Semi-Static Completeness and Model-independent Pricing by Informed Investors

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joint work with Martin Larsson

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Outline

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Model-independent framework:

- $\mathcal{X}$: path-space, $S$: canonical process on $\mathcal{X}$
- $\Psi$: set of claims $\psi$ available for buy-and-hold trading
- $\mathcal{M}$: martingale measures consistent w/ the market price of $\psi$’s
- $\Phi$: a given derivative, robust pricing: $\sup_{Q \in \mathcal{M}} E_Q [\Phi]$
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A central problem in model-independent finance is to prove:

$$\sup_{Q \in \mathcal{M}} \mathbb{E}_Q [\Phi] = \inf \left\{ c \in \mathbb{R} : \Phi \text{ can be hedged pathwise} \right\}$$

starting with initial capital $c$
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- **Note**: $\mathcal{M}$ clearly depends on the underlying filtration, as does the set of available trading strategies.

- **Question**: What can be said about the relation between the super-hedging price and the choice of filtration? In particular, when passing from $\mathcal{F}$ to $\mathcal{G} \supseteq \mathcal{F}$?
Insider information

- Uninformed agent $F \subseteq G$ Informed agent
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- How do things change?

$$\sup_{\mathcal{Q} \in \mathcal{M}} \mathbb{E}_\mathcal{Q} [\Phi] = \inf \left\{ c \in \mathbb{R} : \Phi \text{ can be semi-s.-hedged starting with initial capital } c \right\}$$
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- Informed agent has more trading strategies
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- **Question**: Which measures in $\mathcal{M}(F)$ are still relevant for pricing for the informed agent?
Setup

- \((\Omega, \mathbb{F}, \mathcal{F})\): Filtered measurable space with \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) right-continuous.

\[ \quad \leadsto \text{Later we will consider other filtrations.} \]

- \(S = (S_t)_{0 \leq t \leq T}\): càdlàg \(\mathbb{F}\)-adapted discounted price process of an asset available for dynamic trading. We assume \(S_0 = 0\). (Everything works the same for multiple assets.)

- A risk-free asset with price \(\equiv 1\) available for dynamic trading.

- \(\Psi = \{\psi_1, \ldots, \psi_n\}\) a set of \(\mathcal{F}_T\)-measurable payoffs available for buy-and-hold trading. Today’s price of \(\psi_i\) is zero for each \(i\).
Martingale measures

**Calibrated martingale measures:**

\[ \mathcal{M}(\mathcal{F}) = \left\{ Q \in \mathcal{P}(\mathcal{F}_T) : \begin{array}{l}
S \text{ is an } \mathcal{F} \text{-martingale, } E_Q[S_T^2] < \infty, \\
E_Q[\psi | \mathcal{F}_0] = 0, \ E_Q[\psi^2] < \infty \text{ for all } \psi \in \Psi
\end{array} \right\} \]
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- We want to study \( M(F) \) w.r.to \( F \)
- \( M(F) \) is “huge”

\[ \hookrightarrow \text{Can we reduce to the study of a special subset?} \]
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\[ \mathrm{Can we reduce to the study of a special subset?} \]

\[ \mathrm{For example, if } \mathcal{P}(\mathcal{F}_T) \text{ is endowed with a topology s.t.} \]

\[ \mathcal{M}(\mathbb{F}) \text{ is compact, then} \]

\[ \mathcal{M}(\mathbb{F}) = \text{conv}(\text{ext } \mathcal{M}(\mathbb{F})), \]

where \( \text{ext } \mathcal{M}(\mathbb{F}) \) is the set of all extreme points in \( \mathcal{M}(\mathbb{F}) \).
Extreme points

**Extreme points:** $Q \in \mathcal{M}(\mathbb{F})$ is called an extreme point if

$$Q = \lambda Q^1 + (1 - \lambda) Q^2$$

for $Q^i \in \mathcal{M}(\mathbb{F})$, $\lambda \in (0, 1)$

$$\implies \quad Q^1 = Q^2 = Q$$
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  $$\implies Q^1 = Q^2 = Q$$

- Consider an $\mathcal{F}_T$-measurable payoff $\Phi$ and endow $\mathcal{P}(\mathcal{F}_T)$ with a topology such that
  1. $\mathcal{M}(\mathbb{F})$ is compact and  
  2. $Q \mapsto \mathbb{E}_Q[\Phi]$ is continuous.

  Then  
  $$\sup_{Q \in \mathcal{M}(\mathbb{F})} \mathbb{E}_Q[\Phi] = \sup_{Q \in \text{ext } \mathcal{M}(\mathbb{F})} \mathbb{E}_Q[\Phi].$$
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- $\mathcal{M}(\mathcal{F})$ is compact  
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Then $\sup_{Q \in \mathcal{M}(\mathcal{F})} \mathbb{E}_Q[\Phi] = \sup_{Q \in \text{ext } \mathcal{M}(\mathcal{F})} \mathbb{E}_Q[\Phi]$.

**Note:** The notion of extreme point is purely algebraic, independent of any topology we may put on the space of probability measures.
Example (Discrete time and bounded prices)

- $\Omega = [a, b]^T$, $S$ is the coordinate process,
- each $\omega \mapsto \psi_i(\omega)$ is continuous,
- $F$ is generated by $S$

Then $\mathcal{M}(F)$ is weakly compact.
### Examples

#### Example (Discrete time and bounded prices)

- \( \Omega = [a, b]^T \), \( S \) is the coordinate process,
- each \( \omega \mapsto \psi_i(\omega) \) is continuous,
- \( F \) is generated by \( S \)

Then \( M(F) \) is weakly compact.

#### Example (Continuous time and bounded volatility)

- \( \Omega = \{ \omega \in C_0[0, T] : \langle \omega \rangle \text{ and } \langle \omega \rangle' \text{ exist and are bounded by 1} \} \),
- \( S \) coordinate process, \( \omega \mapsto \psi_i(\omega) \) bounded and continuous,
- \( F \) is generated by \( S \)

Then \( M(F) \) is weakly compact.
Example (Jakubowski topology)

- $\Omega = D_0([0, T], [-1, 1])$ with Jakubowski’s S-topology,
- $S$ is the coordinate process, $\psi_i$ suitable continuity conditions,
- $\mathcal{F}$ is generated by $S$

Semi-static completeness and the Jacod-Yor theorem
The classical Jacod-Yor theorem

- Suppose \( \Psi = \emptyset \) (no static claims).
- For \( Q \in \mathcal{M}(\mathbb{F}) \), by the classical Jacod-Yor (1977) theorem:
  \[
  Q \in \text{ext} \mathcal{M}(\mathbb{F}) \iff L^2(\mathcal{F}_T) = \{ x + (H \cdot S)_T : H \in L^2(S) \} 
  \]
  classical completeness (in \( L^2 \))
- This result can be generalized to the semi-static case.
Generalization of the Jacod-Yor theorem

**Definition**

For $Q \in \mathcal{M}(\mathbb{F})$, we say that **semi-static completeness** holds if any $X \in L^2(\mathcal{F}_T)$ can be represented as

$$X = x + a_1 \psi_1 + \cdots + a_n \psi_n + (H \cdot S)_T$$

for some $x, a_1, \ldots, a_n \in \mathbb{R}$ and $H \in L^2(S)$.

**Notation:**

$$\text{SSC}(\mathbb{F}) = \{ Q \in \mathcal{M}(\mathbb{F}) : \text{semi-static completeness holds} \}$$
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Theorem (semi-static Jacod-Yor theorem)

*The extreme martingale measures are exactly the semi-statically complete models, i.e.*

$$\text{ext } \mathcal{M}(\mathbb{F}) = \text{SSC}(\mathbb{F}).$$
Generalization of the Jacod-Yor theorem

About the proof.

- The proof is very close to the classical case . . .
- . . . but uses duality for random variables \((L^1 - L^\infty)\) instead of processes \((H^1 - BMO)\):
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- . . . but uses duality for random variables \((L^1 - L^\infty)\) instead of processes \((H^1 - \text{BMO})\):

1. Fix \(Q \in \text{ext} \, \mathcal{M}(\mathbb{F})\) and show that this set is dense in \(L^1(\mathcal{F}_T)\)

\[
\left\{ x + \sum_i a_i \psi_i + (H \cdot S)_T : x, a_i \in \mathbb{R}, H \in L^2(S) \right\}.
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\]

2. Prove it is dense and closed in \(L^2(\mathcal{F}_T)\) using Hahn-Banach and a result by Yor (see also Delbaen/Schachermayer, 1999):

**Theorem (Yor (1978))**

Let \(H^n \in L(S)\) be such that \(H^n \cdot S\) is a martingale for each \(n\), and suppose \(\lim_n (H^n \cdot S)_T = X\) in \(L^1\) for some r.v. \(X\). Then there is \(H \in L(S)\) such that \(H \cdot S\) is a martingale with \((H \cdot S)_T = X\).
Generalization of the Jacod-Yor theorem

Remarks.

- Infinitely many $\psi_i$’s would allow to treat the case of a fixed (by the market) marginal law $S_T \sim \mu$
- But the arguments we use in the above proof break down in this case – for the moment we are only able to deal with finitely many $\psi_i$’s
Generalization of the Jacod-Yor theorem

Can we say more?

- In the classical case ($\Psi = \emptyset$), completeness is a strong property — but still allows for many “unstructured” models.
- For instance, completeness holds if $\mathcal{F}$ is generated by $S$, and $S$ is a strong solution to a possibly path-dependent SDE of the form

$$dS_t = \sigma(t; S_u : u \leq t) dW_t$$

where $W$ is a Brownian motion and $\sigma$ is never zero.
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**Question:** Should we expect some additional structure in the semi-static case?
A curious consequence of semi-static completeness

- We are going to show an interesting consequence of semi-static completeness
- For the moment we assume $\Psi = \{\psi\}$
- Fix $Q$ semi-statically complete model
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**Notation:** For any martingale $N$, denote

$$S(N) = \left\{ H \cdot N : H \in L^2(N) \right\}.$$

This is a closed subspace of $H^2$ (stable subspace generated by $N$).
A curious consequence of semi-static completeness

Let \( K \cdot S \) be the orthogonal projection of \( \mathbb{E}_Q[\psi \mid \mathcal{F}_t] \) onto \( S(S) \) and define

\[
M_t = \mathbb{E}_Q[\psi \mid \mathcal{F}_t] - (K \cdot S)_t
\]

**Note**: \( M_T \) is the part of \( \psi \) which is not replicable by trading on \( S \)
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- Then $H \cdot M \perp S(S)$ for any $H \in L^2(M)$
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- By semi-static completeness,
  $$\mathcal{H}^2 = \text{span}\{1\} \oplus \text{span}\{M\} \oplus S(S)$$
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By semi-static completeness,

$$\mathcal{H}^2 = \text{span}\{1\} \oplus \text{span}\{M\} \oplus S(S)$$

Consequently,

$$S(M) = \text{span}\{M\},$$

which is **one-dimensional**!
A curious consequence of semi-static completeness

We will use the following result on $\psi$:

**Lemma**

Let $N$ be a square-integrable martingale null at zero. The following are equivalent:

1. $S(N) = \text{span}\{N\}$
2. $N = N_T 1_{B \times [t^*, T]}$ for some $t^* \in (0, T]$ and some atom $B$ of $\mathcal{F}_{t^*}$
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And the following one on $S$, when $S$ is *continuous*:

**Lemma**

Let $N$ be a continuous local martingale, and let $B$ be an atom of $\mathcal{F}_{t^*}$ for some $t^* \in (0, T]$. Then $N_t = N_0$ on $B$ for all $t < t^*$. 

A curious consequence of semi-static completeness

Therefore, for $S$ continuous, $\Psi = \{\psi\}$, $Q \in \text{SSC}(\mathcal{F})$, we have

$$M = M_T 1_{B \times [t^*, T]}$$

and

$$S_t = S_0 \text{ on } B \text{ for } t \leq t^*$$
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By semi-static completeness,

$$1_B = Q(B) + aM_T + (H \cdot S)_T$$
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$$= Q(B) + (H \cdot S)^{t^*}$$
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By semi-static completeness,

$$1_B = \mathbb{E}_Q\left[ Q(B) + a M_T + (H \cdot S)_T \mid \mathcal{F}_{t^*^-} \right]$$

$$= Q(B) 1_B + (H \cdot S)_{t^*} 1_B$$

$$= Q(B) 1_B \quad \Rightarrow \quad Q(B) = 1.$$
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Semi-static completeness and filtration structure
Fix $Q \in \mathcal{M}(\mathcal{F})$

For $A \in \mathcal{F}_T$, denote by $t(A)$ the first time $A$ becomes measurable,
$$t(A) = \inf\{t \in [0, T] : A \in \mathcal{F}_t\}.$$

**Definition**

An **atomic tree** is a finite collection $T$ of events in $\mathcal{F}_T$ s.t.:

(i) every $A \in T$ is a non-null atom of $\mathcal{F}_{t(A)}$;
(ii) $\forall A, A' \in T$ s.t. $t(A) < t(A')$, either $A \supseteq A'$ or $A \cap A' = \emptyset$;
(iii) $\forall A, A' \in T$ such that $A \supset A'$, $Q(A \setminus A') > 0$;
(iv) the leaves form a partition of $\Omega$ (up to nullsets), and $A$ is an atom of $\mathcal{F}_{t(A')}^-$ whenever $A'$ is a child of $A$.

**leaf:** $A \in T$ s.t. there is no $A' \in T$ s.t. $A' \subsetneq A$;

**dim $T$:** # leaves

**child:** $A'$ is a child of $A$ if $A, A' \in T$ satisfy $A' \subsetneq A$ and there is no $A'' \in T$ such that $A' \subsetneq A'' \subsetneq A$.
Atomic tree
Atomic tree

Remarks.

- $\sigma(T)$ is well-defined. It can be described as $\sigma(T) = \mathcal{F}_{\zeta(T)}$, where the stopping time $\zeta(T)$ is the “end” of the tree:

$$\zeta(T) = \sum_{A \in T \text{ is a leaf}} t(A)1_A.$$

- Note that $\dim T = \dim L^2(\sigma(T))$. 

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- Note that $\dim T = \dim L^2(\sigma(T))$.

**Definition**

We say that $S$ is **complete on** $A \times [t, T]$ for given $t \in [0, T]$ and $A \in \mathcal{F}_t$ if any $X \in L^2(\mathcal{F}_T)$ can be dynamically replicated there:

$$X = x + (H \cdot S)_T \text{ on } A$$

for some $x \in \mathbb{R}$ and some $H \in L^2(S)$ with $H = 0$ on $[0, t]$. 
Recall: $Q \in \mathcal{M}(\mathbb{F})$ is fixed.

**Theorem**

Assume $S$ is continuous. Then semi-static completeness holds if and only if there exists an atomic tree $T$ such that

1. the set $\{\mathbb{E}_Q[\psi_i \mid \sigma(T)]: i = 1, \ldots, n\}$ contains $\dim T - 1$ linearly independent elements,

2. $S$ is complete on $A \times [t(A), T]$ for each leaf $A \in T$.

In this case, $S$ is constant on $[0, \zeta(T)]$ and

$$L^2(\mathcal{F}_T) = \text{span}\{1, \psi\} + S(S) = L^2(\sigma(T)) \oplus S(S).$$
Recall: $Q \in \mathcal{M}(\mathbb{F})$ is fixed.

**Theorem**

Assume $S$ is continuous. Then semi-static completeness holds if and only if there exists an **atomic tree** $T$ such that

1. the set $\{E_Q[\psi_i | \sigma(T)] : i = 1, \ldots, n\}$ contains $\dim T - 1$ linearly independent elements,

2. $S$ is **complete** on $A \times [t(A), T]$ for each leaf $A \in T$.

In this case, $S$ is constant on $[0, \zeta(T)]$ and

$$L^2(\mathcal{F}_T) = \text{span}\{1, \psi\} + S(S) = L^2(\sigma(T)) \oplus S(S).$$

**Remark:** $\psi_i = E_Q[\psi_i | \sigma(T)] + \left(H^i \cdot S\right)_T$, $i = 1, \ldots, n$. orthog. proj.
Semi-static completeness for continuous price processes

The filtration $\mathcal{F}$ under $Q \in \text{SSC}(\mathcal{F})$. Each set of lines emanating from the leaves of $T$ corresponds to a dynamically complete stock price model.
Semi-static completeness for continuous price processes

Example (Semi-statically complete continuous model)

One static claim $\psi = \langle S, S \rangle_T - K$ with zero value at $t = 0$.

- Pick $t^* \in (0, T)$, $\sigma_1, \sigma_2 > 0$ with $\sigma_1 \neq \sigma_2$.
- Set $Q = \lambda Q^1 + (1 - \lambda) Q^2$ where
  $$S_t = \sigma_i W_{t-t^*} 1_{\{t \geq t^*\}}\text{ under } Q^i,$$
  where $W$ is Brownian motion, and $\lambda$ is determined by calibration:
  $$0 = \mathbb{E}_Q[\psi | \mathcal{F}_0] = \lambda \sigma_1^2 (T - t^*) + (1 - \lambda) \sigma_2^2 (T - t^*) - K.$$

- Define $A_i = \{ \partial^+ \langle S, S \rangle_{t^*} = \sigma_i^2 \}$ and set $T = \{ \Omega, A_1, A_2 \}$.
- $T$ is an atomic tree with $\dim T = 2$ and
  $$\mathbb{E}_Q[\psi | \sigma(T)] = \sigma_1^2 (T - t^*) 1_{A_1} + \sigma_2^2 (T - t^*) 1_{A_2} - K \neq 0.$$
- By the theorem, $Q \in \text{SSC}(\mathcal{F})$. 
The leaves $A_1, A_2$ correspond to Bachelier models with volatilities $\sigma_1 > \sigma_2$. Thus the “variance swap” $\psi = \langle S \rangle_T$ is priced differently under the two models, and can be used to hedge against $A_1$ or $A_2$. 
Semi-static completeness for continuous price processes

Example (Semi-statically complete jump model, but no atomic tree)

\[ \psi = [S, S]_T - K \]

\[ S_t = \begin{cases} 
-t & t < \theta \land t^* \\
1 - \theta + f(\theta) W_{t-\theta} & t \geq \theta, \theta < t^* \\
-t^* + 1_{A_1} \sigma_1 W_{t-t^*} + 1_{A_2} \sigma_2 W_{t-t^*} & t \geq t^*, t^* \leq \theta 
\end{cases} \]

with \( \theta \sim \text{Exp}(1) \), \( W, t^*, \sigma_1, \sigma_2 > 0 \) as above, \( f(t) : [0, t^*) \rightarrow \mathbb{R}_+ \).

**Conclusion:** When the asset is allowed to jump, we do not have anymore the tree structure.
Pricing by informed investors
Setup

- $\mathcal{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$: right-continuous filtration (of the informed agent) with
  \[ \mathcal{F}_t \subseteq \mathcal{G}_t, \quad 0 \leq t \leq T. \]
- Access to the same trading instruments: risk-free asset, $S$, $\Psi$
Setup

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- Access to the same trading instruments: risk-free asset, \( S \), \( \Psi \)

- Consider a payoff \( \Phi \). The robust super-hedging price of the informed agent:
  \[ \sup_{Q \in \mathcal{M}(\mathcal{G})} E_Q[\Phi] \]

- As before, we want to study \( \operatorname{ext} \mathcal{M}(\mathcal{G}) \equiv \operatorname{SSC}(\mathcal{G}) \).
Setup

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As before, we want to study \( \text{ext} \mathcal{M}(\mathcal{G}) \equiv \text{SSC}(\mathcal{G}) \).

**Question:** How are \( \text{SSC}(\mathcal{G}) \) and \( \text{SSC}(\mathcal{F}) \) related?
Specification of $\mathcal{G}$: Progressive enlargement with a random time

- Let $\tau$ be a random time: $[0, T] \cup \{\infty\}$-valued random variable.
- Define

$$\mathcal{G}_t = \bigcap_{u > t} \mathcal{F}_u \vee \sigma(u \wedge \tau)$$

Smallest right-continuous filtration that contains $\mathbb{F}$ and makes $\tau$ a stopping time.
Progressive filtration enlargement

**Specification of $\mathcal{G}$**: Progressive enlargement with a random time

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  Smallest right-continuous filtration that contains $\mathcal{F}$ and makes $\tau$ a stopping time.

- For this kind of filtration enlargement there are clear-cut results between $\text{SSC}(\mathcal{G})$ and $\text{SSC}(\mathcal{F})$. 
Theorem

Assume $S$ is continuous and $S(\omega)$ is not constant on $[0, \tau(\omega)]$ for all $\omega \in \Omega$ such that $\tau(\omega) > 0$. Then

$$\text{SSC}(G) = \{ Q \in \text{SSC}(F) : F = G \text{ up to nullsets} \}$$
Progressive filtration enlargement

**Theorem**

Assume $S$ is continuous and $S(\omega)$ is not constant on $[0, \tau(\omega)]$ for all $\omega \in \Omega$ such that $\tau(\omega) > 0$. Then

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In the proof we use the classical Jeulin-Yor theorem.

- Fix $Q \in \text{SSC}(G)$
- Let $Z$ be the Azéma supermartingale associated with $\tau$:
  $$Z_t = Q(\tau > t \mid F_t)$$
  and let $Z = m - a$ be its Doob-Meyer decomposition

**Theorem (Jeulin-Yor (1978))**

*The following process is a $G$-martingale w.r.to $Q$:*  

$$M_t = 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_{s^-}} \, da_s.$$
Progressive filtration enlargement

Sketch of the proof of “⊆”.

- Fix $Q \in \text{SSC}(\mathbb{G})$
- Consider the process $M_t = 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_s^-} da_s$  \hspace{1cm} (1)
Progressive filtration enlargement

Sketch of the proof of “⊆”.

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- Consider the process $M_t = 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} da_s$ \hspace{1cm} (1)
- By semi-static completeness,

  $M = M_0 + V + H \cdot S,$ \hspace{1cm} (2)

  for some $H \in L(S)$ and martingale $V$ with $V_T \in L^2(\sigma(T))$
Progressive filtration enlargement

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- By (1), (2) and continuity of $S$, by considering the jumps of $M$: $\tau = \inf \left\{ t \in [0, T] : \frac{1}{Z_{t-}} \Delta a_t + \Delta V_t = 1 \right\}.$
Progressive filtration enlargement

Sketch of the proof of “⊆”.

- Fix $Q \in \text{SSC}(\mathcal{G})$
- Consider the process $M_t = 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} da_s$ (1)
- By semi-static completeness,
  $M = M_0 + V + H \cdot S,$ (2)
  for some $H \in L(S)$ and martingale $V$ with $V_T \in L^2(\sigma(T))$
- By (1), (2) and continuity of $S$, by considering the jumps of $M$:
  $\tau = \inf \left\{ t \in [0, T] : \frac{1}{Z_{t-}} \Delta a_t + \Delta V_t = 1 \right\}$.
- Also, by assumption $\tau > \sigma := \inf\{t > 0 : S_t \neq S_0\}$
- And $V$ is constant on $\mathcal{F}$, $\infty$ by our Theorem
- Therefore $\tau = \inf \left\{ t \in [0, T] : \frac{1}{Z_{t-}} \Delta a_t = 1 \right\}$ $\mathcal{F}$-stopping time.
Remarks.

- From the proof it is clear that the set equivalence still holds true without any assumption on $S$ when $\Psi = \emptyset$.
- Easy to generalize the theorem for filtration enlargements with multiple random times $\tau_1, \ldots, \tau_m$. Set

$$G_t = \bigcap_{u > t} F_u \vee \sigma(u \wedge \tau_i : i = 1, \ldots, m).$$

**Theorem**

Assume $S$ is continuous and $S(\omega)$ is not constant on $[0, \tau_i(\omega)]$ for all $\omega \in \Omega$ such that $\tau_i(\omega) > 0$, for all $i = 1, \ldots, m$. Then

$$SSC(G) = \{ Q \in SSC(F) : F = G \text{ up to nullsets} \}$$
More general filtration enlargement

Let $\mathcal{H}$ be a filtration and define

$$
\mathcal{G}_t = \bigcap_{u > t} \mathcal{F}_u \vee \mathcal{H}_u.
$$

Assume that, for any $Q \in \mathcal{M}(\mathcal{G})$, there are filtrations $\mathcal{H}^k$ and random times $\tau_k$ such that

1. $\mathcal{H}$ is the progressive enlargement of $\mathcal{H}^k$ with $\tau_1, \ldots, \tau_k$;
2. $S$ is almost surely non-constant on $[0, \tau_k]$ for all $k$;
3. $\mathcal{F}_t = \bigcap_{k \geq 1} \mathcal{F}_t \vee \mathcal{H}^k_t$ up to nullsets for all $t \in [0, T]$.

**Theorem**

*Under the above conditions,*

$$
SSC(\mathcal{G}) = \{ Q \in SSC(\mathcal{F}) : \mathcal{F} = \mathcal{G} \text{ up to nullsets} \}.
$$
Conclusions

- Motivated by robust super-hedging price computation, we study extreme calibrated martingale measures.

- We obtain a semi-static version of the Jacod-Yor theorem.

- Description of semi-statically complete models in terms of dynamically complete models glued together using an atomic tree.

- Application to robust pricing by informed agents:
  Under structural assumptions, informed agents price using only those models that render the additional information uninformative.

- Lots of things remain to be done and appear to be within reach:
  - Infinitely many static claims
  - Better understanding of price processes with jumps
  - Weaker hypotheses with infinitely many random times
  - …
Thank you!