Pathwise analysis and robustness of hedging strategies for path-dependent options

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## Outline

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3. **A Pathwise Approach to Continuous-Time Trading**
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MODEL AMBIGUITY AND HEDGING ISSUES

Classical Framework: Traded assets $X = (X(t))_{t \in [0,T]}$ modeled as $\mathbb{R}_+^d$-valued semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$

- The choice of $\mathbb{P}$ may be challenged ‘a la De Finetti’ (Knightian uncertainty)
  → Our approach: we set up a probability-free financial model
- The gain process is a stochastic integral, thus
  - it is not necessarily defined for a given path/price scenario
  - scenario analysis and stress tests cannot be performed
  → In our setting: for a certain class of trading strategies, the gain process is well-defined path-by-path (as a limit of Riemann sums)
- Robustness analyses are based on the existence of a ‘true model’, and study the performance of a ‘mis-specified model’
  → Our analysis: we study the performance and robustness of hedging strategies in given sets of scenarios.
**Robustness of Hedging Strategies**

Consider a market participant who sells an (exotic) option with payoff $H$ and maturity $T$ on some underlying asset, at a model price given by

$$V(t) = E^Q[H|\mathcal{F}_t]$$

and hedges the resulting Profit/Loss using the hedging strategy derived from the same model (say, Black-Scholes delta hedge). The *actual* dynamics of the underlying asset may, of course, be different from the assumed dynamics.

- How good is the result of the hedging strategy?
- How ‘robust’ is it to model mis-specification?
- How does the hedging error relate to model parameters and option characteristics?
Robustness of Hedging Strategies

El Karoui, Jeanblanc & Shreve (1998) provided an answer to these important questions, for non-path-dependent options, when the underlying dynamics is

$$dS(t) = S(t)r(t)dt + S(t)\sigma(t)dW(t) \quad \text{under } \mathbb{Q}$$

such that $S$ is square-integrable. Then a hedging strategy, computed in a (mis-specified) Markovian model

$$dS(t) = S(t)r(t)dt + S(t)\sigma_0(t, S(t))dW(t)$$

with local volatility $\sigma_0$ leads to a profit

$$\int_0^T \frac{\sigma_0^2(t, S(t)) - \sigma^2(t)}{2} S(t)^2 e^{\int_t^T r(s)ds} \frac{\Gamma(t)}{\sigma^2} \partial_{xx} f(t, S(t)) \, dt$$

where $f$ is the unique solution of the PDE

$$\partial_t f + r(t)x\partial_x^2 f + \sigma_0^2(t, x)x^2\partial_{xx} f/2 = r(t)f \quad f(T, x) = H(x)$$
Notation: Non-Anticipative Functionals

Given \( x \in D([0, T], \mathbb{R}^d) \), for all \( t \in [0, T] \) we denote:

- \( x(t) \in \mathbb{R}^d \) the value of \( x \) at \( t \)
- \( x_t = x(t \wedge \cdot) \in D([0, T], \mathbb{R}^d) \) the path stopped at \( t \)
- \( x_{t-} = x1_{[0,t)} + x(t-)1_{[t,T]} \in D([0, T], \mathbb{R}^d) \)
- for \( \delta \in \mathbb{R}^d \), \( x_t^\delta = x_t + \delta 1_{[t,T]} \in D([0, T], \mathbb{R}^d) \)

We define the **space of stopped paths**:

\[
\Lambda_T := \left( [0, T] \times D([0, T], \mathbb{R}^d) \right) / \sim,
\]

where \((t, x) \sim (t', x') \iff t = t' \text{ and } x_t = x'_t\), and the metric

\[
d_\infty((t, x), (t', x')) = \sup_{u \in [0, T]} |x(u \wedge t) - x'(u \wedge t')| + |t - t'|
\]

**Definition (Non-anticipative functional)**

A **non-anticipative functional** on \( D([0, T], \mathbb{R}^d) \) is a map

\[
F : (\Lambda_T, d_\infty) \to \mathbb{R}.
\]
Notation: Regularity of $F : \Lambda_T \to \mathbb{R}$

A non-anticipative functional $F : \Lambda_T \to \mathbb{R}$ is said to be:

- **continuous at fixed times** if for all $t \in [0, T]$
  
  $$F(t, \cdot) : \left( \left( \{t\} \times D([0, T], \mathbb{R}^d) \right) / \sim, \|\cdot\|_\infty \right) \to \mathbb{R}$$

  is continuous

- **left-continuous**, $F \in C^0_0(\Lambda_T)$, if

  $$\forall (t, x) \in \Lambda_T, \forall \epsilon > 0, \exists \eta > 0 : \forall h \in [0, t], \forall (t - h, x') \in \Lambda_T,$$
  
  $$d_\infty((t, x), (t - h, x')) < \eta \quad \Rightarrow \quad |F_t(x) - F_{t-h}(x')| < \epsilon$$

- **boundedness-preserving**, $F \in B(\Lambda_T)$, if, $\forall K \subset \mathbb{R}^d$ compact,

  $\forall t_0 \in [0, T], \exists C_{K, t_0} > 0$ s.t. $\forall t \in [0, t_0], \forall (t, x) \in \Lambda_T$

  $$x([0, t]) \subset K \Rightarrow |F_t(x)| < C_{K, t_0}$$
**Notation: Differentiability of** $F : \Lambda \rightarrow \mathbb{R}$

### Definition (Horizontal derivative)

A non-anticipative functional $F$ is **horizontally differentiable** at $(t, x) \in \Lambda_T$ if $t \mapsto F(t, x_t)$ is right-differentiable, with derivative denoted by $\mathcal{D}F(t, x)$; if it holds $\forall (t, x) \in \Lambda_T$, then we denote

$$\mathcal{D}F := (\mathcal{D}F(t, \cdot))_{t \in [0, T]}$$

### Definition (Vertical derivative)

A non-anticipative functional $F$ is **vertically differentiable** at $(t, x) \in \Lambda_T$ if the map $\mathbb{R}^d \ni e \mapsto F(t, x_t^e)$ is differentiable at 0; in this case we denote $\nabla_\omega F(t, x) := (\partial_i F(t, x))_{i=1, \ldots, d}$, where

$$\partial_i F(t, x) := \lim_{h \to 0^+} \frac{F(t, x_t^{hei}) - F(t, x_t)}{h}.$$

If this holds for all $(t, x) \in \Lambda_T$, then $\nabla_\omega F := (\nabla_\omega F(t, \cdot))_{t \in [0, T]}$.
Sets of Smooth Non-Anticipative Functionals

**Definition \( (\mathcal{C}^{1,2}_b(\Lambda) \text{ and } \mathcal{C}^{1,2}_{loc}(\Lambda)) \)**

Denote by \( \mathcal{C}^{1,2}_b(\Lambda_T) \) the set of non-anticipative functionals \( F \in \mathbb{C}_i^{0,0}(\Lambda_T) \) such that:

- \( \exists \mathcal{D} F \) continuous at fixed times,
- \( \exists \nabla^j_\omega F \in \mathbb{C}_i^{0,0}(\Lambda_T) \) \( j = 1, 2 \),
- \( \mathcal{D} F, \nabla_\omega F, \nabla^2_\omega F \in \mathcal{B}(\Lambda_T) \).

Denote by \( \mathcal{C}^{1,2}_{loc}(\Lambda_T) \) the set of non-anticipative functionals \( F \in \mathbb{C}_i^{0,0}(\Lambda_T) \) such that there exists a sequence of stopping times \( (\tau_k)_{k \geq 1} \) going to \( \infty \) and a sequence \( (F^k \in \mathcal{C}^{1,2}_b(\Lambda_T))_{k \geq 1} \),

\[
F(t, x_t) = \sum_{k \geq 1} F^k(t, x_t) \mathbb{1}_{[\tau_k(x), \tau_{k+1}(x))}(t) \quad \forall (t, x) \in \Lambda_T
\]
**Quadratic Variation for Càdlàg Paths**

Fixed sequence of time partitions: \( \Pi = \{\pi_n\}_{n \geq 1}, \) 
\[ \pi_n = (t^n_i)_{i=0,...,m(n)}, \quad 0 = t^n_0 < \ldots < t^n_{m(n)} = T, \quad |\pi_n| \xrightarrow{n \to \infty} 0 \]

**Definition (Paths of finite quadratic variation)**

- \( x \in D([0, T], \mathbb{R}) \) is of **finite quadratic variation** along \( \Pi \) if 
  \[ \forall t \in [0, T], \quad [x](t) := \lim_{n \to \infty} \sum_{t^n_i \leq t} (x(t^n_{i+1}) - x(t^n_i))^2 < \infty \]

- \( x \in D([0, T], \mathbb{R}^d) \) is of finite quadratic variation along \( \Pi \) if, \( \forall 1 \leq i, j \leq d, x^i, x^j \) are so. *In this case:* 
  \[ [x]_{i,j}(t) \equiv [x^i, x^j](t) = \frac{1}{2} ([x^i + x^j](t) - [x^i](t) - [x^j](t)) \]

Denote \( Q(U, \Pi) := \{x \in U \subset D([0, T], \mathbb{R}^d), \ x \text{ is of f.q.v. along } \Pi \} \)

For every càdlàg path \( \omega \in Q(D([0, T], \mathbb{R}^d), \Pi) \), we can always assume that \( \sup_{t \in [0, T]} |\Delta \omega(t)| \xrightarrow{n \to \infty} 0. \)
CHANGE OF VARIABLE FORMULA FOR FUNCTIONALS

The piecewise constant approximation

\[ \omega^n := \sum_{i=0}^{m(n)-1} \omega(t^n_{i+1}^n) \mathbb{1}_{[t^n_i, t^n_{i+1}]} + \omega(T) \mathbb{1}_{\{T\}}, \]

converges uniformly to \( \omega \)

Theorem (Cont, Fournié (2010))

If \( F \in \mathbb{C}^{1,2}_{loc}(\Lambda_T) \) and \( \omega \in Q(D([0, T], \mathbb{R}^d), \Pi) \), then the limit

\[ \int_0^T \nabla \omega F_t(\omega_{t-})d\Pi \omega := \lim_{n \to \infty} \sum_{i=0}^{m(n)-1} \nabla \omega F^n_{t_i}(\omega^n_{t_i^-}, \Delta \omega(t^n_i)) \cdot (\omega(t^n_{i+1}) - \omega(t^n_i)) \]

exists and

\[ F_T(\omega_T) = F_0(\omega_0) + \int_0^T \nabla \omega F_t(\omega_{t-})d\Pi \omega \]

\[ + \int_0^T D_t F(\omega_{t-})dt + \frac{1}{2} \int_{(0,T]} \text{tr} \left( \nabla^2 \omega F_t(\omega_{t-})d[\omega]^c(t) \right) \]

\[ + \sum_{t \in (0,T]} (F_t(\omega_t) - F_t(\omega_{t-}) - \nabla \omega F_t(\omega_{t-}) \cdot \Delta \omega(t)) \]
**Functional Itô Formula for Functionals of Semimartingales**

**Theorem (Cont, Fournié (2013))**

If $F \in \mathcal{C}^{1,2}_{loc}(\Lambda_T)$ and $X$ is an $\mathbb{R}^d$-valued semimartingale, then a.s.

$$F(T, X_T) = F(0, X_0) + \int_0^T \nabla_\omega F(t, X_{t-}) \cdot dX$$

$$+ \int_0^T DF(t, X_{t-}) dt + \frac{1}{2} \int_{(0,T]} \text{tr} \left( \nabla^2_\omega F(t, X_{t-}) d[X]^c(t) \right)$$

$$+ \sum_{t \in (0, T]} \left( F(t, X_t) - F(t, X_{t-}) - \nabla_\omega F(t, X_{t-}) \cdot \Delta X(t) \right)$$

In particular, $Y$ defined by $Y(t) = F(t, X_t)$ for all $t \in [0, T]$, is a semimartingale.
A Pathwise Setting for Continuous-Time Trading

Financial market: \((\Omega, \| \cdot \|_\infty)\) metric space, \(\Omega := D([0, T], \mathbb{R}_+^d)\), \(\omega \in \Omega\) is a possible trajectory of the (forward) asset prices, \(\mathcal{F}\) is the Borel \(\sigma\)-field and \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}\) the canonical filtration.

Trading strategies: \((V_0, \phi, \psi)\), where

- \(V_0 : \Omega \rightarrow \mathbb{R}, \mathcal{F}_0\)-measurable (initial investment)
- \(\phi : \mathbb{R}^d\)-valued \(\mathbb{F}\)-adapted càglàd process (asset position)
- \(\psi : \mathbb{R}\)-valued \(\mathbb{F}\)-adapted càglàd process (bond position)

Portfolio value at time \(t \in [0, T]\) along the price path \(\omega \in D([0, T], \mathbb{R}_+^d)\):

\[
V(t, \omega; \phi, \psi) = \phi(t, \omega) \cdot \omega(t) + \psi(t, \omega)
\]

When is a strategy self-financing?
When can we explicitly define its gain?
**Simple Trading Strategies**

**Definition (Simple self-financing trading strategies)**

$$(V_0, \phi, \psi)$$ is a **simple trading strategy** if $\phi \in \Sigma(\mathbb{R}^d, \Pi)$ and $\psi \in \Sigma(\mathbb{R}, \Pi)$, where $\Sigma(U, \Pi) := \bigcup_{n \geq 1} \Sigma(U, \pi^n)$,

$$\Sigma(U, \pi^n) := \left\{ \phi : \forall i = 0, \ldots, m(n) - 1, \exists \lambda_i : \Omega \to U, \quad m(n)\!\!m(n) -1 \quad \mathcal{F}_{t_i^n}\text{-measurable,} \quad \phi(t, \omega) = \sum_{i=0}^{m(n)\!\!m(n)-1} \lambda_i(\omega) \mathbb{1}_{(t_i^n, t_{i+1}^n]} \right\}$$

A simple trading strategy $(V_0, \phi, \psi)$ is a **self-financing** if there exists $n \geq 1$ such that $\phi, \psi \in \Sigma(\pi^n)$ and $\psi(t, \omega) := V_0 - \phi(0+, \omega) \cdot \omega(0) - \sum_{i=1}^{m(n)} \omega(t_i^n \wedge t) \cdot (\phi(t_{i+1}^n \wedge t, \omega) - \phi(t_i^n \wedge t, \omega))$

Equivalently: $V(t, \omega; \phi) = V_0 + G(t, \omega; \phi)$, where

$$G(t, \omega; \phi) := \sum_{i=1}^{m(n)} \phi(t_i^n \wedge t, \omega) \cdot (\omega(t_i^n \wedge t) - \omega(t_{i-1}^n \wedge t))$$
**Definition (Self-financing Trading Strategies on U)**

\((V_0, \phi)\) is a self-financing trading strategy on \(U \subset D([0, T], \mathbb{R}_+^d)\) if there exists a sequence \(\{(V_0, \phi^n, \psi^n), n \in \mathbb{N}\}\) of self-financing simple trading strategies, such that

\[
\forall \omega \in U, \forall t \in [0, T], \quad \phi^n(t, \omega) \xrightarrow{n \to \infty} \phi(t, \omega),
\]

and any of the following equivalent conditions is satisfied:

1. **(I)** \(\exists \mathcal{F}\text{-adapted càdlàg process } G(\cdot, \cdot; \phi), \quad \forall t \in [0, T], \omega \in U \quad G(t, \omega; \phi^n) \xrightarrow{n \to \infty} G(t, \omega; \phi), \quad \Delta G(t, \omega; \phi) = \phi(t, \omega) \Delta \omega(t);\)

2. **(II)** \(\exists \mathcal{F}\text{-adapted càdlàg process } \psi(\cdot, \cdot; \phi), \text{ such that } \forall t \in [0, T], \omega \in U \quad \psi^n(t, \omega) \xrightarrow{n \to \infty} \psi(t, \omega; \phi) \text{ and } \psi(t+, \omega; \phi) - \psi(t, \omega; \phi) = -\omega(t) (\phi(t+, \omega) - \phi(t, \omega));\)

3. **(III)** \(\exists \mathcal{F}\text{-adapted càdlàg process } V(\cdot, \cdot; \phi), \quad \forall t \in [0, T], \omega \in U \quad V(t, \omega; \phi^n) \xrightarrow{n \to \infty} V(t, \omega; \phi), \quad \Delta V(t, \omega; \phi) = \phi(t, \omega) \Delta \omega(t).\)
Gain: Pathwise Construction

Proposition

If there exists $F \in \mathbb{C}_{loc}^{1,2}(\Lambda_T) \cap \mathbb{C}_{r,0}^{r,0}(\Lambda_T)$, $\nabla_\omega F \in \mathbb{C}_{r,0}^{r,0}(\Lambda_T)$,

$$\phi(t, \omega) = \nabla_\omega F(t, \omega_{t^-}) \quad \forall \omega \in Q(\Omega, \Pi), \ t \in [0, T],$$

Then, $\phi$ is the asset position of a self-financing trading strategy on $Q(\Omega, \Pi)$ with gain

$$G(t, \omega; \phi) = \int_0^t \phi(u, \omega_u) d^\Pi \omega$$

$$= \lim_{n \to \infty} \sum_{t_i^n < t} \nabla_\omega F(t_i^n, \omega_{t_i^n^-}) \cdot (\omega(t_{i+1}^n) - \omega(t_i^n))$$

and bond position

$$\psi(t, \omega; \phi) := V_0 - \phi^+(0, \omega) +$$

$$- \lim_{n \to \infty} \sum_{i=1}^{m^n} \omega(t_i^n \wedge t) \cdot (\phi^n(t_{i+1}^n \wedge t, \omega) - \phi^n(t_i^n \wedge t, \omega))$$
Hedging Error and Super-Strategies

Definition (Hedging error in specific scenarios)

The hedging error of a self-financing trading strategy \((V_0, \phi)\) on \(U \subset D([0, T], \mathbb{R}_+^d)\) for a path-dependent derivative with payoff \(H\) in a scenario \(\omega \in U\) is the value

\[
V(T, \omega; \phi) - H(\omega) = V_0(\omega) + G(T, \omega; \phi) - H(\omega).
\]

\((V_0, \phi)\) is called a super-strategy for \(H\) on \(U\) if its hedging error for \(H\) is non-negative on \(U\), i.e.

\[
V_0(\omega) + G(T, \omega; \phi) \geq H(\omega_T) \quad \forall \omega \in U.
\]

Given \(A \in D([0, T], S)\), \(S := \{M \in \mathbb{R}^{d \times d}, M \geq 0\ \text{symmetric}\}\),

\[
Q_A(\Pi) := \left\{ \omega \in Q(\Omega, \Pi) \text{ such that } [\omega](t) = \int_0^t A(s)ds \quad \forall t \in [0, T] \right\}
\]
Proposition (Pathwise replication of exotic derivatives)

If \( H : (\Omega, \| \cdot \|_\infty) \to \mathbb{R} \) is continuous and \( F \in C^{1,2}_{\text{loc}}(\mathcal{W}_T) \cap C^{0,0}(\mathcal{W}_T) \) solves

\[
\begin{aligned}
    &DF(t, \omega_t) + \frac{1}{2} \text{tr} \left( \nabla^2 \omega F(t, \omega_t) \cdot A(t) \right) = 0, \quad t \in [0, T) \\
    &F(T, \omega) = H(\omega), \quad \forall \omega \in Q_A(\Pi)
\end{aligned}
\]

Then, the hedging error of the trading strategy \((F_0(\omega_0), \nabla \omega F)\), self-financing on \( Q(C([0, T], \mathbb{R}^d)) \), in any scenario \( \omega \in Q\tilde{A}(\Pi) \) is

\[
\frac{1}{2} \int_0^T \text{tr} \left( \nabla^2 \omega F(t, \omega_t) \cdot (A(t) - \tilde{A}(t)) \right) \, dt
\]

In particular, in all scenarios \( \omega \in Q_A(\Pi) \), \((F(0, \omega_0), \nabla \omega F)\) replicates \( H \) and its portfolio value at any time \( t \in [0, T] \) is given by \( F(t, \omega_t) \).
A Hedging Formula for Path-Dependent Options

Let \( \Omega := C([0, T], \mathbb{R}_+) \) and \( S \) denote the coordinate process on the canonical space \((\Omega, \mathcal{F}, \mathbb{F})\), i.e. \( S(t, \omega) = \omega(t), \forall \omega \in \Omega, t \in [0, T] \).

**Assumption (Hedger’s model assumption)**

The market participant assumes that the underlying asset price \( S \) evolves according to 
\[
    dS(t) = \sigma(t)S(t)dW(t), 
\]
i.e.

\[
    S(t) = S(0)e^{\int_0^t \sigma(u)dW(u)-\frac{1}{2}\int_0^t \sigma(u)^2du}, \quad t \in [0, T], \tag{1}
\]

where \( W \) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) and the volatility \( \sigma \) is a non-negative \( \mathbb{F} \)-adapted process such that \( \sigma \neq 0 \) \( dt \times d\mathbb{P} \)-almost surely and \( S \) is a square-integrable \( \mathbb{P} \)-martingale.

The discounted *price* at time \( t \) of a path-dependent derivative with payoff \( H(S_T) \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) is given by

\[
    Y(t) = \mathbb{E}^{\mathbb{P}}[H(S_T)|\mathcal{F}_t] \quad \mathbb{P}\text{-a.s.}
\]
Theorem (Universal hedging formula (Cont-Fournié, 2013))

If $\mathbb{E}^P[|H(S_T)|^2] < \infty$, then $Y \in \mathcal{M}^2(F)$ and the following martingale representation formula holds:

$$H(S_T) = Y(0) + \int_0^T \nabla_S Y(u) \cdot dS(u) \quad P\text{-a.s.}$$

Moreover, if $Y \in C_b^{1,2}(S)$, where

$$C_b^{1,2}(S) := \{ Y : \exists F \in C_b^{1,2}(W_T), \ Y(t) = F(t, S_t) \text{ } P\text{-a.s.} \forall t \in [0, T]\},$$

then the hedging strategy for $H$ is pathwise defined by

$$\nabla_S Y = \nabla_\omega F(\cdot, S) \ dt \times dP - a.s.$$  

In case $Y \in \mathcal{M}^2(F)$ but $Y \notin C_b^{1,2}(S)$, the hedging strategy $\nabla_S Y$ for $H$ is a ‘weak vertical derivative’, yet it can be uniformly approximated by regular functionals (i.e. Lu-Cont 2015)
A Universal Pricing Equation

**Theorem (Pricing equation for path-dependent derivatives)**

Consider a path-dependent derivative with payoff $H(S_T)$. If

$$\exists F \in C_0^{1,2}, \quad F(t, S_t) = E^P[H(S_T)|\mathcal{F}_t]$$

such that $DF \in \mathbb{C}_F^{0,0}(\mathcal{W}_T)$, then $F$ is the unique solution of the pricing equation

$$DF(t, \omega_t) + \frac{1}{2} \text{tr} \left( \nabla^2_{\omega} F(t, \omega_t) \cdot \sigma^2(t)\omega^2(t) \right) = 0, \quad (2)$$

with the terminal condition $F(T, \omega) = H(\omega)$, on the topological support of $(S, \mathbb{P})$ in $(C([0, T], \mathbb{R}_+), \|\cdot\|_\infty)$, i.e.

$$\forall \omega \in \text{supp}(S) := \left\{ \omega \in \Omega, \text{ such that } P(S_T \in V) > 0 \right\}$$
Problem:

- The hedger sells a path-dependent option with maturity $T$ and payoff $H(S_T)$ such that $E^\mathbb{P}[|H(S_T)|^2] < \infty$
- He computes the price and hedging strategy according to $\mathbb{P}$
- He trades according to that strategy, but taking in input the realized market prices $\Rightarrow$ in a scenario $\omega$, the final value of the hedging portfolio will differ from $H(\omega_T)$
- What is the performance of the hedging strategy $(Y(0), \nabla_S Y)$ with respect to $H$?
**Definition (Robust delta hedge)**

The delta-hedging strategy $(Y(0), \nabla_S Y)$ is said to be robust for $H$ on $U \subset \Omega$ if it is a super-strategy for $H$ on $U$.

**Notation:** For paths of absolutely continuous finite quadratic variation along $\Pi$, we define the local realized volatility as

$$\sigma^{\text{real}} : [0, T] \times \mathcal{A} \rightarrow \mathbb{R},$$

$$(t, \omega) \mapsto \sigma^{\text{real}}(t, \omega) = \frac{1}{\omega(t)} \sqrt{\frac{d}{dt}[\omega](t)},$$

where

$$\mathcal{A} := \{\omega \in Q(\Omega, \Pi), \ t \mapsto [\omega](t) \text{ is absolutely continuous}\}.$$
The Hedging Error Formula and Robustness

**Proposition (The hedging error formula and robustness of delta-hedging)**

If there exists \( F \in C_b^{1,2}(\mathcal{W}_T) \cap C^{0,0}(\mathcal{W}_T) \) such that \( DF \in C^0_1(\mathcal{W}_T) \), and

\[
F(t, S_t) = \mathbb{E}^P[H(S_T)|\mathcal{F}_t] \quad \text{dt} \times \text{d}P\text{-a.s.}
\]

Then, the hedging error of \((F(0, \cdot), \nabla \omega F)\) for \( H \) is explicitly given by

\[
\frac{1}{2} \int_0^T \left( \sigma(t, \omega)^2 - \sigma^{\text{real}}(t, \omega)^2 \right) \omega^2(t) \nabla^2 \omega F(t, \omega) dt
\]

In particular, if for all \( \omega \in U \) and Lebesgue-a.e. \( t \in [0, T) \)

\[
\nabla^2 \omega F(t, \omega) \geq 0 \quad \text{and} \quad \sigma(t, \omega) \geq \sigma^{\text{real}}(t, \omega)
\]

then the delta hedge for \( H \) is robust on \( U \).
**Proposition (Impact of jumps on delta hedging)**

If there exists a non-anticipative functional $F : \Lambda_T \to \mathbb{R}$ such that

$$F \in C^{1,2}_b(\Lambda_T) \cap C^{0,0}(\Lambda_T), \quad \nabla_\omega F \in C^{0,0}(\Lambda_T), \quad \mathcal{D}F \in C^{1,0}_I(\mathcal{W}_T)$$

$$F(t, S_t) = \mathbb{E}^\mathbb{P}[H(S_T)|\mathcal{F}_t] \quad \text{d}t \times \text{d}\mathbb{P}\text{-a.s.}$$

Then, for any $\omega \in Q(D([0, T], \mathbb{R}_+), \Pi)$ such that $[\omega]^c$ is absolutely continuous, the hedging error of the delta hedge $(F(0, \cdot), \nabla_\omega F)$ for $H$ is explicitly given by

$$\frac{1}{2} \int_0^T \left( \sigma(t, \omega)^2 - \sigma^\text{real}(t, \omega)^2 \right) \omega^2(t) \nabla^2_\omega F(t, \omega) \text{d}t$$

$$- \sum_{t \in (0, T]} (F(t, \omega_t) - F(t, \omega_{t-}) - \nabla_\omega F(t, \omega_{t-}) \cdot \Delta \omega(t)).$$
**Vertical Smoothness**

The previous results extend *El Karoui, Jeanblanc & Shreve (1998)*:
- to path-dependent options
- to a pathwise setting: it gives the *pathwise* P&L of the strategy and removes unnecessary probabilistic assumptions

**Definition (Vertical smoothness)**

A functional \( h : D([0, T], \mathbb{R}) \mapsto \mathbb{R} \) is **vertically smooth on** \( U \subset D([0, T], \mathbb{R}) \) if, \( \forall (t, \omega) \in [0, T] \times U \), the real map

\[
g^h(t, \omega; \cdot) : \mathbb{R} \to \mathbb{R}, \quad e \mapsto h(\omega + e1_{[t, T]})
\]

is of class \( C^2 \) on a neighborhood \( V \) of 0 and there exist \( K, c, \beta > 0 \) such that, for all \( \omega, \omega' \in U \), \( t, t' \in [0, T] \),

\[
\left| \partial_e g^h(e; t, \omega) \right| + \left| \partial_{ee} g^h(e; t, \omega) \right| \leq K, \quad e \in V,
\]

\[
\left| \partial_e g^h(0; t, \omega) \right| - \left| \partial_e g^h(0; t', \omega') \right| + \left| \partial_{ee} g^h(0; t, \omega) - \partial_{ee} g^h(0; t', \omega') \right| \\
\leq c(\|\omega - \omega'\|_\infty + |t - t'|^\beta).
\]
Proposition (Pricing functional: existence and regularity)

Let $H : (D([0, T], \mathbb{R}), \| \cdot \|_{\infty}) \to \mathbb{R}$ a locally-Lipschitz payoff functional such that $\mathbb{E}^P[|H(S_T)|] < \infty$ and define

$$h : (D([0, T], \mathbb{R}) \to \mathbb{R}, \quad h(\omega_T) = H(\exp \omega_T),$$

where $\exp \omega_T(t) := e^{\omega(t)}$ for all $t \in [0, T]$.

If $h$ is vertically smooth on $C([0, T], \mathbb{R}_+)$, then

$$\exists F \in C^0,2_b(\mathcal{W}_T) \cap C^0,0(\mathcal{W}_T)$$

such that

$$F(t, S_t) = \mathbb{E}^P[H(S_T)|\mathcal{F}_t] \quad dt \times d\mathbb{P}\text{-a.s.}$$
Definition (Vertical convexity of non-anticipative functionals)

A non-anticipative functional $G : \Lambda_T \to \mathbb{R}$ is called vertically convex on $U \subset \Lambda_T$ if, for all $(t, \omega) \in U$, there exists a neighborhood $V \subset \mathbb{R}$ of 0 such that the map

$$V \to \mathbb{R}, \quad e \mapsto G\left(t, \omega + e1_{[t,T]}\right)$$

is convex.

Proposition (Propagation of vertical convexity)

Assume that, for all $(t, \omega) \in T \times \text{supp}(S, \mathbb{P})$, there exists an interval $I \subset \mathbb{R}$, $0 \in I$, such that the map

$$\nu^H(\cdot; t, \omega) : I \to \mathbb{R}, \quad e \mapsto \nu^H(e; t, \omega) = H\left(\omega(1 + e1_{[t,T]})\right)$$

is convex. If

$$\exists F \in C^0_b^2(\mathcal{W}_T), \quad F(t, S_t) = \mathbb{E}^\mathbb{P}\left[H(S_T)|\mathcal{F}_t\right] \quad dt \times d\mathbb{P}\text{-a.s.},$$

then $F$ is vertically convex on $T \times \text{supp}(S, \mathbb{P})$. In particular:

$$\nabla^2_\omega F(t, \omega) \geq 0, \quad \forall (t, \omega) \in T \times \text{supp}(S, \mathbb{P}).$$
**Example: Discretely-Monitored Exotic Derivatives with Black-Scholes**

**Lemma (Discretely-monitored path-dependent derivatives with Black-Scholes)**

Let $\sigma : [0, T] \rightarrow \mathbb{R}_+$ such that $\int_0^T \sigma^2(t)dt < \infty$.

Assume that $H : D([0, T], \mathbb{R}_+) \rightarrow \mathbb{R}_+$ and there exist a partition $0 = t_0 < t_1 < \ldots < t_n \leq T$ and a function $h \in C^2_b(\mathbb{R}^n; \mathbb{R}_+)$ such that

$$\forall \omega \in D([0, T], \mathbb{R}_+), \quad H(\omega_T) = h(\omega(t_1), \omega(t_2), \ldots, \omega(t_n)).$$

Then,

$$\exists F \in \mathcal{C}^{1,2}_{loc}(\mathcal{W}_T), \quad F(t, S_t) = \mathbb{E}^P[H(S_T)|\mathcal{F}_t] \quad dt \times dP\text{-a.s.,}$$

and the horizontal and vertical derivatives of $F$ are given in a closed form.
Example: Hedging Asian Options with Black-Scholes

Consider an arithmetic Asian call option

\[ H(S_T) = \left( \frac{1}{T} \int_0^T S(u)du - K \right)^+ \]

and assume \( \sigma : [0, T] \rightarrow \mathbb{R}_+ \) such that \( \int_0^T \sigma^2(t)dt < \infty \).

The pricing functional \( F \) is given by:

\[ F(t, \omega_t) = f(t, a(t), \omega(t)), \quad a(t) = \int_0^t \omega(s)ds, \]

where \( f \in C^{1,1,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap C([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+) \) solves

\[
\begin{cases}
\frac{\sigma^2(t)x^2}{2} \partial^2_{xx} f(t, a, x) + x \partial_a f(t, a, x) + \partial_t f(t, a, x) = 0, \\
f(T, a, x) = g \left( \frac{a}{T} \right)
\end{cases}
\]  

(3)

Note that (3) turns out to be a particular case of the universal pricing equation.
Example: Robustness of Black-Scholes Delta-Hedging for Asian Options

Corollary (Robustness of delta-hedging for Asian options)

If the Black-Scholes volatility term structure over-estimates the local realized volatility on \( A \cap \text{supp}(S, \mathbb{P}) \), i.e.

\[
\sigma(t) \geq \sigma^{\text{real}}(t, \omega) \quad \forall \omega \in A \cap \text{supp}(S, \mathbb{P}),
\]

Then the Black-Scholes-delta hedge for the arithmetic Asian call option is robust on \( A \cap \text{supp}(S, \mathbb{P}) \). Moreover, the hedging error is given by

\[
\frac{1}{2} \int_t^T \left( \sigma(u)^2 - \sigma^{\text{real}}(t, \omega)^2 \right) \omega^2(u) \partial_{xx}^2 f(u, a(u), \omega(u)) \, du
\]

where \( f \) is the solution to the Cauchy problem (3).
**EXAMPLE: HEDGING ASIAN OPTIONS WITH HOBSON-ROGERS**

\[ dS(t) = S(t)\sigma^n(O(t))dW(t), \]

where

\[ dO(t) = \sigma^n(O(t))dW(t) - \frac{1}{2}(\sigma^n(O(t))^2 + \lambda O(t))dt. \]

Consider a geometric Asian call option

\[ H(S_T) = \left(e^{M(T)} - K\right)^+, \quad M(T) = \frac{1}{T} \int_0^T \log S(u)du \]

The pricing functional \( F \) is given by: \( \forall (t, \omega) \in \mathcal{W}_T, \)

\[ F(t, \omega) = u(T - t, \log \omega(t), \log \omega(t) - o(t, \omega), g(t, \omega)), \]

where \( u \) is the classical solution of the following Cauchy problem on \( [0, T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \)

\[
\begin{cases}
\frac{1}{2}\sigma^n(x_1 - x_2)^2(\partial_{x_1x_1}^2 u - \partial_{x_1} u) + \lambda(x_1 - x_2)\partial_{x_2} u + x_1\partial_{x_3} u - \partial_t u = 0, \\
u(0, x_1, x_2, x_3) = \Psi^G(e^{x_1}, \frac{x_3}{T}).
\end{cases}
\]

(4) is another particular case of the universal pricing equation.
HEDGING ASIAN OPTIONS WITH PATH-DEPENDENT MODELS

Corollary

If $\sigma^n(o(t, \omega)) \geq \sigma^{\text{real}}(t, \omega)$ $\forall \omega \in A \cap \text{supp}(S, \mathbb{P})$, then the Hobson-Rogers delta hedge for $H$ is robust on $A \cap \text{supp}(S, \mathbb{P})$. Moreover, the hedging error at maturity is given by

$$\frac{1}{2} \int_0^T \left( \sigma^n(o(t, \omega))^2 - \sigma^{\text{mkt}}(t, \omega)^2 \right) \omega^2(t) \omega_x(t) \log \omega(t) \log \omega(t) - o(t, \omega) g(t, \omega) dt,$$

where $u$ is the solution of the Cauchy problem (4).

Other models that generalize Hobson-Rogers and allow to derive finite-dimensional Markovian representation for $S$ and its arithmetic mean are given by

- Pascucci, Foschi 2006,
- Salvatore, Tankov 2014.

They thus guarantee the existence of a smooth pricing functional for arithmetic Asian options, then robustness of the delta hedge can be proved the same way as we showed in the Black-Scholes and Hobson-Rogers cases.