Nonlinear Price Impact and Portfolio Choice

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Outline

- Motivation:
  Optimal Rebalancing and Execution.
- Model:
  Nonlinear Price Impact.
  Constant investment opportunities and risk aversion.
- Results:
  Optimal policy and welfare. Implications.
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Price Impact and Market Frictions

• Classical theory: no price impact. Same price for any quantity bought or sold. Merton (1969) and many others.

• Bid-ask spread: constant (proportional) “impact”. Price depends only on sign of trade. Constantinides (1985), Davis and Norman (1990), and extensions.

• Price linear in trading rate. Asymmetric information equilibria (Kyle, 1985), (Back, 1992). Quadratic transaction costs (Garleanu and Pedersen, 2013)


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- Price nonlinear in trading rate. 
  Empirical evidence: Hasbrouck and Seppi (2001), Plerou et al. (2002), 
  Lillo et al. (2003), Almgren et al. (2005).

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  Portfolio choice?
Portfolio Choice with Frictions

- With constant investment opportunities and constant relative risk aversion:
  - Classical theory: hold portfolio weights constant at Merton target.
  - Proportional bid-ask spreads: hold portfolio weight within buy and sell boundaries (no-trade region).
  - Linear impact: trading rate proportional to distance from target.
  - Rebalancing rule for nonlinear impact?
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• Inputs
  • Price exogenous. Geometric Brownian Motion.
  • Constant relative risk aversion and long horizon.
  • Nonlinear price impact:
    trading rate one-percent higher means impact $\alpha$-percent higher.

• Outputs
  • Optimal trading policy and welfare.
  • High liquidity asymptotics.
  • Linear impact and bid-ask spreads as extreme cases.

• Focus is on temporary price impact:
  • No permanent impact as in Huberman and Stanzl (2004)
  • No transient impact as in Obizhaeva and Wang (2006) or Gatheral (2010).
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Market

- Brownian Motion $(W_t)_{t \geq 0}$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$.
- Best quoted price of risky asset. Price for an infinitesimal trade.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

- Trade $\Delta \theta$ shares over time interval $\Delta t$. Order filled at price

$$\tilde{S}_t(\Delta \theta) := S_t \left(1 + \lambda \left| \frac{S_t \Delta \theta_t}{X_t \Delta t} \right|^\alpha \sgn(\dot{\theta}) \right)$$

where $X_t$ is investor’s wealth. Proxies total market’s wealth.
- $\lambda$ measures illiquidity. $1/\lambda$ market depth. Like Kyle’s (1985) lambda.
- Price worse for larger quantity $|\Delta \theta|$ or shorter execution time $\Delta t$.
  Price linear in quantity, inversely proportional to execution time.
- Impact of dollar trade $S_t \Delta \theta$ declines as large investor’s wealth increases.
- Makes model scale-invariant.
  Doubling wealth, and all subsequent trades, doubles final payoff exactly.
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Alternatives?

- Alternatives: quantities $\Delta \theta$, or share turnover $\Delta \theta/\theta$. Consequences?
  - Quantities ($\Delta \theta$):
    \[
    \tilde{S}_t(\Delta \theta) := S_t + \lambda \frac{\Delta \theta}{\Delta t}
    \]
  - Price impact independent of price. Not invariant to stock splits!
  - Suitable for short horizons (liquidation) or mean-variance criteria.
  - Share turnover:
    Stationary measure of trading volume (Lo and Wang, 2000). Observable.
    \[
    \tilde{S}_t(\Delta \theta) := S_t \left(1 + \lambda \frac{\Delta \theta}{\theta_t \Delta t}\right)
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  - Problematic. Infinite price impact with cash position.
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Wealth and Portfolio

- Continuous time: cash position

\[ dC_t = -S_t \left( 1 + \lambda \left| \frac{\dot{\theta} S_t}{X_t} \right|^\alpha \text{sgn}(\dot{\theta}) \right) d\theta_t = - \left( \frac{S_t \dot{\theta} t}{X_t} + \lambda \left| \frac{\dot{\theta} S_t}{X_t} \right|^{1+\alpha} \right) X_t dt \]

- Trading volume as wealth turnover \( u_t := \frac{\dot{\theta} S_t}{X_t} \).
  Amount traded in unit of time, as fraction of wealth.

- Dynamics for wealth \( X_t := \theta_t S_t + C_t \) and risky portfolio weight \( Y_t := \frac{\theta_t S_t}{X_t} \)

\[ \frac{dX_t}{X_t} = Y_t(\mu dt + \sigma dW_t) - \lambda |u_t|^{1+\alpha} dt \]

\[ dY_t = (Y_t(1 - Y_t)(\mu - Y_t \sigma^2) + (u_t + \lambda Y_t |u_t|^{1+\alpha})) dt + \sigma Y_t(1 - Y_t) dW_t \]

- Illiquidity...

- ...reduces portfolio return \((-\lambda u_t^{1+\alpha})\).
  Turnover effect quadratic: quantities times price impact.

- ...increases risky weight \(\lambda Y_t u_t^{1+\alpha}\). Buy: pay more cash. Sell: get less.
  Turnover effect linear in risky weight \(Y_t\). Vanishes for cash position.
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  ...increases risky weight \((\lambda Y_t u_t^{1+\alpha})\). Buy: pay more cash. Sell: get less.
  Turnover effect linear in risky weight \( Y_t \). Vanishes for cash position.
Wealth and Portfolio

- Continuous time: cash position
  
  \[ dC_t = -S_t \left( 1 + \lambda \left| \frac{\dot{\theta}_t S_t}{X_t} \right|^\alpha \right) d\theta_t = - \left( \frac{S_t \dot{\theta}_t}{X_t} + \lambda \left| \frac{\dot{\theta}_t S_t}{X_t} \right|^{1+\alpha} \right) X_t dt \]

- Trading volume as wealth turnover \( u_t := \frac{\dot{\theta}_t S_t}{X_t} \).
  Amount traded in unit of time, as fraction of wealth.

- Dynamics for wealth \( X_t := \theta_t S_t + C_t \) and risky portfolio weight \( Y_t := \frac{\theta_t S_t}{X_t} \)
  
  \[
  \frac{dX_t}{X_t} = Y_t(\mu dt + \sigma dW_t) - \lambda |u_t|^{1+\alpha} dt \\
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## Admissible Strategies

### Definition

Admissible strategy: process \((u_t)_{t \geq 0}\), adapted to \(\mathcal{F}_t\), such that system

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has unique solution satisfying \(X_t \geq 0\) a.s. for all \(t \geq 0\).

- Contrast to models without frictions or with transaction costs: control variable is not risky weight \(Y_t\), but its “rate of change” \(u_t\).
- Portfolio weight \(Y_t\) is now a *state variable*.
- Illiquid vs. perfectly liquid market. Steering a ship vs. driving a race car.
- Frictionless solution \(Y_t = \frac{\mu}{\gamma \sigma^2}\) unfeasible. A still ship in stormy sea.
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Objective

- Investor with relative risk aversion $\gamma$.
- Maximize equivalent safe rate, i.e., power utility over long horizon:
  $$\max_u \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E} \left[ X_T^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$$
  
- Tradeoff between speed and impact.
- Optimal policy and welfare.
- Implied trading volume.
- Dependence on parameters.
- Asymptotics for small $\lambda$.
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Verification

Theorem

If $\frac{\mu}{\gamma \sigma^2} \in (0, 1)$, then the optimal wealth turnover and equivalent safe rate are:

$$\hat{u}(y) = \left| \frac{q(y)}{(\alpha + 1) \lambda (1 - y q(y))} \right|^{1/\alpha} \text{sgn}(q(y))$$  

$$\text{EsR}_{\gamma}(\hat{u}) = \beta$$

where $\beta \in (0, \frac{\mu^2}{2 \gamma \sigma^2})$ and $q : [0, 1] \mapsto \mathbb{R}$ are the unique pair that solves the ODE

$$- \hat{\beta} + \mu y - \gamma \frac{\sigma^2}{2} y^2 + y(1 - y)(\mu - \gamma \sigma^2 y) q$$

$$+ \frac{\alpha}{(\alpha + 1)^{1+1/\alpha}} \frac{|q|^{\alpha+1}}{(1 - y q)^{1/\alpha}} \lambda^{-1/\alpha} + \frac{\sigma^2}{2} y^2 (1 - y)^2 (q' + (1 - \gamma)q^2) = 0$$

$$q(0) = \lambda \frac{1}{\alpha+1} (\alpha + 1) \frac{1}{\alpha+1} \left( \frac{\alpha+1}{\alpha} \beta \right)^{\frac{\alpha}{\alpha+1}}, \quad \frac{\alpha}{(\alpha+1)^{1+1/\alpha}} \frac{|q(1)|^{\alpha+1}}{(1 - q(1))^{1/\alpha}} \lambda^{-1/\alpha} = \hat{\beta} - \mu + \gamma \frac{\sigma^2}{2}$$

- License to solve an ODE of Abel type. Function $q$ and scalar $\beta$ not explicit.
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**Asymptotics**

**Theorem**

$c_\alpha$ and $s_\alpha$ unique pair that solves

\[
 s'(z) = z^2 - c - \alpha(\alpha + 1)^{-(1 + 1/\alpha)}|s(z)|^{1 + 1/\alpha} \quad \lim_{z \to \pm \infty} \frac{|s_\alpha(z)|}{|z|^{2\alpha/(\alpha + 1)}} = (\alpha + 1)^{-\alpha/(\alpha + 1)}
\]

Set $l_\alpha := \left[\left(\frac{\sigma^2}{2}\right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4\right]^\frac{\alpha + 1}{\alpha + 3}$, $A_\alpha = \left(\frac{2l_\alpha}{\gamma \sigma^2}\right)^{1/2}$, $B_\alpha = l_\alpha^{-\frac{\alpha}{\alpha + 1}}$.

Asymptotic optimal strategy and welfare:

\[
 \hat{u}(y) = -\left|\frac{s_\alpha\left(\lambda - \frac{1}{\alpha + 3}(y - \bar{Y})/A_\alpha\right)}{B_\alpha(\alpha + 1)}\right|^{1/\alpha} \text{sgn}(y - \bar{Y})
\]

\[
 \text{EsR}_\gamma(\hat{u}) = \frac{\mu^2}{2\gamma \sigma^2} - c_\alpha l_\alpha \lambda \frac{2}{\alpha + 3} + o(\lambda^{\frac{2}{\alpha + 3}})
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- Implications?
Asymptotics

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• Implications?
Trading Rate ($\mu = 8\%$, $\sigma = 16\%$, $\lambda = 0.1\%$, $\gamma = 5$)

Trading rate (vertical) against current risky weight (horizontal) for $\alpha = 1/8, 1/4, 1/2, 1$. Dashed lines are no-trade boundaries ($\alpha = 0$).
Trading Policy

- Trade towards $\bar{Y}$. Buy for $y < \bar{Y}$, sell for $y > \bar{Y}$.
- Trade faster if market deeper. Higher volume in more liquid markets.
- Trade slower than with linear impact near target. Faster away from target. With linear impact trading rate proportional to displacement $|y - \bar{Y}|$.
- As $\alpha \downarrow 0$, trading rate:
  - vanishes inside no-trade region
  - explodes to $\pm \infty$ outside region.
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Welfare

- Welfare cost of friction:

\[ c_\alpha \left[ \left( \frac{\sigma^2}{2} \right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4 \right]^{\frac{\alpha+1}{\alpha+3}} \lambda^{\frac{2}{\alpha+3}} \]

- Last factor accounts for effect of illiquidity parameter.
- Middle factor reflects volatility of portfolio weight.
- Constant \( c_\alpha \) depends on \( \alpha \) alone. No explicit expression for general \( \alpha \).
- Exponents \( 2/(\alpha + 3) \) and \( (\alpha + 1)/(\alpha + 3) \) sum to one. Geometric average.
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Universal Constant $c_\alpha$

$c_\alpha$ (vertical) against $\alpha$ (horizontal).
Proposition

Rescaled portfolio weight $Z_s^\lambda := \lambda^{-\frac{1}{\alpha+3}} \left( Y_{\lambda^2/(\alpha+3)} - \bar{Y} \right)$ converges weakly to the process $Z_0^0$, defined by

$$dZ_0^0 = v_\alpha(Z_0^0) ds + \bar{Y}(1 - \bar{Y})\sigma dW_s$$

$$v_\alpha(z) := -\left| \frac{B_\alpha s_\alpha(z/A_\alpha)}{(\alpha + 1)} \right|^{1/\alpha} \text{sgn}(z)$$

- “Nonlinear” stationary process. Ornstein-Uhlenbeck for linear impact.
- No explicit expression for drift – even asymptotically.
- Long-term distribution?
Portfolio Dynamics

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Long-term weight ($\mu = 8\%, \sigma = 16\%, \gamma = 5$)

Density (vertical) of the long-term density of rescaled risky weight $Z^0$ (horizontal) for $\alpha = 1/8, 1/4, 1/2, 1$. Dashed line is uniform density ($\alpha \rightarrow 0$).
Linear Impact ($\alpha = 1$)

- Solution to

$$s'(z) = z^2 - c - \alpha(\alpha + 1)^{-\left(1+1/\alpha\right)}|s(z)|^{1+1/\alpha}$$

is $c_1 = 2$ and $s_1(z) = -2z$.

- Optimal policy and welfare:

$$\hat{u}(y) = \sigma \sqrt{\frac{\gamma}{2\lambda}} (\bar{Y} - y) + O(1)$$

$$\text{EsR}_{\gamma}(\hat{u}) = \frac{\mu^2}{2\gamma \sigma^2} - \sigma^3 \sqrt{\frac{\gamma}{2}} \bar{Y}^2 (1 - \bar{Y})^2 \lambda^{1/2} + O(\lambda)$$
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Transaction Costs \((\alpha \downarrow 0)\)

- Solution to

\[
s'(z) = z^2 - c - \alpha(\alpha + 1)^{-1+1/\alpha}|s(z)|^{1+1/\alpha}
\]

converges to \(c_0 = (3/2)^{2/3}\) and

\[
s_0(z) := \lim_{\alpha \to 0} s_\alpha(z) = \begin{cases} 
1, & z \in (-\infty, -\sqrt{c_0}], \\
\frac{z^3}{3} - c_0z, & z \in (-\sqrt{c_0}, \sqrt{c_0}), \\
-1, & z \in [\sqrt{c_0}, +\infty).
\end{cases}
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\[
Y_\pm = \frac{\mu}{\gamma \sigma^2} \pm \left( \frac{3}{4\gamma} \bar{Y}^2 (1 - \bar{Y})^2 \right)^{1/3} \varepsilon^{1/3}
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- Compare to transaction cost model (Gerhold et al., 2014).
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Trading Volume and Welfare

- **Expected Trading Volume**

\[ |ET| := \lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{u}_\lambda(Y_t)| \, dt = K_\alpha \left[ \left( \frac{\sigma^2}{2} \right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4 \right]^{\frac{1}{\alpha+3}} \lambda^{-\frac{1}{\alpha+3}} + o(\lambda^{-\frac{1}{\alpha}}) \]

- Define welfare loss as decrease in equivalent safe rate due to friction:

\[ \text{LoS} = \frac{\mu^2}{2\gamma \sigma^2} - \text{EsR}_\gamma(\hat{u}) \]

- Zero loss if no trading necessary, i.e. \( \bar{Y} \in \{0, 1\} \).

- Universal relation:

\[ \text{LoS} = N_\alpha \lambda |ET|^{1+\alpha} \]

  where constant \( N_\alpha \) depends only on \( \alpha \).

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  Superlinear effect with liquidity (price times quantity).
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Hacking the Model ($\alpha > 1$)

- Empirically improbable. Theoretically possible.
- Trading rates below one cheap. Above one expensive.
- As $\alpha \uparrow \infty$, trade at rate close to one. Compare to Longstaff (2001).
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Neither a Borrower nor a Shorter Be

**Theorem**

If \( \frac{\mu}{\gamma \sigma^2} \leq 0 \), then \( Y_t = 0 \) and \( \hat{u} = 0 \) for all \( t \) optimal. Equivalent safe rate zero.

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- If Merton investor shorts, keep all wealth in safe asset, but do not short.
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- Portfolio choice for a risk-neutral investor!
- Corner solutions. But without constraints?
- Intuition: the constraint is that wealth must stay positive.
- Positive wealth does not preclude borrowing with block trading, as in frictionless models and with transaction costs.
- Block trading unfeasible with price impact proportional to turnover. Even in the limit.
- Phenomenon disappears with exponential utility.
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Control Argument

- Value function $v$ depends on (1) current wealth $X_t$, (2) current risky weight $Y_t$, and (3) calendar time $t$.

$$
dv(t, X_t, Y_t) = v_t dt + v_x dX_t + v_y dY_t + \frac{v_{xx}}{2} d\langle X \rangle_t + \frac{v_{yy}}{2} d\langle Y \rangle_t + v_{xy} d\langle X, Y \rangle_t
$$

$$
= v_t dt + v_x (\mu X_t Y_t - \lambda X_t |u_t|^{\alpha+1}) dt + v_x X_t Y_t \sigma dW_t
+ v_y (Y_t (1 - Y_t)(\mu - Y_t \sigma^2) + u_t + \lambda Y_t |u_t|^{\alpha+1}) dt + v_y Y_t (1 - Y_t) \sigma dW_t
+ \left( \frac{\sigma^2}{2} v_{xx} X_t^2 Y_t^2 + \frac{\sigma^2}{2} v_{yy} Y_t^2 (1 - Y_t)^2 + \sigma^2 v_{xy} X_t Y_t^2 (1 - Y_t) \right) dt,
$$

- Maximize drift over $u$, and set result equal to zero:

$$
n_t + y (1 - y)(\mu - \sigma^2 y) v_y + \mu x y v_x + \frac{\sigma^2 y^2}{2} (x^2 v_{xx} + (1 - y)^2 v_{yy} + 2x(1 - y) v_{xy})
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Homogeneity and Long-Run

- Homogeneity in wealth \( v(t, x, y) = x^{1-\gamma} v(t, 1, y) \).
- Guess long-term growth at equivalent safe rate \( \beta \), to be found.
- Substitution \( v(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} e^{(1-\gamma)(\beta(T-t)+\int_y^y q(z)dz)} \) reduces HJB equation

\[
-\beta + \mu y - \gamma \frac{\sigma^2}{2} y^2 + q y (1-y)(\mu - \gamma \sigma^2 y) + \frac{\sigma^2}{2} y^2(1-y)^2 (q' + (1-\gamma)q^2)
+ \max_u \left( -\lambda |u|^{\alpha+1} + (u + \lambda y |u|^{\alpha+1})q \right) = 0,
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- Maximum for \( |u(y)| = \left| \frac{q(y)}{(\alpha+1)\lambda(1-yq(y))} \right|^{1/\alpha} \).
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- \( \beta = \frac{\mu^2}{2\gamma \sigma^2} \), \( q = 0 \), \( y = \frac{\mu}{\gamma \sigma^2} \) corresponds to Merton solution.
- Classical model as a singular limit.
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\]

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\[
-\beta + \mu y - \gamma \frac{\sigma^2}{2} y^2 + y(1-y)(\mu - \gamma \sigma^2 y)q + \frac{\alpha}{(\alpha + 1)^{1+1/\alpha}} \frac{|q|^\alpha}{(1-yq)^{1/\alpha}} \lambda^{-1/\alpha} + \frac{\sigma^2}{2} y^2(1-y)^2(q' + (1-\gamma)q^2) = 0.
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$$-\beta + \mu y - \gamma \frac{\sigma^2}{2} y^2 + qy(1-y)(\mu - \gamma \sigma^2 y) + \frac{\sigma^2}{2} y^2(1-y)^2(q' + (1-\gamma)q^2)$$

$$+ \max_u (-\lambda |u|^{\alpha+1} + (u + \lambda y|u|^{\alpha+1})q) = 0,$$

- Maximum for $|u(y)| = \left| \frac{q(y)}{(\alpha+1)\lambda(1-yq(y))} \right|^{1/\alpha}$.
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- $\beta = \frac{\mu^2}{2\gamma\sigma^2}$, $q = 0$, $y = \frac{\mu}{\gamma\sigma^2}$ corresponds to Merton solution.
- Classical model as a singular limit.
Homogeneity and Long-Run

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Asymptotics away from Target

- Guess that \( q(y) \to 0 \) as \( \lambda \downarrow 0 \). Limit equation:

\[
\frac{\gamma \sigma^2}{2} (\bar{Y} - y)^2 = \lim_{\lambda \to 0} \frac{\alpha}{\alpha + 1} (\alpha + 1)^{-1/\alpha} |q|^{\frac{\alpha + 1}{\alpha}} \lambda^{-1/\alpha}.
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- Expand equivalent safe rate as \( \beta = \frac{\mu^2}{2\gamma \sigma^2} - c(\lambda) \)

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- Plug expansion in HJB equation

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q^{(1)}(y) = \lambda^{\frac{1}{\alpha+1}} (\alpha + 1)^{\frac{1}{\alpha+1}} \left( \frac{\alpha + 1}{\alpha} \frac{\gamma \sigma^2}{2} \right)^{\frac{\alpha}{\alpha+1}} |\bar{Y} - y|^{\frac{2\alpha}{\alpha+1}} \text{sgn}(\bar{Y} - y).
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Asymptotics close to Target

- Zoom in around target weight $\bar{Y}$.
- Guess $c(\lambda) := \frac{\mu^2}{2\gamma\sigma^2} - \beta = \bar{c}\lambda^{\frac{2}{\alpha+3}}$. Set $y = \bar{Y} + \lambda^{\frac{1}{\alpha+3}}z$, $r_\lambda(z) = q_\lambda(y)\lambda^{-\frac{3}{\alpha+3}}$
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Issues

• How to make argument rigorous?
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Verification

Lemma

Let $q$ solve the HJB equation, and define $Q(y) = \int_{y}^{0} q(z)dz$. There exists a probability $\hat{P}$, equivalent to $P$, such that the terminal wealth $X_T$ of any admissible strategy satisfies:

$$E[X_T^{1-\gamma}]^{1\gamma} \leq e^{\beta T + Q(y)} E_{\hat{P}}[e^{-(1-\gamma)Q(Y_T)}]^{1\gamma},$$

and equality holds for the optimal strategy.

- Solution of HJB equation yields asymptotic upper bound for any strategy.
- Upper bound reached for optimal strategy.
- Valid for any $\beta$, for corresponding $Q$.
- Idea: pick largest $\beta^*$ to make $Q$ disappear in the long run.
- A priori bounds:

$$\beta^* < \frac{\mu^2}{2\gamma\sigma^2}$$  \hspace{1cm} \text{(frictionless solution)}$$

$$\max\left(0, \mu - \frac{\gamma}{2}\sigma^2\right) < \beta^*$$  \hspace{1cm} \text{(all in safe or risky asset)}
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Let \( q \) solve the HJB equation, and define \( Q(y) = \int_y^\infty q(z)dz \). There exists a probability \( \hat{P} \), equivalent to \( P \), such that the terminal wealth \( X_T \) of any admissible strategy satisfies:

\[
E[X_T^{1-\gamma}]^{\frac{1}{1-\gamma}} \leq e^{\beta T + Q(y)} E_{\hat{P}}[e^{-(1-\gamma)Q(Y_T)}]^{\frac{1}{1-\gamma}},
\]

and equality holds for the optimal strategy.

- Solution of HJB equation yields asymptotic upper bound for any strategy.
- Upper bound reached for optimal strategy.
- Valid for any \( \beta \), for corresponding \( Q \).
- Idea: pick largest \( \beta^* \) to make \( Q \) disappear in the long run.
- A priori bounds:

\[
\beta^* < \frac{\mu^2}{2\gamma\sigma^2} \quad \text{(frictionless solution)}
\]

\[
\max\left(0, \mu - \frac{\gamma}{2}\sigma^2\right) < \beta^* \quad \text{(all in safe or risky asset)}
\]
Theorem

Assume $0 < \frac{\mu}{\gamma \sigma^2} < 1$. There exists $\beta^*$ such that HJB equation has solution $q(y)$ with positive finite limit in 0 and negative finite limit in 1.

- for $\beta > 0$, there exists a unique solution $q_{0,\beta}(y)$ to HJB equation with positive finite limit in 0.
- for $\beta > \mu - \frac{\gamma \sigma^2}{2}$, there exists a unique solution $q_{1,\beta}(y)$ to HJB equation with negative finite limit in 1.
- there exists $\beta_u$ such that $q_{0,\beta_u}(y) > q_{1,\beta_u}(y)$ for some $y$;
- there exists $\beta_l$ such that $q_{0,\beta_l}(y) < q_{1,\beta_l}(y)$ for some $y$;
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- Boundary conditions are natural!
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Explosion with Leverage

Lemma

If $Y_t$ that satisfies $Y_0 \in (1, +\infty)$ and

$$dY_t = Y_t(1 - Y_t)(\mu dt - Y_t\sigma^2 dt + \sigma dW_t) + u_t dt + \lambda Y_t |u_t|^{1+\alpha} dt$$

explodes in finite time with positive probability.

Lemma

Let $\tau$ be the exploding time of $Y_t$. Then wealth $X_\tau = 0$ a.s on $\{\tau < +\infty\}$.

- Feller's criterion for explosions.
- No strategy admissible if it begins with levered or negative position.
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