Derivative Pricing using a Reduced Basis Method for Parameter Functions

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Overview

**Objective:** fast derivative pricing

- Solving a pricing model for different parameters and payoff functions
- Application in model calibration process

**Method:** reduced basis method

- Model Reduction Method
- Partial Differential Equations
- Parameter dependent problems

**Example:** Heston model

- 2D parabolic PDE
- Affine model
Problem formulation

Consider the Itô diffusion
\[
dX_t = b(X_t)dt + \sigma(X_t)dB_t
\]
and the associated operator
\[
\mathcal{A}v(x) = \sum_i b_i(x) \frac{\partial v}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{i,j}(x) \frac{\partial^2 v}{\partial x_i \partial x_j}.
\]

Feynman-Kac Formula

If \( v(t, x) \) is the solution of
\[
\frac{\partial v}{\partial t} = \mathcal{A}v \quad t > 0, \quad \text{with} \quad v(T, x) = f(x)
\]
then
\[
v(t, x) = E(f(X_T)|X_t = x).
\]
Problem formulation - Heston model

- European options
- Strike price $K$
- Underlying asset and volatility modelled stochastically $\rightarrow$ **2D problem**
- Existence of closed form solution


**Heston - SDE**

$$dS_t = rS_t dt + \sqrt{v_t} S_t dz_1(t), \quad dv_t = \kappa [\theta - v_t] dt + \sigma \sqrt{v_t} dz_2(t)$$

- $r$: return rate of the asset
- $\sigma$: vol. of vol.
- $\theta$: long term variance
- $\kappa$: mean reversion rate to $\theta$
- $z_1, z_2$ Wiener processes with correlation $\rho$
Heston - PDE

- Transformation in time $t \rightarrow T - t$

Expected value given as solution of parabolic PDE in $z = (S, \nu)$

$$\frac{d}{dt} u - \sum_{i,j=1}^{2} A_{i,j} u_{z_i, z_j} - \sum_{i=1}^{2} b_i u_{z_i} + cu = 0$$

$$u(0) = \mu_0 \quad \text{(payoff)}$$

$$A = \frac{1}{2} \nu \begin{pmatrix} S^2 & \rho \sigma S \\ \rho \sigma S & \sigma^2 \end{pmatrix}, \quad b = \begin{pmatrix} Sr \\ \kappa \theta - \kappa \nu \end{pmatrix}, \quad c = r.$$ 

- Parameter $\rho, \sigma, \kappa, \theta$
- $r$ is riskless interest rate ($\approx 0$)
- Parameter function $\mu_0$
Space-Time Variational Formulation

- Weak formulation of PDE in space AND time
- Convenient for reduced basis method
- Solution $u(t, x)$ is a function in space AND time

Hilbert spaces $V \hookrightarrow H$ dense

$$X := \{ u \in L_2(0, T; V) : u' \in L_2(0, T; V') \}$$
$$Y := L_2(0, T; V) \times H.$$  

For $A \in \mathcal{L}(V, V')$, $g \in L_2(0, T; V')$ find $u \in X \subset L_2(0, T; V)$

$$u'(t) + Au(t) = g \text{ in } L_2(0, T; V'), \quad u(0) = u_0. \quad (1)$$

$$\int_I \langle u'(t), v_1(t) \rangle_{V' \times V} dt + \int_I a(u(t), v_1(t)) dt + (u(0), v_2)_H$$

$$= \int_I \langle g(t), v_1(t) \rangle_{V' \times V} dt + (u_0, v_2)_H \quad \forall \ v = (v_1, v_2) \in Y. \quad (2)$$

Reduced Basis Method

Two introductions (2015)

For $\mu = (\mu_0, \mu_1) \in \mathcal{D}$

find $u(\mu) \in \mathcal{X}$ such that $b(u(\mu), v; \mu) = f(v; \mu)$ for all $v = (v_1, v_2) \in \mathcal{Y}$

$$b(u(\mu), v; \mu) = \int_I \langle \dot{u}(\mu)(t), v_1(t) \rangle_{V^\prime \times V} dt + \int_I a(u(\mu)(t), v_1(t); \mu_1) dt + (u(\mu)(0), v_2)_H, \quad f(v; \mu) = (\mu_0, v_2)_H.$$  

Standard methods (FEM, FV, FD, ...) $\Rightarrow B_N(\mu) u_N(\mu) = f_N(\mu)$ ($N$ large)

$\Rightarrow$ computationally expensive for

- multi-query (e.g. calibration/optimisation),
- real time (e.g. pricing)
- parameter variations for every computation.

Approximate $\mathcal{X}_N \subset M = \{ u(\mu) | \mu \in \mathcal{D} \} \subset \mathcal{X} \Rightarrow B_N u_N = f_N$ ($N \ll N$).
Reduced Basis Method

- reduces large linear equation system to small one
- error analysis available
- best approximation results
Reduced Basis Method

- Offline:
  - For sample set \( \{ \mu_1, \ldots, \mu_N \} \) (Greedy procedure) pre-compute (FEM) “snapshots”
  \[ u^N(\mu_1), \ldots, u^N(\mu_N) \]
  - \( X_N = \text{span}\{u^N(\mu_1), \ldots, u^N(\mu_N)\} \) (Reduced Basis Space) and test space \( Y_N \)

- Online: (complexity independent on \( N \))
  - For new \( \mu \notin \{ \mu_1, \ldots, \mu_N \} \) solve
  \[ u_N(\mu) \in X_N : b(u_N(\mu), v; \mu) = f(v) \quad \forall v \in Y_N \]

Greedy procedure: \( M_{\text{train}} \subset D \). Choose a first \( \mu: \quad u^N(\mu), X_1 = \{u^N(\mu)\} \).

WHILE error > tol

\[ \mu^* := \arg \max_{\mu \in M_{\text{train}}} \text{error}(u^N(\mu), u_N(\mu)) : \quad u^N(\mu^*), X_{N+1} := X_N \cup \{u^N(\mu^*)\}. \]
Space-Time Reduced Basis Method


- Online: one $N \times N$ linear system (no time stepping!)

- Good error estimator
  - residual based
  - can be efficiently computed
  - to be used in the Greedy procedure offline
  - control of the error online

$$
\beta(\mu) \| u - u_N \|_X \leq \sup_{v \in Y} \frac{b(u - u_N, v; \mu)}{\| v \|_Y} = \sup_{v \in Y} \frac{f(v) - b(u_N, v; \mu)}{\| v \|_Y}
$$

- Greedy works for $\mu \in \mathbb{R}^p$

Reduced basis depends on payoff $\mu_0$!
Space-Time RBM for Parameter Functions

AM and Urban, *A reduced basis method for parabolic PDEs with parameter functions and application to option pricing* (to appear).

\[ \mu \in H, \quad \text{Hilbert space} \]

Reduced basis for \( D_0 \times D_1 \subset H \times \mathbb{R}^p \) parameter domain?

\[
b(u(\mu), v; \mu) := \int_I \langle \dot{u}(\mu)(t), v_1(t) \rangle_{V' \times V} dt + \int_I a(u(\mu)(t), v_1(t); \mu_1) dt + (u(\mu)(0), v_2)_H,
\]

\[
f(v; \mu) := \int_I \langle g(t), v_1(t) \rangle_{V' \times V} dt + (\mu_0, v_2)_H.
\]

Assumption: \( D^\mathcal{L} \subset D \) finite description: \( \mu_0^\mathcal{L} = \sum_{\ell=1}^\mathcal{L} d_\ell(\mu_0) \delta_\ell \)

Split error estimator:

\[
\| u_N^\mathcal{N} (\mu) - u_N(\mu) \|_X \leq \frac{\| f(\cdot; \mu) - b(u_N(\mu), \cdot; \mu) \|_{V'}}{\beta_{LB}} = \Delta_N(\mu) \leq \Delta_0^N(\mu) + \Delta_1^N(\mu),
\]

\[
\Delta_0^N(\mu) := \frac{\| u_N(\mu)(0) - \mu_0 \|_H}{\beta_{LB}}, \quad \Delta_1^N(\mu) := \frac{\| g_1(\cdot) - b_1(u_N(\mu), \cdot; \mu) \|_{(L^2(I; V))'}}{\beta_{LB}}.
\]
Space-Time RBM for Parameter Functions - Offline Phase

- \( X^N = E_{\text{time}} \otimes V_{\text{space}}, \quad Y^N = (F_{\text{time}} \otimes V_{\text{space}}) \times V_{\text{space}} \)

\[ u(\mu) = u^0(\mu_0) + w(\mu) \]

1st part \((\Delta_{N_0}^0 < \text{tol}_0)\):

- Only parameter function \( \mu_0 \)
- Find good sample \( \mu_0^1, \ldots, \mu_0^{N_0} \) (POD, Greedy)
- RB space \( H_{N_0} = \{h^1, \ldots, h^{N_0}\} \subset H \rightarrow \{\sigma^1\} \otimes H_{N_0} \subset X^N \)

Compute \( h_i \) for \( i = 1, \ldots, N_0 \):

\[ (h_i, \phi_j)_H = (\mu_0^i, \phi_j)_H \forall \phi_j \in V_{\text{space}}, \quad u_0^i := \sigma^1 \otimes h_i \]

2nd part \((\Delta_{N_1}^1 < \text{tol}_1)\):

- Find good samples \( \{\mu^1, \ldots, \mu^{N_1}\} \subset D_0 \times D_1 \) (Evolution Greedy)
- RB space \( \mathbb{W}_{N_1} := \{w_1, \ldots, w_{N_1}\} \)

Compute \( w_j \) for \( \mu^j = (\mu_0^j, \mu_1^j) \):

\[ b_1(w_j, z; \mu) = g_1(z) - b_1(u_{N_0}^0(\mu_0^j), z; \mu) \quad \forall \ z \in F_{\text{time}} \otimes V_{\text{space}}, \]
Space-Time RBM for Parameter Functions - Online Phase

For new \((\mu_0, \mu_1) \in D\).

1. Find \(\tilde{h}_{N_0}(\mu_0) \in H_{N_0}: (\tilde{h}_{N_0}(\mu_0), h^i)_H = (\mu_0, h^i)_H \quad \forall h^i \in H_{N_0}\)

2. Find \(w_{N_1}^1(\mu) \in W_{N_1}: b_1(w_{N_1}^1(\mu), v; \mu_1) = g_1(v) - b_1(\sigma^1 \otimes \tilde{h}_{N_0}(\mu_0), v; \mu_1) \quad \forall v \in Y_{N_1}^1\)

3. \(u_N(\mu) = \sigma^1 \otimes \tilde{h}_{N_0}(\mu_0) + w_{N_1}^1(\mu), \quad N = N_0 + N_1\)

- Construct reduced test space \(Y_N^1(\mu)\) for (inf-sup) stability (supremizers).
Numerical Experiments

- Heston model: $V = H_0^1(\Omega, \omega)$, $H = L_2(\Omega, \omega)$ (weighted Sobolev Spaces)

- $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^2$

- $\Omega_1 := (0, 140) \cup [140, \infty)$, $\Omega_2 := (0, 1) \cup [1, \infty)$

- $\mu_1 = \rho, \kappa = 0.3313, \sigma = 0.6083, \theta = 0.1914$ and $r = 0$

- Parameter domain $[-0.5, 0.5] =: \mathcal{D}_1$

- $T = 1, \beta_{LB} = 0.003$

- $\mathcal{D}_0 \approx \mathcal{D}_0^L = \{\delta_1, \ldots, \delta_L\}$ pw. linear on $\mathcal{T}_{\Omega_1} = \{0, 70, 80, 90, 100, 110, 200\} \ (L = 7)$, const. on $\Omega_2$ (exact for $K = 70, 80, \ldots$)

All RB calculations were implemented in RBmatlab, see http://www.morepas.org.

1st part: Initial Value: POD $\rightarrow \{h_1, \ldots, h_7\}$
2nd part: Evolution Greedy

\[ M_{\text{train}} := \{ h_1, \ldots, h_7 \} \times \{ x_k = -0.5 + k \Delta s : k = 0, \ldots, 11, \Delta s = \frac{1}{11} \} \]
\[ \subset \text{span}\{\delta_1, \ldots, \delta_7\} \times [-0.5, 0.5] \]

Evol. Greedy: Maximum error over iterations - error estimate \( \Delta_{N_1}^1 \) and true error.

Model Calibration

Market price observations $V_1, \ldots, V_M$ for pairs $(T_i, \mu^i_0), i = 1, \ldots, M$

$$\min_{\mu_1} \sum_{i=1}^{M} (V_i - V(S, \nu, 0; T_i, (\mu^i_0, \mu_1)))^2$$

given $V(S, \nu, 0; T_i, (\mu^i_0, \mu_1)) = u((S, \nu), T_i; (\mu^i_0, \mu_1))$.

PDE-constraint optimisation using RBM


PDE-constraint optimisation with Space-Time RBM

Parameter functions:

\[
\begin{aligned}
&\min_{\mu_1} J(u^N(\mu_0, \mu_1)) \\
&\text{s.t. } u^N(\mu_0, \mu_1) \text{ solves PDE in space-time variational formulation}
\end{aligned}
\]

\[
\begin{aligned}
&\min_{\mu_1} J(u_N(\mu_0, \mu_1)) \\
&\text{s.t. } u_N(\mu_0, \mu_1) = u^0(\mu_1) + w^1(\mu_0, \mu_1) \text{ solution of two reduced problems}
\end{aligned}
\]

- Parameter dependence \(\mu_0(\mu_1)\) possible
- \(\|u^N(\mu) - u_N(\mu)\|_X \leq \text{RB-tol} \) and \(|J(u^N) - J(u_N)| \leq \text{tol}\)
- Gradient via sensitivity PDE

RB only as good as the high dimensional discretisation \(X^N\) and \(D_0^C\)!
A first Experiment

- **Matlab**: `lsqnonlin`: trust region, gradient approximated with finite differences
- NO sensitivity PDE
- Data: Call option on DAX, different time to maturity ($\leq 1$ year), different strike prices
- Calibration over $\rho$ only
- Initial points in $[-0.5,0.5]$

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<th>time</th>
<th>number of function calls</th>
<th>residual</th>
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<td>RB-Sol./ Call</td>
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- Speedup factor 4
Conclusion

- (parabolic) PDE
- No closed form solution needed
- Initial value is parameter
- Space-time RBM approach: two reduced basis’
- Speedups in e.g. calibration
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Thank you for your attention!