Robust strategies, pathwise Itô calculus, and generalized Takagi functions

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In mainstream finance, the price evolution of a risky asset is usually modeled as a stochastic process defined on some probability space.

**Problem**
- only one single trajectory of the asset price process is observable
- there are no repeated “experiments”
- the price evolution typically lacks stationarity

It follows that the law of the stochastic process cannot be measured accurately by means of statistical observation. We are facing **model ambiguity**.

Practically important consequence: **model risk**

Occam’s razor: do without a probability space
1. Continuous-time finance without probability

Let $X_t$, $0 \leq t \leq T$, be the price evolution of a risky asset. We assume for simplicity that $X$ is a continuous function and that there is a riskless asset with prices $B_t = 1$.

Trading strategy $(\xi, \eta)$:
- $\xi_t$ shares of the risky asset
- $\eta_t$ shares of the riskless asset at time $t$.

Portfolio value at time $t$:

$$V_t = \xi_t X_t + \eta_t$$
Key notion for continuous-time finance: self-financing strategy

If trading is only possible at times $0 = t_0 < t_1 < \cdots < t_N = T$, a strategy $(\xi, \eta)$ is self-financing if and only if

$$V_{t_i} = V_0 + \sum_{k=1}^{i} \xi_{t_{k-1}}(X_{t_k} - X_{t_{k-1}}), \quad i = 1, \ldots, N$$

Now let $(\mathcal{T}_n)_{n \in \mathbb{N}}$ be a refining sequence of partitions (i.e., $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots$ and $\text{mesh}(\mathcal{T}_n) \to 0$). Then $(\xi, \eta)$ can be called self-financing if we may pass to the limit in (1). That is,

$$V_t = V_0 + \int_0^t \xi_s \, dX_s, \quad 0 \leq t \leq T,$$

where the integral should be understood as the limit of the corresponding Riemann sums:

$$\int_0^t \xi_s \, dX_s = \lim_{n \to \infty} \sum_{s \in \mathcal{T}_n, s \leq t} \xi_s(X_{s'} - X_s)$$

(Here, $s'$ denotes the successor of $s$ in $\mathcal{T}_n$).
A special strategy

Here we give a version of an argument from Föllmer (2001)

**Proposition 1.** For $K \in \mathbb{R}$ let

$$\xi_t = 2(X_t - K) \quad 0 \leq t \leq T.$$ 

Then $\int_0^t \xi_t \, dX_t$ exists for all $t$ as the limit of Riemann sums if and only if the quadratic variation of $X$,

$$\langle X \rangle_t := \lim_{N \uparrow \infty} \sum_{s \in T_N, s \leq t} (X_{s'} - X_s)^2,$$

exists for all $t$. In this case

$$\int_0^t \xi_s \, dX_s = (X_t - K)^2 - (X_0 - K)^2 - \langle X \rangle_t$$
For $K = X_0$

**Proposition 1.** Let

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$$\int_0^t \xi_s \, dX_s = (X_t - X_0)^2 - \langle X \rangle_t$$
We always have $\langle X \rangle_t = 0$ if $X$ is of bounded variation or Hölder continuous for some exponent $\alpha > 1/2$ (e.g., fractional Brownian motion with $H > 1/2$).

Otherwise, the quadratic variation $\langle X \rangle$ depends strongly on the choice of $(\mathbb{T}_n)$. Indeed, for instance it is well known that for any continuous function $X$ there exists a refining sequence of partitions along which $\langle X \rangle_t = 0$ (e.g., Freedman (1983)).
If $\langle X \rangle_t$ exists and is continuous in $t$, Itô’s formula holds in the following strictly pathwise sense (Föllmer 1981):

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s) \, d\langle X \rangle_s$$

where

$$\int_0^t f'(X_s) \, dX_s = \lim_{n \uparrow \infty} \sum_{s \in T_n, s \leq t} f'(X_s)(X_s' - X_s)$$

is sometimes called the Föllmer integral and $\int_0^t f''(X_s) \, d\langle X \rangle_s$ is a standard Riemann Stieltjes integral.

This formula was extended by Dupire (2009) and Cont & Fournié (2010) to a functional context.
Incomplete list of financial applications of pathwise Itô calculus

- Strictly pathwise approach to Black–Scholes formula (Bick & Willinger 1994)
- Robustness of hedging strategies and pricing formulas for exotic options (A.S. & Stadje 2007, Cont & Riga 2015)
- Model-free replication of variance swaps (e.g., Davis et al. (2010))
- CPPI strategies (A.S. 2014)
- Functional and pathwise extension of the Fernholz–Karatzas stochastic portfolio theory (A.S., Speiser & Voloshchenko 2015)

The key to many of these results is the following associativity property of the Föllmer integral:

\[ \int_0^t \eta_s \, d\left( \int_0^s \xi_r \, dX_r \right) = \int_0^t \eta_s \xi_s \, dX_s \]

2. In search of a class of test integrators

Let’s fix the sequence of dyadic partitions,

\[ \mathbb{T}_n := \{ k2^{-n} \mid k = 0, \ldots, 2^n \}, \quad n = 1, 2, \ldots \]

**Goal:** Find a rich class of functions \( x \in C[0, 1] \) that admit a nontrivial continuous quadratic variation along \( (\mathbb{T}_n) \).

Of course this is true for the sample paths of Brownian motion or other continuous semimartingales—as long as these sample paths do not belong to a certain nullset \( A \).

But \( A \) is not explicit, and so it is not possible to tell whether a specific realization \( x \) of Brownian motion does indeed admit the quadratic variation \( \langle x \rangle_t = t \) along \( (\mathbb{T}_n)_{n \in \mathbb{N}} \).

Moreover, this selection principle for functions \( x \) lets a probabilistic model enter through the backdoor...
A result of N. Gantert

Recall that the *Faber–Schauder functions* are defined as

\[
e_{0,0}(t) := (\min\{t, 1-t\})^+ \quad e_{m,k}(t) := 2^{-m/2} e_{0,0}(2^m t - k)
\]

Functions \(e_{n,k}\) for \(n = 0\), \(n = 2\), and \(n = 5\)
Every function $x \in C[0, 1]$ with $x(0) = x(1) = 0$ can be represented as

$$x = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}$$

where

$$\theta_{m,k} = 2^{m/2} \left( 2x \left( \frac{2k + 1}{2m+1} \right) - x \left( \frac{k}{2m} \right) - x \left( \frac{k + 1}{2m} \right) \right).$$

Gantert (1991, 1994) showed that

$$\langle x \rangle^n_t := \sum_{s \in \mathbb{T}_n, s \leq t} (x(s') - x(s))^2$$

can be computed for $t = 1$ as

$$\langle x \rangle^n_1 = \frac{1}{2^n} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} \theta_{m,k}^2.$$
By letting

$$\mathcal{X} := \left\{ x \in C[0,1] \mid x = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k} \text{ for coefficients } \theta_{m,k} \in \{-1, +1\} \right\}$$

(which is easily shown to be possible) we hence get a class of functions with $$\langle x \rangle_1 = 1$$ for all $$x \in \mathcal{X}$$.

As a matter of fact:

**Proposition 2.** Every $$x \in \mathcal{X}$$ has the quadratic variation $$\langle x \rangle_t = t$$ along $$(\mathbb{T}_n)$$. 

Link to the Takagi function and its generalizations

The specific function

\[ \hat{x} := \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} e_{m,k} \]

has some interesting properties.
The function \( \hat{x} \) is closely related to the celebrated Takagi function,

\[
\tau = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} 2^{-m/2} e_{m,k}
\]

which was first found by Takagi (1903) and rediscovered many times (e.g., by van der Waerden (1930), Hildebrandt (1933), Tambs–Lyche (1942), and de Rham (1957)).

Our class \( \mathcal{X} \) has a nonempty intersection with the “Takagi class” introduced by Hata & Yamaguti (1984) and is a subset of the class of generalized Takagi functions studied by Allaart (2009).
Functions in $\mathcal{X}$ for various (deterministic) choices of $\theta_{m,k} \in \{-1, 1\}$
Similarities with sample paths of a Brownian bridge

Plots of $x \in \mathcal{X}$ when the $\theta_{m,k}$ form a $\{-1, +1\}$-valued i.i.d. sequence

- Lévy–Ciesielski construction of the Brownian bridge
- Quadratic variation
- Nowhere differentiability (de Rham 1957, Billingsley 1982, Allaart 2009)
- Hausdorff dimension of the graph of $\hat{x}$ is $\frac{3}{2}$ (Ledrappier 1992)
The class of functions with quadratic variation is not a vector space

**Proposition 3.** Consider the function $y \in \mathcal{H}$ defined through $\theta_{m,k} = (-1)^m$. Then

$$
\lim_{n \to \infty} \langle \hat{x} + y \rangle^{2n}_t = \frac{4}{3} t \quad \text{and} \quad \lim_{n \to \infty} \langle \hat{x} + y \rangle^{2n+1}_t = \frac{8}{3} t
$$

The function $\hat{x} + y$ with $\langle \hat{x} + y \rangle^7$ and $\langle \hat{x} + y \rangle^8$
A function $z \notin \mathcal{X}$ with exactly three distinct accumulation points for $\langle z \rangle_t^n$
The maximum of $\hat{x}$

Kahane (1959) showed that the maximum of the Takagi function is $\frac{3}{2}$. For $\hat{x}$, we need different arguments.

Functions $\hat{x}^n(t) := \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} e_{m,k}(t)$ and their maxima on $[0, \frac{1}{2}]$
The preceding plot suggests the recursions

\[ t_{n+1} = \frac{t_n + t_{n-1}}{2} \quad \text{and} \quad M_{n+1} = \frac{M_n + M_{n-1}}{2} + 2^{-\frac{n+2}{2}} \]

These are solved by

\[ t_n = \frac{1}{3}(1-(-1)^n2^{-n}) \quad \text{and} \quad M_n = \frac{1}{3}\left(2+\sqrt{2}+(-1)^{n+1}2^{-n}(\sqrt{2}-1)\right) - 2^{-n/2} \]

By sending \( n \uparrow \infty \), we obtain:

**Theorem 1.** The uniform maximum of functions in \( X \) is attained by \( \hat{x} \) and given by

\[ \max_{x \in X} \max_{t \in [0,1]} |x(t)| = \max_{t \in [0,1]} \hat{x}(t) = \frac{1}{3}(2 + \sqrt{2}). \]

Maximal points are \( t = \frac{1}{3} \) and \( t = \frac{2}{3} \).
Corollary 1. The maximal uniform oscillation of functions in $\mathcal{X}$ is

\[
\max_{x \in \mathcal{X}} \max_{s,t \in [0,1]} |x(t) - x(s)| = \frac{1}{6} (5 + 4\sqrt{2})
\]

where the respective maxima are attained at $s = 1/3$, $t = 5/6$, and

\[
x^* := e_{0,0} + \sum_{m=1}^{\infty} \left( \sum_{k=0}^{2m-1-1} e_{m,k} - \sum_{\ell=2m-1}^{2m-1} e_{m,\ell} \right)
\]
Uniform moduli of continuity

Kahane (1959), Kôno (1987), Hata & Yamaguti (1984), and Allaart (2009) studied moduli of continuity for (generalized) Takagi functions. However, their arguments are not applicable to the functions in $\mathcal{X}$.

Let

$$\omega(h) := \left(1 + \frac{1}{\sqrt{2}}\right)h2^{-\left\lfloor -\log_2 h \right\rfloor/2} + \frac{1}{3}(\sqrt{8} + 2)2^{-\left\lfloor -\log_2 h \right\rfloor/2}$$

Then $\omega(h) = O(\sqrt{h})$ as $h \downarrow 0$. More precisely,

$$\liminf_{h \downarrow 0} \frac{\omega(h)}{\sqrt{h}} = 2\sqrt{\frac{4}{3} + \sqrt{2}} \quad \limsup_{h \downarrow 0} \frac{\omega(h)}{\sqrt{h}} = \frac{1}{6}(11 + 7\sqrt{2})$$
Theorem 2 (Moduli of continuity).

(a) The function $\hat{x}$ has $\omega$ as its modulus of continuity. More precisely,

$$\limsup_{h \downarrow 0} \max_{0 \leq t \leq 1-h} \frac{|\hat{x}(t + h) - \hat{x}(t)|}{\omega(h)} = 1$$

(b) An exact uniform modulus of continuity for functions in $\mathcal{X}$ is given by $\sqrt{2}\omega$. That is,

$$\limsup_{h \downarrow 0} \sup_{x \in \mathcal{X}} \max_{0 \leq t \leq 1-h} \frac{|x(t + h) - x(t)|}{\omega(h)} = \sqrt{2}$$

Moreover, the above supremum over functions $x \in \mathcal{X}$ is attained by the function $x^*$ in the sense that

$$\limsup_{h \downarrow 0} \max_{0 \leq t \leq 1-h} \frac{|x^*(t + h) - x^*(t)|}{\omega(h)} = \sqrt{2}$$
The Faber–Schauder development of $x^*$ is plotted individually for generations $m \leq n - 1$ (with $n = 3$ here).

The aggregated development over all generations $m \geq n$ corresponds to a sequence of rescaled functions $\hat{x}$.

$$\omega(h) = \left(1 + \frac{1}{\sqrt{2}}\right)h2^{-\lfloor-\log_2 h\rfloor/2} + \frac{1}{3}(\sqrt{8} + 2)2^{-\lfloor-\log_2 h\rfloor/2}$$

linear part self-similar part
Consequences

- Functions in $\mathcal{X}$ are uniformly Hölder continuous with exponent $\frac{1}{2}$
- Functions in $\mathcal{X}$ have a finite 2-variation and hence can serve as integrators in rough path theory
- $\mathcal{X}$ is a compact subset of $C[0, 1]$
Thank you


