Discretizing the Heston Model: An Analysis of the Weak Convergence Rate

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Joint work with Martin Altmayer

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Part I: The Numerical Problem

Weak approximation of the Heston model
Multidimensional Heston model

Asset prices \((\text{Heston}, 1993)\)

\[ S_t^i = \exp(X_t^i), \quad t \in [0, T], \quad i = 1, \ldots, d \]

where

\[
\begin{align*}
\text{d}X_t^i &= \left(b_i - \frac{1}{2}V_t^i\right)\text{d}t + \sqrt{V_t^i}\text{d}B_t^i, \quad X_0^i = x_0^i \in \mathbb{R} \\
\text{d}V_t^i &= \kappa_i(\lambda_i - V_t^i)\text{d}t + \theta_i \sqrt{V_t^i}\text{d}W_t^i, \quad V_0^i = v_0^i > 0
\end{align*}
\]

where \(b_i \in \mathbb{R}, \kappa_i, \lambda_i, \theta_i > 0\), correlated Brownian motions \(B^i, W^i\)

Popular, but difficult model

- Moment explosions possible, i.e. \(\mathbb{E}|S_T^i|^p\) maybe infinite, depending on the parameters (Andersen, Piterbarg, 2007; Lions, Musiela, 2007)
- SDE with non-Lipschitz coefficients: no standard results from numerical analysis applicable
- Volatility process \(V^i\): CIR process, positive sample paths
Numerical problem

Weak convergence analysis

Task Find implementable approximation $x_T^\Delta$ such that

$\lim_{\Delta \to 0} E[f(x_T^\Delta)] = E[f(X_T)]$ for $f \in \mathcal{F}_{\text{conv}}$

and

$|E[f(x_T^\Delta)] - E[f(X_T)]| \leq c_f \Delta^\alpha$ for $f \in \mathcal{F}_{\text{rate},\alpha}$

with

(a) $\mathcal{F}_{\text{conv}}, \mathcal{F}_{\text{rate},\alpha} \subset \{ f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}; \text{measurable} \}$

(b) convergence rate $\alpha > 0$

as large as possible

Note $f = \text{payoff} \circ \exp$; SDE for $(X, V)$: log-Heston SDE
Known results

- Replace $\nu$ by $|\nu|$ or $\nu^+$ and discretize with Euler scheme:
  
  \[
  (\text{conv}) \quad \mathcal{F} = \{ f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}; \text{bounded}, \lambda^d - \text{a.s. continuous}\}
  \]

  Higham, Mao, 2005; Lord et al, 2010; ...

- No mathematical results for (rate) available, “only” simulation studies
  Kahl, Jäckel, 2006; Andersen, 2008; Lord et al, 2010; ...

  Convergence rate seems to deteriorate if $2\kappa_i \lambda_i / \theta_i^2 \ll 1$

Remarks

For $d = 1$:

- Exact simulation of $X_T$ possible (Broadie, Kaya, 2006)

- Alfonsi, 2005 & 2010: weak convergence analysis for CIR
Part II: A Warning

Shortcomings of the Euler scheme under non-standard assumptions
Moment explosions

Heston-3/2-model: volatility process

\[ dZ_t = 1.2Z_t(0.8 - Z_t)dt + Z_t^{3/2}dW_t, \quad Z_0 = z_0 = 0.5 \]

Euler scheme

\[ z_{k+1} = z_k + 1.2z_k(0.8 - z_k)\Delta + |z_k|^{3/2}\Delta_k W \]

with \( \Delta = T/n, \Delta_k W = W_{(k+1)T/n} - W_{kT/n} \)

Euler based standard Monte Carlo estimator

\[ \hat{P}_{\Delta,N} = \frac{1}{N} \sum_{i=1}^{N} |z_T^{\Delta,(i)}| \]

for \( \mathbb{E}|Z_T| \) where \( z_T^{\Delta,(i)} \) iid copies of \( z_T^{\Delta} = z_n \)
Moment explosions

For \( T = 4 \): \( E|Z_4| = 0.5662... \)

Simulation study for \( \hat{p}_{\Delta,N} \):

<table>
<thead>
<tr>
<th>repetitions / stepsize</th>
<th>( \Delta = 2^0 )</th>
<th>( 2^{-2} )</th>
<th>( 2^{-4} )</th>
<th>( 2^{-6} )</th>
<th>( 2^{-8} )</th>
<th>( 2^{-10} )</th>
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<tbody>
<tr>
<td>( N = 10^3 )</td>
<td>6.3272</td>
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<td>Inf</td>
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<td>0.5535</td>
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<td>( 10^6 )</td>
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<td>( 10^7 )</td>
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</tr>
</tbody>
</table>

In fact:

\[
\lim_{\Delta \to 0} E|Z_T^\Delta| = \infty
\]

Hutzenthaler et al, 2011, also for more general SDEs with super-linear coefficients

**In our context**  no superlinear coeff’s, but exponentially growing payoffs
Weak order: Always $\alpha = 1$?

Kebaier, 2005

\[ \begin{align*}
\text{d}X_t &= -\frac{X_t}{2} \text{d}t - Y_t \text{d}W_t, \\
\text{d}Y_t &= -\frac{Y_t}{2} \text{d}t + X_t \text{d}W_t
\end{align*} \]

with $(x_0, y_0) = (\cos \theta, \sin \theta)$ (solution paths lie on unit circle!)

Euler scheme

\[ \begin{align*}
    x_{k+1} &= x_k - \frac{x_k}{2} \Delta - y_k \Delta W, \\
    y_{k+1} &= y_k - \frac{y_k}{2} \Delta + x_k \Delta W
\end{align*} \]

Let $\alpha \in [1/2, 1]$, $f_\alpha(x, y) = |x^2 + y^2 - 1|^{2\alpha}$. Then

\[ \Delta^{-\alpha} \cdot \left| E f_\alpha(x_T^\Delta, y_T^\Delta) - E f_\alpha(X_T, Y_T) \right| \to \text{const} \]

with $x_T^\Delta = x_n$, $y_T^\Delta = y_n$

**In our context** Log-Heston SDE takes values in a half space, payoffs typically non-smooth
Part III: Our Results

Weak convergence analysis for a combined Euler & drift-implicit Milstein scheme
Numerical method

Euler scheme for log-asset price:

\[ x_{k+1}^i = x_k^i + \left( b_i - \frac{1}{2} v_k^i \right) \Delta + \sqrt{v_k^i} \Delta_k B_i \]

Drift-implicit Milstein scheme for volatility:

\[ v_{k+1}^i = v_k^i + \kappa_i (\lambda_i - v_{k+1}^i) \Delta + \theta \sqrt{v_k^i} \Delta_k W_i + \frac{\theta_i^2}{4} (|\Delta_k W_i|^2 - \Delta) \]

Günther et al, 2008

Note

\[ v_{k+1}^i = \frac{1}{1 + \kappa_i \Delta} \left( \left( \sqrt{v_k^i} + \frac{\theta_i}{2} \Delta_k W_i \right)^2 + \left( \kappa_i \lambda_i - \frac{\theta_i^2}{4} \right) \Delta \right) \]

Thus: \( v_k^i \) well defined and positivity preserving for \( \kappa_i \lambda_i / \theta_i^2 \geq 1/4 \)
Weak convergence

Notation/Assumption: $\rho_i = \frac{1}{t} E B_t^i W_t^i, \ t > 0$

Theorem (Altmayer, N, 2015)

Let $\min_{i=1,...,d} \kappa_i \lambda_i / \theta_i^2 > 1/4$.

(i) (conv) holds for

$$\mathcal{F} = \{ f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}; \ \lambda^d - a.s. continuous, \ E f(X_T) = \infty \}$$

(ii) If $\max_{i=1,...,d} \rho_i < 0$, then (conv) holds for

$$\mathcal{F} = \left\{ f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}; \ \lambda^d - a.s. continuous, \ f(x) = O(\sum_{i=1}^d \exp(x_i)) \right\}$$

Remarks

- Case (i): moment explosions are recovered
- Case (ii) applies e.g. to basket calls
Weak convergence

Theorem (Altmayer, N, 2015)

Let \( \min_{i=1,...,d} \kappa_i \lambda_i / \theta_i^2 > 1/4 \).

(i) \((\text{conv})\) holds for

\[
\mathcal{F} = \{ f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}; \lambda^d - \text{a.s. continuous}, \mathbf{E}f(X_T) = \infty \}
\]

(ii) If \( \max_{i=1,...,d} \rho_i < 0 \), then \((\text{conv})\) holds for

\[
\mathcal{F} = \{ f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}; \lambda^d - \text{a.s. continuous}, f(x) = \mathcal{O}\left( \sum_{i=1}^{d} \exp(x_i) \right) \}
\]

Proof

- Step 1: \( x_n \to X_T \) in probability, e.g. using Yamada functions for CIR
- Step 2, case (i): Fatou’s lemma
- Step 2, case (ii): dominated convergence via representation

\[
x_n^i \leq \text{const} - \left( \frac{1}{2} - \frac{\rho_i \kappa_i}{\theta_i} \right) \sum_{k=0}^{n-1} \nu_k \Delta - \sum_{k=0}^{n-1} \frac{\rho_i \theta_i}{4} (\Delta_k W_i)^2 + \sum_{k=0}^{n-1} \sqrt{\nu_k \Delta_k} (B_i - \rho_i W_i)
\]

and conditional normality
Convergence rate

Theorem (Altmayer, N, 2015)

Let \( d = 1, \, \kappa \lambda / \theta^2 > 1, \, \varepsilon > 0 \). Then:
(rate) for all \( \alpha \in (0, 1) \) and \( \mathcal{F} = \{ f \in C^{2,\varepsilon}(\mathbb{R}; \mathbb{R}_{\geq 0}); \, f \text{ compact support} \} \)

Proof Tools:

(1) classical Kolmogorov PDE approach, i.e. \( u(t, x, v) = E f(X^{x,v}_{T-t}) \) and

\[
|Ef(X_T) - Ef(x_n)| \leq \sum_{k=1}^{n} |Eu(k\Delta, x_k, v_k) - Eu((k-1)\Delta, x_{k-1}, v_{k-1})|
\]

(Talay, Tubaro, 1990)

(2) a-priori estimates for log-Heston PDE (Fehan, Pop, 2013) on \( \mathbb{R} \times \mathbb{R}_{>0} \)

(3) bound for inverse moment of \( v_k \) \( \rightarrow \) condition on \( \kappa \lambda / \theta^2 \)

(4) tail estimates for CIR and numerical scheme

(5) Malliavin integration by parts

Crucial for (2), (3), (5): positivity preserving numerical scheme for CIR

Numerical tests Weak convergence order one also under less restrictive assumptions on \( \kappa \lambda / \theta^2 \) and \( \mathcal{F} \) \( \rightarrow \) (as always for CIR / log-Heston ...)
Summary

Results

Combined Euler & drift-implicit Milstein for log-Heston SDE:

- Weak convergence for $d > 1$ for large class of testfunctions under mild assumptions on the parameters
- Weak convergence order of (almost) one for $d = 1$, $\kappa \lambda / \theta^2 > 1$ and $C^{2,\varepsilon}_c$-test functions

Related work

- Multilevel quadrature and Malliavin smoothing in the Heston model
- Altmayer, 2015: weak convergence order one for measurable and bounded payoffs if $\kappa \lambda / \theta^2 > 7/4$

Work in progress / Open problems

Weak convergence order for more irregular payoffs / under weaker condition on $\kappa \lambda / \theta^2$ for $d > 1$