Option pricing in a quadratic variance swap model

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Motivation:

- More and more instruments traded on variance
- Affine models fail to reproduce large increases in variance
- Quadratic models are an appealing alternative to jumps in variance
- Index options useful in portfolio allocation
- But not straightforward to price.

In this talk:

1. Provide a short introduction to quadratic variance swap models
2. Derive an option pricing method.
Realized variance

- Filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})\).

- The \(\mathbb{Q}\)-dynamics of the index are specified as

\[
\frac{dS_t}{S_{t^-}} = r_t \, dt + \sigma_t \, dB_t + \int_{\mathbb{R}} \xi \left( \chi(dt, d\xi) - \nu_t^Q(d\xi)dt \right),
\]

- Let \(t = t_0 < t_1 < \cdots < t_n = T\). The annualized realized variance is given as:

\[
RV(t, T) = \frac{252}{n} \sum_{i=1}^{n} \left( \log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2
\]

\[
\rightarrow \frac{252}{n} \text{QV}(t, T)
\]

\[
= \frac{252}{n} \int_t^T \sigma_s^2 \, ds + \int_t^T \int_{\mathbb{R}} (\log(1 + \xi))^2 \chi(ds, d\xi).
\]
Variance swaps

- A variance swap initiated at $t$ with maturity $T$, or term $T - t$, pays the difference between the annualized realized variance $RV(t, T)$ and the variance swap rate $VS(t, T)$ fixed at $t$. No arbitrage implies that

$$VS(t, T) = \frac{1}{T - t} \mathbb{E}^Q [QV(t, T) | \mathcal{F}_t] = \frac{1}{T - t} \mathbb{E}^Q \left[ \int_t^T \nu_s^Q \, ds \mid \mathcal{F}_t \right],$$

where the $Q$-spot variance process is

$$\nu_t^Q = \sigma_t^2 + \int_{\mathbb{R}} (\log(1 + \xi))^2 \nu_t^Q(d\xi).$$
Quadratic variance swap models

- $X$ is a diffusion process in $\mathbb{R}^m$. Under $\mathbb{Q}$, it satisfies:

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t.$$ 

- $X$ is quadratic if its drift and diffusion functions are linear and quadratic in the state variable:

$$\mu(x) = b + \beta x,$$

$$\Sigma(x)\Sigma(x)^T = a + \sum_{k=1}^{m} \alpha^k x_k + \sum_{k,l=1}^{m} A^{kl} x_k x_l.$$ 

- A quadratic variance swap model is obtained under the assumption that the $\mathbb{Q}$-spot variance is a quadratic function of the latent state variable $X_t$:

$$\nu^Q_t = g(X_t) = \phi + \psi^T X_t + X_t^T \pi X_t$$

for $\phi \in \mathbb{R}$, $\psi \in \mathbb{R}^m$, and $\pi \in \mathbb{S}^m$. 

Option pricing in a quadratic variance swap model
The quadratic variance swap model admits a quadratic term structure:

\[(T - t) \text{VS}(t, T) = \Phi(T - t) + \Psi(T - t)^T X_t + X_t^T \Pi(T - t) X_t = G(T - t, X_t)\]

where \(\Phi, \Psi\) and \(\Pi\) satisfy the linear system of ODEs

\[
\begin{align*}
\frac{d\Phi(\tau)}{d\tau} &= \phi + b^T \Psi(\tau) + \text{tr}(a\Pi(\tau)) & \Phi(0) &= 0 \\
\frac{d\Psi(\tau)}{d\tau} &= \psi + \beta^T \Psi(\tau) + 2\Pi(\tau)b + \alpha \cdot \Pi(\tau) & \Psi(0) &= 0 \\
\frac{d\Pi(\tau)}{d\tau} &= \pi + \beta^T \Pi(\tau) + \Pi(\tau)\beta + A \cdot \Pi(\tau) & \Pi(0) &= 0
\end{align*}
\]

where \((\alpha \cdot \Pi)_k = \text{tr}(\alpha^k \Pi)\) and \((A \cdot \Pi)_{kl} = \text{tr}(A^{kl} \Pi)\).
A Bivariate Model Specification

- Assume $X_t = (X_{1t}, X_{2t})^T$:
  
  \[ dX_{1t} = (b_1 + \beta_{11}X_{1t} + \beta_{12}X_{2t})dt + \sqrt{a_1 + \alpha_1X_{1t} + A_1X_{1t}^2}dW_{1t}^* \]
  
  \[ dX_{2t} = (b_2 + \beta_{22}X_{2t})dt + \sqrt{a_2 + \alpha_2X_{2t} + A_2X_{2t}^2}dW_{2t}^* \]

- Spot variance quadratic in $X_{1t}$:
  
  \[ v_t = \phi_0 + \psi_0X_{1t} + X_{1t}\pi_0X_{1t}. \]

- **Interpretation**: variance mean-reverts to a stochastic level
  
  \[ \frac{b_1 + \beta_{12}X_{2t}}{|\beta_{11}|}. \]

- Explicit representation of forward variance $f(t, T)$. 
Estimation results

T = 2 months

- Model-based VS rates
- Actual VS rates

Option pricing in a quadratic variance swap model
Modelling assumptions

- The price process jumps by a deterministic size $\xi > -1$. Jumps are driven by a Poisson process:

$$\frac{dS_t}{S_{t-}} = r \, dt + \sigma(X_t) \mathbf{R}(X_t) \mathbf{d}W_t + \xi \left(dN_t - \nu^Q(X_t) \, dt\right).$$

The $\mathbb{Q}$-spot variance is given by:

$$\nu_t^Q = g(X_t) = \sigma(X_t)^2 + (\log(1 + \xi))^2 \nu^Q(X_t).$$

- $\nu^Q(x) = \nu^Q \sigma(x)^2$,

$$\sigma(x)^2 = \frac{g(x)}{1 + (\log(1 + \xi))^2 \nu^Q} = K_{\sigma^2} g(x).$$
Option pricing

Main steps:

- Derive the moments of the log price process $L_t = \log S_t$.

- Edgeworth expansion of the characteristic function of $L_T | \mathcal{F}_{t_0}$:

  \[
  \mathbb{E}_t^Q [e^{zL_T}] = \exp \left( \sum_{n=1}^{\infty} C_n \frac{z^n}{n!} \right) = \exp \left( C_1 z + C_2 \frac{z^2}{2} \right) \left( 1 + C_3 \frac{z^3}{3!} + O(z^4) \right).
  \]

Decomposition of $L_t = \log S_t$

For $t \geq t_0 \geq 0$, $L_t = Y_t + X_{3t}$ where

$$Y_t = \int_{t_0}^{t} \left( (\log(1 + \xi) - \xi)\nu^Q(X_s) - \frac{1}{2}\sigma(X_s)^2 \right) ds$$

$$= K_Y \int_{t_0}^{t} g(X_s) ds,$$

where

$$K_Y = -\frac{(\xi - \log(1 + \xi))\nu^Q + 1/2}{1 + (\log(1 + \xi))^2\nu^Q} < 0.$$

$$dX_{3t} = r \, dt + \sigma(X_t)R(X_t)^\top dW_t + \log(1 + \xi)(dN_t - \nu^Q(X_t) dt), \quad X_{3t_0} = \log S_{t_0}.$$
Analysis of $X_t$

- Define the jump-diffusion process $X_t = (X_{1t}, X_{2t}, X_{3t})^T$. Its diffusion matrix $A(x)$ is

\[
A(x) = \begin{pmatrix}
1 + A_1 x_1^2 & 0 & R_1(x) \sigma(x) \sqrt{1 + A_1 x_1^2} \\
0 & x_2 + A_2 x_2^2 & 0 \\
R_1(x) \sigma(x) \sqrt{1 + A_1 x_1^2} & 0 & \sigma(x)^2
\end{pmatrix}.
\]

- We want $q_0, q_1$ and $q_2$ such that

\[
R_1(x) \sigma(x) \sqrt{1 + A_1 x_1^2} = R_1(x_1) \sqrt{K_{\sigma^2} g(x_1) \sqrt{1 + A_1 x_1^2}} = q_0 + q_1 x_1 + q_2 x_1^2.
\]

- To capture the leverage effect, $R_1(x_1) \approx -\text{sign} (\psi + 2\pi X_{1t}) \times 0.7$. Hence we choose $q_0, q_1$ and $q_2$ to match the highest order terms of:

\[
(q_0 + q_1 x_1 + q_2 x_1^2)^2 \approx 0.7^2 K_{\sigma^2} g(x_1)(1 + A_1 x_1^2).
\]

- With this specification, $X_t$ is a quadratic jump-diffusion process and hence is polynomial preserving.
Moments of $X_T$


**Conditional moments of $X_T$**

Let $D = \frac{(3+N)(2+N)(1+N)}{6}$ denote the dimension of the space of polynomials in $X_T$ of degree $N$ or less. The $D$-row vector of the mixed $\mathcal{F}_{t_0}$-conditional moments of $X_T$ of order $N$ or less with $T \geq t_0$ is given by

$$
\left(1, \mathbb{E}^Q[X_{1T}|\mathcal{F}_{t_0}], \ldots, \mathbb{E}^Q[X_{2T}X_{3T}^{N-1}|\mathcal{F}_{t_0}], \mathbb{E}^Q[X_{3T}^N|\mathcal{F}_{t_0}]\right)
$$

$$
= \left(1, X_{1t_0}, \ldots, X_{2t_0}X_{3t_0}^{N-1}, X_{3t_0}^N\right) e^{\tilde{B}(T-t_0)},
$$

where $\tilde{B}$ is an upper block triangular $D \times D$ matrix and $e^{\tilde{B}(T-t_0)}$ denotes the matrix exponential of $\tilde{B}(T-t_0)$. 
Moments of $X_T$

**Proof:** The generator of $X_t$ is given by:

$$Af(x) = \begin{pmatrix} \beta_{11} x_1 + \beta_{12} x_2 \\
 b_2 + \beta_{22} x_2 \\
 r \end{pmatrix}^T \nabla_x f(x) + \frac{1}{2} \sum_{i,j=1}^{3} A_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

$$+ \left( f(x + \log(1 + \xi)e_3) - f(x) - \nabla_x f(x)^T e_3 \log(1 + \xi) \right) \nu^Q(x),$$

where $e_3 = (0, 0, 1)^T \Rightarrow A$ polynomial preserving.

The mixed conditional moments satisfy the backward Kolmogorov equation. Solve the PDE by guessing that the solution is a polynomial in $X_{t_0}$ of degree $N$. Apply $A$ to the mixed powers $1, x_1, ..., x_2 x_3^{N-1}, x_3^N$ and collect terms.
Moments of $L_T$

- Powers of $L_T$ are obtained from

$$L^n_T = (Y_T + X_{3T})^n = \sum_{k=0}^{n} \binom{n}{k} Y^n_T X_{3T}^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} K^n_Y(k!) \int_{t_0}^{T} \int_{t_1}^{T} \ldots \int_{t_{k-1}}^{T} g(X_{t_1}) \ldots g(X_{t_k}) dt_k \ldots dt_1 X_{3T}^{n-k}.$$ 

- Moments of $L_T$ calculated using nested conditional expectations:

$$\mathbb{E}_t^Q \left[ g(X_{t_1}) \ldots g(X_{t_k}) X_{3T}^{n-k} \right] = \mathbb{E}_t^Q \left[ g(X_{t_1}) \ldots g(X_{t_k}) \mathbb{E}_t^Q \left[ X_{3T}^{n-k} \right] \right]$$

$$= \mathbb{E}_t^Q \left[ g(X_{t_1}) \ldots g(X_{t_{k-1}}) \mathbb{E}_t^{Q} \left[ g(X_{t_k}) P_0(t_k, X_{t_k}) \right] \right]$$

$$= P_k(t_k, t_{k-1}, \ldots, t_0, X_{t_0}).$$
Conclusion

- We develop a quadratic variance swap model which is tractable and parsimonious in the number of parameters.

- Variance swap rates are available in closed-form, up to the resolution of ODEs.

- We derive an pricing methodology for European index options, which uses the polynomial preserving property of quadratic jump-diffusions to approximate the characteristic function of the log price.

Thank you for your attention!