^*, x^*) = (V, g)$, two dimensional version of the Skohorod lemma to a diffusion reflected at the boundary in horizontal directions. (Burdzy and Toby (1995), Annals of Prob.)
Solving the free-boundary problem... and the control problem

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The boundary $g$ is an increasing function defined as the unique solution to $g'(y) = f(y, g(y))$ with terminal condition $g(1) = x_1$. The optimal time to default is the hitting time of $y^{**}$ where $g(y^{**}) = 0$. 

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We develop a simple dynamic model of a firm facing both solvency and liquidity risks.

1.5-dimensional stochastic control problem that we solve quasi-explicitly.

The dividend boundary $x^*$ is continuously increasing in the belief about the profitability of the firm and converges to the JS dividend boundary.

Next: Costly external funding $\rightarrow$ optimal issuance.