Term Structure Modelling beyond the Intensity Paradigm

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Motivation

The model for the default time

Defaultable Term Structure Modelling

Generalized Merton Models

Affine models
Term structure models (risk-free, credit risky, electricity, foreign exchange) are typically build from a family of fundamental instruments which offer a (random) payment $H_T$ at maturity $T \geq t$.

Under a suitable no-arbitrage criterion (NAFL) there exists an equivalent (local) martingale measure with respect to a certain numéraire, say $S^0$.

With sufficient integrability,

$$P(t, T) = E_Q \left[ \frac{S^0_t}{S^0_T} H_T | \mathcal{F}_t \right].$$
If prices are positive and absolutely continuous, then

\[ P(t, T) = e^{-\int_t^T f(t,u)du} \]  

(Heath-Jarrow-Morton).

It is quite typical in financial markets that fundamental decisions (ECB-interest rates, regulation in electricity markets, planned outtakes of power plants, dividend payments, etc.) occur at predictable times (not totally inaccessible).

This may effect either the numéraire itself or the predictable projection of \( H_T \), such that (1) may lead to arbitrage possibilities.
In credit risk (say with $S^0 \equiv 1$)

$$P(t, T) = E_Q[\mathbbm{1}_{\{\tau > T\}} | \mathcal{F}_t],$$

with default time $\tau$, i.e. $H_T = \mathbbm{1}_{\{\tau > T\}}$.

Inspired by the Poisson process one typically assumes that

$$H_t^p = \int_0^{t \wedge \tau} \lambda(s) ds$$

which implies that $\tau$ is totally inaccessible and hence an absolutely continuous term structure.

Then $P(\tau = t) = 0$ holds for all $t \geq 0$ which contradicts empirical evidence.

Typical structural models show a totally different behaviour.
Merton (1974)

Simple liability structure: default happens at maturity $T$ of the issued bond if the firm value is not sufficient to cover the liabilities, i.e.

$$P(\tau = T) > 0.$$  

- We have a similar situation in many structural models, where default has a positive probability for occurring at a pre-specified times, such as coupon dates.
- Those times may be random, but are always predictable.
The default of Greece on 1st of July is a prime example of such a case.
Bélanger et al. (2004) take this as motivation and study a first-passage time model for the default time which contains the Merton-model.

As an example we could consider a generalized intensity-based model where

\[ H_t^\rho = \int_0^{t \wedge \tau} \lambda(s) dA(s) \]

with an increasing, deterministic process \( A \). If \( \lambda > 0 \), we have a positive default probability a times \( t \) with \( \Delta A(t) \neq 0 \).
Consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, Q^*)\) with a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions, \(Q^* \sim P\).

The default time \(\tau\) is an \(\mathbb{F}\)-stopping time,

Denote the default indicator process \(H\) by

\[
H_t = \mathbb{1}_{\{t \geq \tau\}}, \quad t \geq 0
\]

\(H\) is a submartingale, such that by the Doob-Meyer decomposition

\[
M_t = H_t - H_t^p
\]

is a true martingale with an increasing, predictable process \(H^p\).
Simplifying assumption

Assume that

$$H_t^P = \int_0^t h_s ds + \int_0^t \int_{\mathbb{R}} x \Gamma(ds, dx), \quad t \geq 0$$

with \( \Gamma(dt, dx) = \sum_{s > 0} 1_{\{\Delta H_s^P \neq 0\}} \delta_{(s, \Delta H_s^P)}(dt, dx) \) being a predictable integer-valued random measure.

Intuition: when \( \Delta H_t^P > 0 \), there is a positive probability that the company defaults at time \( t \). We call such times possible default dates. The set of possible default dates is a thin set which we denote by \( \{U_1, U_2, \ldots \} \).
We call a random time $U$ **announced** by $S$ if $S$ is an $\mathbb{F}$-stopping time with $S < U$ almost surely and $U$ is $\mathcal{F}_S$-measurable.
We make the following assumptions.

**(A1)** The process $h$ is progressively measurable and locally integrable,

$$\int_0^{T^*} |h_s| \, ds < \infty, \quad Q^*\text{-a.s.}$$

**(A2)** Each possible default date $U_i$ is announced by $S_i$, $1 \leq i \leq N$. Moreover, there are positive random variables $\Gamma_1, \Gamma_2, \ldots$ such that each $\Gamma_i$ is $\mathcal{F}_{S_i}$-measurable. The random measure $\Gamma(ds, dx)$ is given by

$$\Gamma([0, t], dx) = \sum_{i=1}^N 1_{\{U_i \leq t\}} \delta_{\Gamma_i}(dx).$$

Note that, under *(A2)*, the random measure $\Gamma$ is predictable.
Defaultable Term Structure Modelling

- At time $t$, all times announced up to $t$ should be taken into account:

$$\mu_t(du) := \sum_{S_i \leq t} \delta_{U_i}(du).$$

- We consider defaultable bond prices given by

$$P(t, T) = 1_{\{\tau > t\}} \exp \left( - \int_t^T f(t, u) du - \int_t^T g(t, u) \mu_t(du) \right)$$

$$= 1_{\{\tau > t\}} \exp \left( - \int_t^T f(t, u) du - \sum_{i: S_i \leq t} 1_{\{T_i \in (t, T]\}} g(t, U_i) \right)$$

(4)

for $0 \leq t \leq T \leq T^*$
Assume that the processes $f$ and $g$ satisfy

$$f(t, T) = f(0, T) + \int_0^t a(u, T)du + \int_0^t b(u, T) \cdot dW_u$$  \hspace{1cm} (5)$$

$$g(t, T) = g(0, T) + \int_0^t \alpha(u, T)du + \int_0^t \beta(u, T) \cdot dW_u$$  \hspace{1cm} (6)$$

with an $n$-dimensional $Q^*$-Brownian motion $W$. 
(B1) the initial forward curve is measurable, and integrable on \([0, T^*]\):
\[
\int_0^{T^*} |f(0, u)| + |g(0, u)| du < \infty, \quad Q^*-\text{a.s.},
\]

(B2) the drift parameters \(a(\omega, s, t)\) and \(\alpha(\omega, s, t)\) are \(\mathbb{R}\)-valued \(\mathcal{O} \otimes \mathcal{B}\)-measurable and
\[
\int_0^{T^*} \int_0^{T^*} |a(s, t)| ds \, dt < \infty, \quad Q^*-\text{a.s.},
\]
\[
\sup_{s, t \leq T^*} |\alpha(s, t)| < \infty, \quad Q^*-\text{a.s.},
\]

(B3) the volatility parameter \(b(\omega, s, t)\) is \(\mathbb{R}^n\)-valued, \(\mathcal{O} \otimes \mathcal{B}\)-measurable, and
\[
\sup_{s, t \leq T^*} \|b(s, t)\| < \infty, \quad Q^*-\text{a.s.},
\]
while \(\beta(\omega, s, t)\) is \(\mathbb{R}^n\)-valued, \(\mathcal{O} \otimes \mathcal{B}\)-measurable, and square integrable on \([0, T^*]\):
\[
\int_0^{T^*} \int_0^{T^*} \|\beta(s, t)\|^2 ds \, dt < \infty, \quad Q^*-\text{a.s.}
\]

(B4) we assume that \(\mu(dt, du) = \sum_{i=1}^{n} \delta(s_i, u_i)(dt, du)\) has an absolutely continuous compensator \(\nu_t(du)dt\) and
\[
\int_0^{T^*} \int_0^{T^*} |e^{-g(t, u)} - 1| \nu(t, du)dt < \infty, \quad Q^*-\text{a.s.}
\]
Moreover \(Q^*(\tau = S_i) = 0\) for all \(i \geq 1\).
Absence of Arbitrage

\[ \bar{a}(t, T) = \int_t^T a(t, u) du, \quad \bar{b}(t, T) = \int_t^T b(t, u) du, \]
\[ \bar{\alpha}(t, T) = \int_t^T \alpha(t, u) \mu_t(du), \quad \bar{\beta}(t, T) = \int_t^T \beta(t, u) \mu_t(du). \]

**Theorem**

Assume that (A1)-(A2) and (B1)-(B4) hold. Then \( Q^* \) is an ELMM if and only if the following two conditions hold:

\[
\int_0^t f(s, s) ds + \sum_{U_i \leq t} g(U_i, U_i) = \int_0^t (r_s + h_s) ds - \sum_{U_i \leq t} \log(1 - \Gamma_i), \tag{7}
\]
\[
\bar{a}(t, T) + \bar{\alpha}(t, T) = \frac{1}{2} \| \bar{b}(t, T) + \bar{\beta}(t, T) \|^2 + \int_t^T \left( e^{-g(t, u)} - 1 \right) \nu(t, du), \tag{8}
\]

\( 0 \leq t \leq T \leq T^*, \ dQ^* \otimes dt\)-almost surely on \( \{ t < \tau \} \).
Example

Consider $\lambda > 0$, $0 < u_1 < \cdots < u_N$, and positive random variables $\lambda'_1, \ldots, \lambda'_N$. Set

$$\Lambda_t = \lambda t + \sum_{u_i \leq t} \lambda'_i.$$  

Let $E$ be a standard exponential random variable, independent from $\Lambda$, and set

$$\tau = \inf\{ t \geq 0 : \Lambda_t \geq E \}.$$  

Then $\Delta H^P_{u_i} > 0$ because $u_i$ is a possible default date:

$$Q^*(\tau = u_i) = Q^*(\lambda'_i \geq E) = \mathbb{E}^*[1 - \exp(-\lambda'_i)].$$  

If $\Lambda$ is deterministic and $r = 0$, we obtain

$$P(t, T) = Q^*(\tau > T|\tau > t) = 1_{\{\tau > t\}} \exp \left( -\lambda(T - t) - \sum_{u_i \in (t, T]} \lambda'_i \right).$$  

Note that

$$H^P_t = \lambda(t \wedge \tau) + \sum_{i : u_i \leq (t \wedge \tau)} (1 - e^{-\lambda'_i}).$$
The setup simplifies if the defaultable dates are deterministic:

- Consider a set $\mathcal{U} = \{u_1, u_2, \ldots \} \subset \mathbb{R}_{>0}$ such that any time outside $\mathcal{U}$ is totally inaccessible for the default time $\tau$, i.e. $P(\tau = t) = 0$ for all $t \notin \mathcal{U}$.
- Defaultable bond prices given by (!)

$$P_M(t, T) = 1_{\{\tau > t\}} \exp \left( - \int_t^T f(t, u) \nu(du) \right), \quad 0 \leq t \leq T \leq T^*; \quad (9)$$

with

$$\nu(du) = du + \sum_{i \geq 1} \delta_{u_i}(du).$$

(A2′) Assume $P(\tau = t) = 0$ for all $t \notin \mathcal{U}$ and that there are random variables $0 \leq \Gamma_1, \Gamma_2, \ldots$ such that $\Gamma_i$ is $\mathcal{F}_{u_i-}$-measurable. The predictable random measure $\Gamma = \sum_{u_i \in \mathcal{U}} \delta_{(u_i, \Gamma_i)}$ is finite, that is $\Gamma([0, T^*], \mathbb{R}) < \infty$, $Q^*$-a.s.

A model satisfying (A2′) will be called generalized Merton model.
Set

\[
\bar{a}(t, T) = \int_t^T a(t, u) \nu(du), \quad \bar{b}(t, T) = \int_t^T b(t, u) \nu(du).
\]  \hspace{1cm} (10)

Theorem

Assume that \((A1),(A2')\) and \((B1')-(B3')\) hold. Then \(Q^*\) is an ELMM if and only if the following two conditions hold:

\[
\int_0^t f(s, s) \nu(ds) = \int_0^t r_s ds + \int_0^t h(s) ds - \sum_{i: u_i \leq t} \log(1 - \Gamma_i), \quad (11)
\]

\[
\bar{a}(t, T) = \frac{1}{2} \parallel \bar{b}(t, T) \parallel^2, \quad (12)
\]

for \(0 \leq t \leq T \leq T^* \) \(dQ^* \otimes dt\)-almost surely on \(\{t < \tau\}\).
It turns out that a generalized Merton model has the default compensator

\[
H^p_t = \int_0^{t \wedge \tau} h(s) ds + \sum_{u_i \leq (t \wedge \tau)} \Gamma_i = \int_0^{t \wedge \tau} h'(s) dA(s),
\]

thus - it can be viewed also as a generalized intensity-based model.
We assume $\mathcal{U} = \{u_1, \ldots, u_n\}$ and $r_t = 0$.

The idea is to consider an affine process $X$ and

$$
H^p_t = \int_0^t \left( \alpha_0(s) + \langle \beta_0(s), X_s \rangle \right) ds + \sum_{i=1}^n 1_{\{t \geq u_i\}} \left( \alpha_i + \langle \beta_i, X_{u_i} \rangle \right). \quad (13)
$$
Consider a state space in canonical form \( D = \mathbb{R}^m \times \mathbb{R}^n \) for integers \( m, n \geq 0 \) with \( m + n = d \) and a \( d \)-dimensional Brownian motion \( W \). Let \( \mu \) and \( \sigma \) be defined on \( D \) by

\[
\mu(x) = \mu_0 + \sum_{i=1}^{d} x_i \mu_i, \tag{14}
\]

\[
\frac{1}{2} \sigma(x) \top \sigma(x) = \sigma_0 + \sum_{i=1}^{d} x_i \sigma_i, \tag{15}
\]

where \( \mu_0, \mu_i \in \mathbb{R}^d, \sigma_0, \sigma_i \in \mathbb{R}^{d \times d} \), for all \( i \in \{1, \ldots, d\} \).

Then the unique strong solution of

\[
dX_t = \mu(X_t) dt + \sigma(X_t) dW_t, \quad X_0 = x \tag{16}
\]

is an affine process \( X \) on the state space \( D \).
Definition

We call a bond-price model **affine** if there exist functions \( \phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \), \( \psi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d \) such that

\[
P^M(t, T) = 1_{\{\tau > t\}} e^{-\phi(t, T) - \langle \psi(t, T), X_t \rangle}, \quad 0 \leq t \leq T.
\] (17)

- We consider a generalized Merton model with \( \mathcal{U} = \{u_1, \ldots, u_n\} \).
- For the càd-functions \( \phi, \psi \) we denote by \( \partial_t \) the partial derivative from the right-hand side.
Proposition

Assume that $\alpha_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\beta_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ are measurable and bounded functions such that $\alpha_0(s) + \langle \beta_0(s), x \rangle \geq 0$ for $s \geq 0$, $x \in D$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\beta_1, \ldots, \beta_n \in \mathbb{R}^d$ satisfy $\alpha_i + \langle \beta_i, x \rangle \geq 0$ for all $1 \leq i \leq n$ and $x \in X$. Moreover,

$$
\phi(T, T) = 0
$$

$$
-\partial_t \phi(t, T) = \alpha_0(t) + \langle \mu_0, \psi(t, T) \rangle - \langle \psi(t, T), \sigma_0 \psi(t, T) \rangle, \quad t \neq u_i
$$

$$
\Delta \phi(u_i, T) = \alpha_i
$$

$$
\psi(T, T) = 0
$$

$$
-\partial_t \psi_k(t, T) = \beta_{0,k}(t) + \langle \mu_k, \psi(t, T) \rangle - \langle \psi(t, T), \sigma_k \psi(t, T) \rangle, \quad t \neq u_i
$$

$$
\Delta \psi_k(u_i, T) = \beta_{k,i}.
$$

Then, the affine model given by (13) and (17) satisfies NAFL.
Example

In the one-dimensional case we consider \( X \) given as solution of

\[
    dX_t = (\mu_0 + \mu_1 X_t) dt + \sigma \sqrt{X_t} dW_t, \quad t \geq 0.
\]

We assume that \( \mathcal{U} = \{1\} \) and choose \( \nu(du) = \delta_{\{1\}}(du) \). Moreover, let \( \alpha_0 = 0 \), \( \beta_0 = 1 \) as well as \( \alpha_1 = 0 \) and \( \beta_1 \geq 0 \), such that

\[
    H^p_t = \int_0^t X_s ds + 1_{\{t \geq 1\}} \beta_1 X_1.
\]

Hence the probability of default at 1 given the information until time 1—is \( \beta_1 X_1 \).
An arbitrage-free model can be obtained as follows:

\[ L_1(t) = 2(e^{\theta t} - 1), \quad L_2(t) = \theta(e^{\theta t} + 1) + \mu_1(e^{\theta t} - 1), \]
\[ L_3(t) = \theta(e^{\theta t} + 1) - \mu_1(e^{\theta t} - 1), \quad L_4(t) = \sigma^2(e^{\theta t} - 1). \]

Let

\[ A_0(s) = \frac{2\mu_0}{\sigma^2} \log \left( \frac{2\theta e^{\frac{(\sigma - \mu_1)t}{2}}}{L_3(t)} \right), \quad B_0(s) = -\frac{L_1(t)}{L_3(t)} \]

such that with \( A(t, T) = A_0(T - t) \) and \( B(t, T) = B_0(T - t) \) for \( 0 \leq t \leq T < 1 \), the conditions of Proposition 5 hold. Similarly, for \( 1 \leq t \leq T \) and \( T \geq 1 \), choosing \( A(t, T) = A_0(T - t) \) and \( B(t, T) = B_0(T - t) \) implies again the validity of the Riccati equations.
On the other hand, for $0 \leq t < 1$ and $T \geq 1$ we set

$$u(T) = B(1-, T) = B(1, T) - \psi_1 = B_0(T - t) - \psi_1,$$

and let

$$A(t, T) = \frac{2\mu_0}{\sigma^2} \log \left( \frac{2\theta e^{(\sigma - \mu_1)(1-t)}}{L_3(1-t) - L_4(1-t)u(T)} \right)$$

$$B(t, T) = -\frac{L_1(1-t) - L_2(1-t)u(T)}{L_3(1-t) - L_4(1-t)u(T)}.$$

In this case $\Delta A(1, T) = 0$ and $\Delta B(1, T) = \psi_1$. \qed
Jointly with Martin Keller-Ressel we study the following class of affine processes (talk on Wednesday):

**Definition**

A semimartingale $X$ is called **affine** if it is adapted and there exist $C$ and $C^d$-valued càdlàg functions $\phi(s, t, u)$ and $\psi(s, t, u)$, respectively, such that

$$
E \left[ e^{\langle u, X_t \rangle} \mid F_s \right] = \exp \left( \phi(s, t, u) + \langle \psi(s, t, u), X_s \rangle \right)
$$

(18)

holds for all $u \in i\mathbb{R}^d$ and $0 \leq s \leq t$.

This definition does not require **stochastic continuity** for an affine process which is an assumption in the treatments Filipović (2005), Keller-Ressel et al. (2011).
Considering semimartingales which are not stochastically continuous requires a modification of the HJM-approach to term structure models.

- Drift conditions can be obtained leading to arbitrage-free models.
- Affine models which are not stochastically continuous have still a high degree of tractability.
