Asymptotic behaviour of multivariate default probabilities and default correlations under stress

7th General AMaMeF and Swissquote Conference  
EPFL, Lausanne

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7 September 2015
Stress testing of bank portfolios

- Calculate risk measures (expected loss, value-at-risk, economic capital) and regulatory capital under adverse market conditions

- Stress tests are typically conducted within models

- Crucial inputs of any portfolio model:
  - Distribution assumption on portfolio constituents, e.g.
    - normally distributed asset returns
    - fat-tailed asset returns
  - Dependence assumption among portfolio constituents, e.g. correlation
Stress testing of bank portfolios

- **Questions:**
  - What is the **model behaviour** under stress?
  - What are **model side effects** when stress testing?

- In a series of papers we investigate these questions:
Overview

- Structural credit portfolio models and stress testing
- Distribution of model variables
- Asset correlations under stress
- Default probabilities and default correlations under stress
- Risk measures
- Conclusion
Overview

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Structural credit model

- **Merton-model** (Merton, 1974) links default of firm to relationship between assets and liabilities

- Counterparty \( i \) in default at \( T \)...
  - ... if asset value \( Z_i \) ...
  - ... below debt value \( D_i \)

- Default event: \( \{ Z_i < D_i \} \)

- Portfolio loss:
  \[
  L := \sum_{i=1}^{d} \ell_i \cdot 1\{Z_i \leq D_i\},
  \]
  with \( \ell_i \) loss-at-default of counterparty \( i \)
Credit portfolio risk

- Risk concentrations, correlations:

\[
Z_i = \sqrt{R_i^2} \sum_{j=1}^{m} w_{ij} X_j + \sqrt{1 - R_i^2} \varepsilon_i, \quad i = 1, \ldots, d,
\]

where

- \( X_1, \ldots, X_m \): systematic factors or risk factors,
- \( \varepsilon_i \): firm-specific factor
Credit portfolio risk measures

- **Expected loss:**  
  \[ \mathbb{E}(L) = \sum_{i=1}^{d} l_i \cdot P(Z_i \leq D_i) \]

- **Value-at-risk (at level \( \beta \)):** \( \beta \)-quantile of \( L \):  
  \[ \text{VaR}_\beta(L) = \inf \{ x \in \mathbb{R} : P(L \leq x) \geq \beta \} \]

- **Default correlations** as measure of dependence:  
  \[
  \text{Corr}(\mathbf{1}_{\{Z_i \leq D_i\}}, \mathbf{1}_{\{Z_j \leq D_j\}}) = \frac{P(Z_i \leq D_i, Z_j \leq D_j) - p_i p_j}{\sqrt{p_i(1 - p_i)p_j(1 - p_j)}},
  \]

  where \( p_i := P(Z_i \leq D_i), \ i = 1, \ldots, d \)
Stress testing

- Translate stress scenario into constraints on risk factors
- Truncate risk factor variable $Z_0$ (which is typically one of the systematic factors $X_i$):
  \[ Z_0 \leq C, \quad C \in \mathbb{R} \text{ stress level} \]
- Portfolio risk is evaluated under $P(\cdot | Z_0 \leq C)$
- Consistent framework that associates severity of stress scenario with probability of stress scenario
- See e.g. Bonti et al. (2006); Duellmann and Erdelmeier (2009); Kalkbrener and Packham (2015a)
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Elliptically distributed random variables

- Standard approach in credit risk portfolio modelling: risk factors and asset variables *normally distributed*

- Generalisations:
  - normal variance mixture distribution
  - elliptical distribution

- Cover variety of light-tailed to heavy-tailed distribution
Elliptically distributed random variables

- Let random vector $\mathbf{Z} = (Z_0, \ldots, Z_d)^T$ follow elliptical distribution with representation
  
  $$\mathbf{Z} \sim \mathcal{G}A\mathbf{U},$$

  where
  
  - $G > 0$ is a scalar random variable, the mixing variable,
  - $A$ is a deterministic $(d + 1) \times (d + 1)$ matrix with $AA^T := \Sigma$, which in turn is a $(d + 1) \times (d + 1)$ nonnegative definite symmetric matrix of rank $d + 1$,
  - $\mathbf{U}$ is a $(d + 1)$-dimensional random vector uniformly distributed on the unit sphere $S_{d+1} := \{ \mathbf{z} \in \mathbb{R}^{d+1} : \mathbf{z}^T \mathbf{z} = 1 \}$,
  - $\mathbf{U}$ is independent of $G$. 

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Asset correlations under stress (Kalkbrener and Packham, 2015a):

- multivariate normal or $t$-distribution: explicit formulas for asset correlations under stress

\[ \text{Corr}^C(Z_i, Z_j) \]

- Normal variance mixture distribution:

\[
\lim_{C \to -\infty} \text{Corr}^C(Z_i, Z_j) = \frac{\rho_i \rho_j + (\rho_{ij} - \rho_i \rho_j) (\alpha - 1)}{\sqrt{(\rho_i^2 + (1 - \rho_i^2)(\alpha - 1))(\rho_j^2 + (1 - \rho_j^2)(\alpha - 1))}},
\]

with $\alpha > 2$ the tail index of asset returns and risk factor and $\rho_i = \rho_{0i}$
Asset correlations

- In a typical scenario, assets de-correlate with increasing stress and increasing tail index.
- Helps explain why the relative impact of stress on EL of portfolio is stronger than on VaR.

left: Conditional asset correlations; right: Asymptotic asset correlations.
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Conclusion
• Fréchet max domain / regular variation:
  ▶ $RV_\alpha$: regularly varying functions with index $\alpha \in \mathbb{R}$
  ▶ $G$ in Fréchet max domain iff $P(G > \cdot) \in RV_{-\alpha}$ for some $\alpha > 0$

• Gumbel max domain / rapid variation:
  ▶ $RV_{-\infty}$: rapidly varying functions
  ▶ If $G$ in Gumbel max-domain, then $P(G > \cdot) \in RV_{-\infty}$
Preliminaries

- Fréchet max domain / regular variation:
  - $RV_{\alpha}$: regularly varying functions with index $\alpha \in \mathbb{R}$
  - $G$ in Fréchet max domain iff $P(G > \cdot) \in RV_{-\alpha}$ for some $\alpha > 0$

- Gumbel max domain / rapid variation:
  - $RV_{-\infty}$: rapidly varying functions
  - If $G$ in Gumbel max-domain, then $P(G > \cdot) \in RV_{-\infty}$

- Random vector $Z = GAU$ standardised, so that $\Sigma = AA^T$ is the correlation matrix

- Correlations are positive, i.e., $\rho_{ij} > 0$

- $A_{i,:}$: $i$-th row of $A$

- $F_U$: uniform distribution on $S_{d+1}$
Default probabilities

Theorem (Packham et al. (2016))

(i) If $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$, then

$$\lim_{C \to -\infty} \mathbf{P}(Z_1 \leq D_1, \ldots, Z_d \leq D_d | Z_0 \leq C) = \int_{\mathbf{u} \in S_{d+1}, A_0 \mathbf{u} > 0, \ldots, A_d \mathbf{u} > 0} (A_0 \mathbf{u})^\alpha dF_U(\mathbf{u}) \left( \int_{\mathbf{u} \in S_{d+1}, A_0 \mathbf{u} > 0} (A_0 \mathbf{u})^\alpha dF_U(\mathbf{u}) \right)^{-1}.$$

(ii) If $\mathbf{P}(G > \cdot) \in RV_{-\infty}$, then

$$\lim_{C \to -\infty} \mathbf{P}(Z_1 \leq D_1, \ldots, Z_d \leq D_d | Z_0 \leq C) = 1.$$

- Asymptotic default probabilities do not depend on default thresholds.
Default probabilities

Theorem (Packham et al. (2016))

Let $\mathbb{P}(G > \cdot) \in RV_{-\alpha}$.

(i) For $d = 1$,

$$\lim_{C \to -\infty} \mathbb{P}(Z_1 \leq D_1 | Z_0 \leq C) = t_{\alpha+1} \left( \frac{\sqrt{\alpha + 1} \rho_{01}}{\sqrt{1 - \rho_{01}^2}} \right) \in [1/2, 1).$$

(ii) For $d = 2$,

$$\lim_{C \to -\infty} \mathbb{P}(Z_1 \leq D_1, Z_2 \leq D_2 | Z_0 \leq C)$$

$$= \frac{1}{2} t_{\alpha+1} \left( \frac{\sqrt{(\alpha + 1) t}}{\sqrt{1 - t^2}} \right) + \text{some long integrals}.$$
• Asymptotic univariate PD’s (left) and bivariate PD’s (right) as a function of the tail index.

• Correlations: $\rho_{01} = \rho_{02} = \rho$ and $\rho_{12} = \rho^2$. 
Default correlations

- Regularly varying case easily calculated from previous Theorem.

Theorem

Let $P(G > \cdot) \in RV_{-\infty}$. Then,

$$\lim_{C \to -\infty} Corr^C(\mathbf{1}_{Z_1 \leq D_1}, \mathbf{1}_{Z_2 \leq D_2}) = 0,$$

where $Corr^C$ denotes the correlation under $P(\cdot|Z_0 \leq C)$. 
Default probabilities vs. tail dependence

- **Asymptotic default probability:**
  \[
  \lim_{C \to -\infty} P(Z_1 \leq D | Z_0 \leq C).
  \]

- **Coefficient of (lower) tail dependence:**
  \[
  \lambda_l(Z_0, Z_1) := \lim_{C \to -\infty} P(Z_1 \leq C | Z_0 \leq C).
  \]

- In the light-tailed case, tail dependence is 0, which is in contrast to the asymptotic default probability

- **Tail dependence function** that captures both:
  \[
  \lambda(Z_0, Z_1, x) := \lim_{C \to -\infty} P(Z_1 \leq x \cdot C | Z_0 \leq C), \quad x \in \mathbb{R}.
  \]
Relation to tail dependence

- Closed formula for tail dependence function $\lambda(Z_0, Z_1, x)$ in paper
- Special cases:
  - stressed PD’s: $x = 0$
  - tail dependence: $x = 1$
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Risk measures

- Risk measures for portfolio consisting of 60 homogeneous counterparties, each with a PD of 1%.
- Left: Value-at-risk at 99% confidence level
- Right: Expected loss
Risk measures

- Risk measures for portfolio consisting of 60 homogeneous counterparties, each with a PD of 1%.
- Economic Capital (VaR-EL)
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Conclusion

• Stress tests are an integral part of risk management and banking supervision, ...

• ... and the analysis and understanding of risk model behaviour under stress has become ever more important.

• We analyse asset correlations, default probabilities and default correlations under stress in a generalised Merton-type credit portfolio setup covering light- and heavy-tailed distributions.

• It turns out that the model behaviour under stress depends on the heaviness of the tails of the risk factors.

• Contrary to popular belief, light-tailed models show a higher impact in extreme stress scenarios.

• We use our results to study the implications for credit reserves and capital requirements under stress.
References


Thank you!

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