Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting

Thomas Kruse

based on joint work with Stefan Ankirchner, Monique Jeanblanc and Alexandre Popier

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Lausanne

Financial support from the French Banking Federation through the Chaire Markets in Transition is gratefully acknowledged.
Case study: Sell $x$ shares of Adidas within $T$ minutes using market orders.
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Assumption (Almgren&Chriss):

$$S_{\text{mid}} - S_{\text{real}} = \eta z$$

- $\eta$: amount sold at time $t$
- $z$: price impact factor
Case study: Sell $x$ shares of Adidas within $T$ minutes using market orders.

Assumption (Almgren&Chriss):

$$S_t^{\text{mid}} - S_t^{\text{real}} = \eta z$$

$z$: amount sold at time $t$
$\eta$: price impact factor
Stochastic Liquidity

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**Bid/Ask Orders**

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Case study: Sell $x$ shares of Adidas within $T$ seconds using market orders.

Assumption (Almgren&Chriss):

$$S_t^{\text{mid}} - S_t^{\text{real}} = \eta_t z$$

$z$: amount sold at time $t$
$(\eta_t)$: price impact process
The model: Trading rates determine remaining position

- $T < \infty$: time horizon
- $x \in \mathbb{R}$: initial position
- $X_t$: position size at time $t \in [0, T]$: 
  
  $$dX_t = x + \int_0^t \alpha_s ds + \int_0^t \beta_s dN_s$$

  - $\alpha_t$: trading rate at time $t$.
  - $\beta_t$: amount placed as a passive order at time $t$
  - $N$: Poisson process with intensity $\mu > 0$

- **Constraint:** $X_T = 0$ on a set $S \in \mathcal{F}_T$. 
A reduced form model à la Almgren & Chriss

\[ E \left[ \int_0^T \left( \eta_t |\alpha_t|^p + \lambda_t |\beta_t|^p + \gamma_t |X_t|^p \right) dt + \xi 1_{S^c} |X_T|^p \right] \longrightarrow \min \]

- \( p > 1 \) (\( q \) its Hölder conjugate)
- \( (\eta_t), (\lambda_t), (\gamma_t) \): nonnegative, progressively measurable
- \( \xi \): nonnegative, \( \mathcal{F}_T \)-measurable random variable
- stochastic basis \((\Omega, \mathcal{F}, P, (\mathcal{F}_t))\) satisfying usual conditions
Related literature

- Schied 2013: Solves a variant of this problem in a Markovian framework using superprocesses

- Graewe, Horst, Séré 2015: Show smoothness of the value function in a Markovian framework

- Graewe, Horst, Qiu 2014: Analyze both Markovian and non-Markovian dependence of the coefficients by means of BSPDEs
A maximum principle

Let \((Y, \psi, M)\) satisfy

\[ dY_t = \left( (p - 1) \frac{Y_t^q}{\eta_{t-1}} + \Theta(t, Y_t, \psi_t) - \gamma_t \right) dt + \psi_t \, d\tilde{N}_t + dM_t \]

with \(\Theta\) Lipschitz continuous in \(y\) and \(\psi\)

\(M\) is a local martingale orthogonal to \(\tilde{N}\)

\(\lim_{t \to T} Y_t = \xi 1_{S^c} + \infty 1_S\).

Then the process given by

\[
X_t^* = x \exp \left[ - \int_0^t \left( \frac{Y_u}{\eta_u} \right)^{q-1} \, du \right] \exp \left[ (q - 1) \int_0^t \ln \left( \frac{\lambda_u}{Y_u - + \psi_u} \right) \, dN_u \right].
\]

is optimal and the value function is given by \(v(t, x) = Y_t x^p\).
BSDEs with singular terminal condition

Consider the BSDE

\[ dY_t = -f(t, Y_t, \psi_t) dt + \int_{\mathcal{Z}} \psi_t(z) \tilde{\pi}(dz, dt) + dM_t \]

\[ Y_T = \xi \]

where \( \tilde{\pi} \) is a compensated Poisson measure and \( P[\xi = \infty] > 0 \).

Central assumptions on \( f \):

- **monotonicity in \( y \):**
  
  \[ (f(t, y, \psi) - f(t, y', \psi))(y - y') \leq \chi(y - y')^2. \]

- **Lipschitz continuity in \( \psi \):**
  
  \[ |f(t, y, \psi) - f(t, y, \varphi)| \leq K\|\psi - \varphi\|_{L^2}. \]

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Singular BSDEs and applications to position targeting
Consider the BSDE

\[ dY_t = -f(t, Y_t, \psi_t)dt + \int Z \psi_t(z)\tilde{\pi}(dz, dt) + dM_t \]

\[ Y_T = \xi \]

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Central assumptions on \( f \):

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  \]

- **Lipschitz continuity in \( \psi \):**
  \[
  |f(t, y, \psi) - f(t, y, \varphi)| \leq K\|\psi - \varphi\|_{L^2}.\]

- **at least polynomial growth in \( y \):**
  \[
  -f(t, y, \psi) \geq \frac{1}{\eta_t}|y|^q - f(t, 0, \psi), \quad y \geq 0, \quad q > 1.
  \]
Approximation from below

Consider the BSDE

\[ dY_t^L = -f(t, Y_t^L, \psi_t)dt + \int_Z \psi_t^L(z)\tilde{\pi}(dz, dt) + dM_t^L \]

\[ Y_T^L = \xi \land L \]

**Theorem**

For every \( L > 0 \) there exists a solution \((Y^L, \psi^L, M^L)\) to (1) satisfying the estimate

\[ Y_t^L \leq \frac{K}{(T-t)^p} \left[ E \left( \int_t^T \left( \eta_s^{p-1} + (T-s)^p f(s, 0, 0)^+ \right)' ds \bigg| \mathcal{F}_t \right) \right]^{1/l}. \]
Existence and Minimality

**Theorem**
There exists a process \((Y, \psi, M)\) such that for every \(t < T\) and as \(L \uparrow \infty\)

- \(Y_t^L \uparrow Y_t\) a.s.
- \(\psi^L \rightarrow \psi\) in \(L_\pi([0, t])\)
- \(M^L \rightarrow M\) in \(\mathcal{M}^l([0, t])\).

The process \((Y, \psi, M)\) satisfies

\[
dY_t = -f(t, Y_t, \psi_t)dt + \int_\mathcal{Z} \psi_t(z)\tilde{\pi}(dz, dt) + dM_t
\]

on \([0, t)\) and \(\lim \inf_{t \uparrow T} Y_t \geq \xi\). Moreover, \(Y\) is minimal.
Assume that

\[ E \left[ \int_0^T \gamma_t^2 \, dt \right] < \infty, \quad E \left[ \int_0^T \eta_t^2 \, dt \right] < \infty \quad \text{and} \quad E \left[ \int_0^T \frac{1}{\eta_t^{q-1}} \, dt \right] < \infty. \]

**Corollary**

There exists a minimal supersolution \((Y, \psi, M)\) to

\[ dY_t = \left( (p - 1) \frac{Y_t^q}{\eta_t^{q-1}} + \Theta(t, Y_t, \psi_t) - \gamma_t \right) \, dt + \psi_t \, d\tilde{N}_t + dM_t \quad (2) \]

with \( \lim \inf_{t \to T} Y_t \geq \xi 1_{S^c} + \infty 1_S \).
Optimal controls

\[ E \left[ \int_0^T \left( \eta_t |\alpha_t|^p + \lambda_t |\beta_t|^p + \gamma_t |X_t|^p \right) dt + \xi 1_{S^c} |X_T|^p \right] \rightarrow \min \]

**Theorem**

The process given by

\[ X_t^* = x \exp \left[ - \int_0^t \left( \frac{Y_u}{\eta_u} \right)^{q-1} du \right] \exp \left[ (q - 1) \int_0^t \ln \left( \frac{\lambda_u}{Y_u + \psi_u} \right) dN_u \right]. \]

is optimal and the value function is given by \( v(t, x) = Y_t x^p. \)

The proof is based on a penalization argument.
Replace the deterministic time horizon $T$ by a stopping time $\tau$. 

Consider the BSDE 

$$dY_t = Y^2_t dt + dM_t$$ 

$Y_\tau = \infty$ 

with $E[1_\tau] = \infty$. 

Consider first the terminal condition $Y_L^{\tau} = L$. Then one can show that $Y_L^0 \geq E[1_\tau + 1/L]$. 

In particular $\lim \inf L \to \infty Y_L^0 \geq E[1_\tau] = \infty$. 

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Singular BSDEs and applications to position targeting
Random execution period

Replace the deterministic time horizon \( T \) by a stopping time \( \tau \).

\textbf{Example}

\textit{Consider the BSDE}

\[
dY_t = Y_t^2 dt + dM_t \\
Y_\tau = \infty
\]

\textit{with} \( E \left[ \frac{1}{\tau} \right] = \infty \).

\textit{Consider first the terminal condition} \( Y_\tau^L = L \). \textit{Then one can show that}

\[
Y_0^L \geq E \left[ \frac{1}{\tau + 1/L} \right].
\]

\textit{In particular}

\[
\liminf_{L \to \infty} Y_0^L \geq E \left[ \frac{1}{\tau} \right] = \infty.
\]
Let $\Gamma$ be a diffusion in $\mathbb{R}^d$

$$d\Gamma_t = b(\Gamma_t)dt + \sigma(\Gamma_t)dW_t$$

with $\sigma$ being uniformly elliptic. Let $D \subset \mathbb{R}^d$ be open and bounded with $C^2$-boundary. Define

$$\tau = \tau_D = \inf\{t \geq 0, \quad \Gamma_t \notin D\}.$$

Consider the BSDE

$$dY_t = -f(t, Y_t, \psi_t)dt + \int_Z \psi_t(z)\tilde{\pi}(dz, dt) + dM_t$$

$$Y_\tau = \xi$$
Random execution period

\[ dY_t = -f(t, Y_t, \psi_t)dt + \int_{\mathcal{Z}} \psi_t(z)\tilde{\pi}(dz, dt) + dM_t \]

Consider first the terminal condition \( Y_T^L = \xi \land L \).

**Theorem**

For every \( L > 0 \) there exists a solution \((Y^L, \psi^L, M^L)\) satisfying the estimate

\[ Y_t^L \leq \frac{C}{\text{dist}(\Gamma_t \land \tau)^{p-1}}. \]
Random execution period

\[ dY_t = -f(t, Y_t, \psi_t)dt + \int_\mathcal{Z} \psi_t(z) \tilde{\pi}(dz, dt) + dM_t \]

Consider first the terminal condition \( Y_{\tau}^{L} = \xi \wedge L \).

**Theorem**

*For every \( L > 0 \) there exists a solution \((Y^L, \psi^L, M^L)\) satisfying the estimate*

\[ Y_t^L \leq \frac{C}{\text{dist}(\Gamma_{t \wedge \tau})^{p-1}}. \]

Then we obtain existence of a minimal supersolution to the singular BSDE and optimal controls as before.
Processes with uncorrelated multiplicative increments

**Definition**

$\eta$ has uncorrelated multiplicative increments (umi) if

$$E \left[ \frac{\eta_t}{\eta_s} \mid \mathcal{F}_s \right] = E \left[ \frac{\eta_t}{\eta_s} \right]$$

for all $s \leq t < T$. 
Definition

\( \eta \) has uncorrelated multiplicative increments (umi) if

\[
E \left[ \frac{\eta_t}{\eta_s} \mid \mathcal{F}_s \right] = E \left[ \frac{\eta_t}{\eta_s} \right]
\]

for all \( s \leq t < T \).

Examples

- \( \eta \) is deterministic
- \( \eta \) is a martingale
- \( d\eta_t = \mu(t)\eta_t dt + \sigma(t, \eta_t)dW_t \)
- \( \eta_t = e^{Z_t} \) where \( Z \) is a Lévy process
Assume $\gamma = 0$ and $\mu = 0$.

**Proposition**

Suppose that $\eta$ has umi, then

$$Y_t = \frac{1}{\left( \int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds \right)^{p-1}}$$

is the minimal solution to (2) with singular terminal condition. The deterministic control

$$X_t = x \frac{1}{\int_0^T \frac{1}{E[\eta_s]^{q-1}} ds} \int_t^T \frac{1}{E[\eta_s]^{q-1}} ds$$

is optimal. In particular, if $p = 2$, then $\dot{X}_t = -c \frac{1}{E[\eta_t]}$. 
Assume $\gamma = 0$ and $\mu = 0$.

**Proposition**

Suppose that $\eta$ has umi, then

$$Y_t = \frac{1}{\left( \int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds \right)^{p-1}}$$

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is optimal. In particular, if $p = 2$, then $\dot{X}_t = -c \frac{1}{E[\eta_t]}$.

Vice versa, assume that the optimal control $X_t = xe^{-\int_0^t (\frac{Y_s}{\eta_s})^{q-1} ds}$ is deterministic. Then $\eta$ has umi.
Extensions

- Include directional views for the price process
- Incorporate volume uncertainty
Thank you!