Linear-Rational Term-Structure Models

Anders Trolle
(joint with Damir Filipović and Martin Larsson)

Ecole Polytechnique Fédérale de Lausanne
Swiss Finance Institute

AMaMeF and Swissquote Conference, September 9, 2015
Near-zero short-term interest rates
Contribution

- Existing models that respect zero lower bound (ZLB) on interest rates face limitations:
  - Shadow-rate models do not capture volatility dynamics
  - Multi-factor CIR and quadratic models do not easily accommodate unspanned factors and swaption pricing

- We develop a new class of **linear-rational** term structure models
  - Respects ZLB on interest rates
  - Easily accommodates unspanned factors affecting volatility and risk premia
  - Admits semi-analytical solutions to swaptions

- Extensive empirical analysis
  - Parsimonious model specification has very good fit to interest rate swaps and swaptions since 1997
  - Captures many features of term structure, volatility, and risk premia dynamics.
Contribution

- Existing models that respect zero lower bound (ZLB) on interest rates face limitations:
  - Shadow-rate models do not capture volatility dynamics
  - Multi-factor CIR and quadratic models do not easily accommodate unspanned factors and swaption pricing

- We develop a new class of **linear-rational** term structure models
  - Respects ZLB on interest rates
  - Easily accommodates unspanned factors affecting volatility and risk premia
  - Admits semi-analytical solutions to swaptions

- Extensive empirical analysis
  - Parsimonious model specification has very good fit to interest rate swaps and swaptions since 1997
  - Captures many features of term structure, volatility, and risk premia dynamics.
Contributions

- Existing models that respect zero lower bound (ZLB) on interest rates face limitations:
  - Shadow-rate models do not capture volatility dynamics
  - Multi-factor CIR and quadratic models do not easily accommodate unspanned factors and swaption pricing
- We develop a new class of **linear-rational** term structure models
  - Respects ZLB on interest rates
  - Easily accommodates unspanned factors affecting volatility and risk premia
  - Admits semi-analytical solutions to swaptions
- Extensive empirical analysis
  - Parsimonious model specification has very good fit to interest rate swaps and swaptions since 1997
  - Captures many features of term structure, volatility, and risk premia dynamics.
Outline

The linear-rational framework

The Linear-Rational Square-Root (LRSQ) model

Empirical analysis
Outline

The linear-rational framework

The Linear-Rational Square-Root (LRSQ) model

Empirical analysis
Linear-rational framework and bond pricing

- **State-price density, \( \zeta_t \)**

\[
\Pi(t, T) = \frac{1}{\zeta_t} \mathbb{E}_t[\zeta_T C_T]
\]

- **\( m \)-dimensional factor process, \( Z_t \), with linear drift given by**

\[
dZ_t = \kappa(\theta - Z_t)dt + dM_t,
\]

for some \( \kappa \in \mathbb{R}^{m \times m} \), \( \theta \in \mathbb{R}^m \), and some martingale \( M_t \)

- **\( \zeta_t \) given by**

\[
\zeta_t = e^{-\alpha t} \left( \phi + \psi^\top Z_t \right),
\]

for some \( \phi \in \mathbb{R} \) and \( \psi \in \mathbb{R}^m \) such that \( \phi + \psi^\top z > 0 \) for all \( z \in E \), and some \( \alpha \in \mathbb{R} \)

- **Conditional expectations:**

\[
\mathbb{E}_t[Z_T] = \theta + e^{-\kappa(T-t)}(Z_t - \theta)
\]

- **Price of zero-coupon bond:**

\[
P(t, t + \tau) = \frac{(\phi + \psi^\top \theta)e^{-\alpha \tau} + \psi^\top e^{-(\alpha + \kappa)\tau}(Z_t - \theta)}{\phi + \psi^\top Z_t}
\]
State-price density, $\zeta_t$

$$\Pi(t, T) = \frac{1}{\zeta_t} \mathbb{E}_t[\zeta_T C_T]$$

$m$-dimensional factor process, $Z_t$, with linear drift given by

$$dZ_t = \kappa (\theta - Z_t) dt + dM_t,$$

for some $\kappa \in \mathbb{R}^{m \times m}$, $\theta \in \mathbb{R}^m$, and some martingale $M_t$

$\zeta_t$ given by

$$\zeta_t = e^{-\alpha t} (\phi + \psi^T Z_t),$$

for some $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^m$ such that $\phi + \psi^T z > 0$ for all $z \in E$, and some $\alpha \in \mathbb{R}$

Conditional expectations:

$$\mathbb{E}_t[Z_T] = \theta + e^{-\kappa(T-t)}(Z_t - \theta)$$

Price of zero-coupon bond:

$$P(t, t + \tau) = \frac{(\phi + \psi^T \theta)e^{-\alpha \tau} + \psi^T e^{-(\alpha + \kappa) \tau} (Z_t - \theta)}{\phi + \psi^T Z_t}$$
Linear-rational framework and bond pricing

- **State-price density,** $\zeta_t$
  \[
  \Pi(t, T) = \frac{1}{\zeta_t} \mathbb{E}_t[\zeta_T C_T]
  \]

- **$m$-dimensional factor process,** $Z_t$, with linear drift given by
  \[
  dZ_t = \kappa(\theta - Z_t)dt + dM_t,
  \]
  for some $\kappa \in \mathbb{R}^{m \times m}$, $\theta \in \mathbb{R}^m$, and some martingale $M_t$

- $\zeta_t$ given by
  \[
  \zeta_t = e^{-\alpha t} (\phi + \psi^T Z_t),
  \]
  for some $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^m$ such that $\phi + \psi^T z > 0$ for all $z \in E$, and some $\alpha \in \mathbb{R}$

- **Conditional expectations:**
  \[
  \mathbb{E}_t[Z_T] = \theta + e^{-\kappa(T-t)}(Z_t - \theta)
  \]

- **Price of zero-coupon bond:**
  \[
  P(t, t + \tau) = \frac{(\phi + \psi^T \theta)e^{-\alpha \tau} + \psi^T e^{-(\alpha + \kappa) \tau}(Z_t - \theta)}{\phi + \psi^T Z_t}
  \]
Interest rates and the zero lower bound

- **Short rate:**

\[ r_t = -\partial_T \log P(t, T)|_{T=t} = \alpha - \frac{\psi^T \kappa (\theta - Z_t)}{\phi + \psi^T Z_t} \]

- Define

\[ \alpha^* = \sup_z \frac{\psi^T \kappa (\theta - z)}{\phi + \psi^T z} \quad \text{and} \quad \alpha_* = \inf_z \frac{\psi^T \kappa (\theta - z)}{\phi + \psi^T z} \]

- Set \( \alpha = \alpha^* \) so that

\[ r_t \in [0, \alpha^* - \alpha_*] \]

- \( \alpha^* \) and \( \alpha_* \) are finite if \( z \in \mathbb{R}_d^+ \) and all components of \( \psi \) are strictly positive

- Range is parameter dependent, verify that range is wide enough

- If eigenvalues of \( \kappa \) have nonnegative real part then \( \alpha \) is the infinite-maturity ZCB yield
Interest rates and the zero lower bound

- **Short rate:**

  \[ r_t = -\partial_T \log P(t, T)|_{T=t} = \alpha - \frac{\psi^\top \kappa(\theta - Z_t)}{\phi + \psi^\top Z_t} \]

- **Define**

  \[ \alpha^* = \sup_z \frac{\psi^\top \kappa(\theta - z)}{\phi + \psi^\top z} \quad \text{and} \quad \alpha_* = \inf_z \frac{\psi^\top \kappa(\theta - z)}{\phi + \psi^\top z} \]

- **Set** \( \alpha = \alpha^* \) so that

  \[ r_t \in [0, \alpha^* - \alpha_*] \]

- \( \alpha^* \) and \( \alpha_* \) are finite if \( z \in \mathbb{R}^d_+ \) and all components of \( \psi \) are strictly positive

- Range is parameter dependent, verify that range is wide enough

- If eigenvalues of \( \kappa \) have nonnegative real part then \( \alpha \) is the infinite-maturity ZCB yield
Short rate:

\[ r_t = -\partial_T \log P(t, T)|_{T=t} = \alpha - \frac{\psi^\top \kappa (\theta - Z_t)}{\phi + \psi^\top Z_t} \]

Define

\[ \alpha^* = \sup_z \frac{\psi^\top \kappa (\theta - z)}{\phi + \psi^\top z} \text{ and } \alpha_* = \inf_z \frac{\psi^\top \kappa (\theta - z)}{\phi + \psi^\top z} \]

Set \( \alpha = \alpha^* \) so that

\[ r_t \in [0, \alpha^* - \alpha_*] \]

\( \alpha^* \) and \( \alpha_* \) are finite if \( z \in \mathbb{R}_+^d \) and all components of \( \psi \) are strictly positive.

Range is parameter dependent, verify that range is wide enough.

If eigenvalues of \( \kappa \) have nonnegative real part then \( \alpha \) is the infinite-maturity ZCB yield.
Interest rate swaps

- Exchange a stream of fixed-rate for floating-rate payments
- Consider a tenor structure

\[ T_0 < T_1 < \cdots < T_n, \quad T_i - T_{i-1} \equiv \Delta \]

- At \( T_i, \ i = 1 \ldots n \):
  - pay \( \Delta k \), for fixed rate \( k \)
  - receive floating LIBOR \( \Delta L(T_{i-1}, T_i) = \frac{1}{P(T_{i-1}, T_i)} - 1 \)

- Value of payer swap at \( t \leq T_0 \)

\[ \Pi_t^{\text{swap}} = P(t, T_0) - P(t, T_n) - \Delta k \sum_{i=1}^{n} P(t, T_i) \]

- Fixed leg

- Floating leg

- Forward swap rate \( S_t = \frac{P(t, T_0) - P(t, T_n)}{\Delta \sum_{i=1}^{n} P(t, T_i)} \)
Swaptions

- **Payer swaption** = option to enter the swap at $T_0$ paying fixed, receiving floating

- Payoff at expiry $T_0$ of the form

$$C_{T_0} = (\prod_{T_0}^{\text{swap}})^+ = \left( \sum_{i=0}^{n} c_i P(T_0, T_i) \right)^+ = \frac{1}{\zeta_{T_0}} p_{\text{swap}}(Z_{T_0})^+$$

for the explicit linear function

$$p_{\text{swap}}(z) = \sum_{i=0}^{n} c_i e^{-\alpha T_i} \left( \phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i-T_0)}(z - \theta) \right)$$

- Swaption price at $t \leq T_0$ is given by

$$\Pi_t^{\text{swap}} = \frac{1}{\zeta_t} \mathbb{E}[\zeta_{T_0} C_{T_0} | \mathcal{F}_t] = \frac{1}{\zeta_t} \mathbb{E}_t \left[ p_{\text{swap}}(Z_{T_0})^+ \right]$$

- Efficient swaption pricing via Fourier transform . . .!
Swaptions

- **Payer swaption** = option to enter the swap at $T_0$ paying fixed, receiving floating

- Payoff at expiry $T_0$ of the form

$$C_{T_0} = (\Pi_{T_0}^{\text{swap}})^+ = \left( \sum_{i=0}^{n} c_i P(T_0, T_i) \right)^+ = \frac{1}{\zeta_{T_0}} p_{\text{swap}}(Z_{T_0})^+$$

for the explicit linear function

$$p_{\text{swap}}(z) = \sum_{i=0}^{n} c_i e^{-\alpha T_i} \left( \phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i-T_0)}(z - \theta) \right)$$

- Swaption price at $t \leq T_0$ is given by

$$\Pi_t^{\text{swap}} = \frac{1}{\zeta_t} \mathbb{E}[\zeta_{T_0} C_{T_0} | \mathcal{F}_t] = \frac{1}{\zeta_t} \mathbb{E}_t [p_{\text{swap}}(Z_{T_0})^+]$$

- Efficient swaption pricing via **Fourier transform** . . . !
Fourier transform

Define

\[ \hat{q}(x) = \mathbb{E}_t \left[ \exp (x \ p_{\text{swap}}(Z_{T_0})) \right] \]

for every \( x \in \mathbb{C} \) such that the conditional expectation is well-defined.

Then

\[ \Pi_t^{\text{swaption}} = \frac{1}{\zeta_t \pi} \int_0^{\infty} \text{Re} \left[ \frac{\hat{q}(\mu + i\lambda)}{(\mu + i\lambda)^2} \right] d\lambda \]

for any \( \mu > 0 \) with \( \hat{q}(\mu) < \infty \).

\( \hat{q}(x) \) has semi-analytical solution in LRSQ model.
The linear-rational framework

The Linear-Rational Square-Root (LRSQ) model

Empirical analysis
Objective: A model with joint factor process \((Z_t, U_t)\), where

- \(Z_t\): \(m\) term structure factors
- \(U_t\): \(n \leq m\) USV factors

Denoted \(LRSQ(m,n)\)

Based on a \((m + n)\)-dimensional square-root diffusion process \(X_t\) taking values in \(\mathbb{R}^{m+n}_+\) of the form

\[
dX_t = (b - \beta X_t) \, dt + \text{Diag} \left( \sigma_1 \sqrt{X_{1t}}, \ldots, \sigma_{m+n} \sqrt{X_{m+n,t}} \right) \, dB_t,
\]

Define \((Z_t, U_t) = SX_t\) as linear transform of \(X_t\) with state space \(E = S(\mathbb{R}^{m+n}_+)\)

Need to specify a \((m + n) \times (m + n)\)-matrix \(S\) such that

- the implied term structure state space is \(E = \mathbb{R}^m_+\)
- the drift of \(Z_t\) does not depend on \(U_t\), while \(U_t\) feeds into the martingale part of \(Z_t\)
S given by

\[ S = \begin{pmatrix} \text{Id}_m & A \\ 0 & \text{Id}_n \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} \text{Id}_n \\ 0 \end{pmatrix}. \]

\[ \beta \] chosen upper block-triangular of the form

\[ \beta = S^{-1} \begin{pmatrix} \kappa & 0 \\ 0 & A^\top \kappa A \end{pmatrix} S = \begin{pmatrix} \kappa & \kappa A - AA^\top \kappa A \\ 0 & A^\top \kappa A \end{pmatrix} \]

for some \( \kappa \in \mathbb{R}^{m \times m} \)

\[ b \] given by

\[ b = \beta S^{-1} \begin{pmatrix} \theta \\ \theta_U \end{pmatrix} = \begin{pmatrix} \kappa \theta - AA^\top \kappa A \theta_U \\ A^\top \kappa A \theta_U \end{pmatrix} \]

for some \( \theta \in \mathbb{R}^m \) and \( \theta_U \in \mathbb{R}^n \).
Resulting joint factor process \((Z_t, U_t)\):

\[
\begin{align*}
\mathrm{d}Z_t &= \kappa (\theta - Z_t) \, \mathrm{d}t + \sigma (Z_t, U_t) \, \mathrm{d}B_t \\
\mathrm{d}U_t &= A^\top \kappa A (\theta_U - U_t) \, \mathrm{d}t + \text{Diag} \left( \sigma_{m+1} \sqrt{U_{1t}} \, \mathrm{d}B_{m+1,t}, \ldots, \sigma_{m+n} \sqrt{U_{nt}} \, \mathrm{d}B_{m+n,t} \right),
\end{align*}
\]

with dispersion function of \(Z_t\) given by

\[
\sigma(z, u) = (\text{Id}_m, A) \text{Diag} \left( \sigma_1 \sqrt{z_1 - u_1}, \ldots, \sigma_{m+n} \sqrt{u_n} \right).
\]

Example: \(LRSQ(1,1)\)

\[
\begin{align*}
\mathrm{d}Z_{1t} &= \kappa_{11} (\theta_1 + \theta_2 - Z_{1t}) \, \mathrm{d}t + \sigma_1 \sqrt{Z_{1t}} - U_{1t} \, \mathrm{d}B_{1t} + \sigma_2 \sqrt{U_{1t}} \, \mathrm{d}B_{2t} \\
\mathrm{d}U_{1t} &= \kappa_{22} (\theta_2 - U_{1t}) \, \mathrm{d}t + \sigma_2 \sqrt{U_{1t}} \, \mathrm{d}B_{2t}
\end{align*}
\]
## Linear-rational vs. exponential-affine framework

<table>
<thead>
<tr>
<th></th>
<th>Exponential-affine</th>
<th>Linear-rational</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short rate</td>
<td>affine</td>
<td>LR</td>
</tr>
<tr>
<td>ZCB price</td>
<td>exponential-affine</td>
<td>LR</td>
</tr>
<tr>
<td>ZCB yield</td>
<td>affine</td>
<td>log of LR</td>
</tr>
<tr>
<td>Coupon bond price</td>
<td>sum of exponential-affines</td>
<td>LR</td>
</tr>
<tr>
<td>Swap rate</td>
<td>ratio of sums of exponential-affines</td>
<td>LR</td>
</tr>
<tr>
<td>ZLB</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>USV</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Cap/floor valuation</td>
<td>semi-analytical</td>
<td>semi-analytical</td>
</tr>
<tr>
<td>Swaption valuation</td>
<td>approximate</td>
<td>semi-analytical</td>
</tr>
<tr>
<td>Linear state inversion</td>
<td>ZCB yields</td>
<td>bond prices or swap rates</td>
</tr>
</tbody>
</table>
The linear-rational framework

The Linear-Rational Square-Root (LRSQ) model

Empirical analysis
Data and estimation approach

- Panel data set of swaps and swaptions
- Swap maturities: 1Y, 2Y, 3Y, 5Y, 7Y, 10Y
- Swaptions expiries: 3M, 1Y, 2Y, 5Y
- 866 weekly observations, Jan 29, 1997 – Aug 28, 2013
- Estimation approach: Quasi-maximum likelihood in conjunction with the unscented Kalman Filter

Panel A1: Swap data
Panel B1: Swaption data
Panel A2: Swap fit, LRSQ(3,3)
Panel B2: Swaption fit, LRSQ(3,3)
Panel A3: Swap RMSE, LRSQ(3,3)
Panel B3: Swaption RMSE, LRSQ(3,3)

Figure 2: Data and fit
Panel A1 shows time series of the 1-year, 5-year, and 10-year swap rates (displayed as thick light-grey, thick dark-grey, and thin black lines, respectively). Panel B1 shows time series of the normal implied volatilities on three “benchmark” swaptions: the 3-month option on the 2-year swap, the 2-year option on the 2-year swap, and the 5-year option on the 5-year swap (displayed as thick light-grey, thick dark-grey, and thin black lines, respectively). Panels A2 and B2 show the fit to swap rates and implied volatilities, respectively, in case of the LRSQ(3,3) specification. Panels A3 and B3 show time series of the root-mean-squared pricing errors (RMSE) of swap rates and implied volatilities, respectively, in case of the LRSQ(3,3) specification. The units in Panels B1, B2, A3, and B3 are basis points. The grey areas mark the two NBER-designated recessions from March 2001 to November 2001 and from December 2007 to June 2009, respectively. Each time series consists of 866 weekly observations from January 29, 1997 to August 28, 2013.
Model specifications

- Model specifications (always 3 term structure factors)
  - $LRSQ(3,1)$: volatility of $Z_{1t}$ containing an unspanned component
  - $LRSQ(3,2)$: volatility of $Z_{1t}$ and $Z_{2t}$ containing unspanned components
  - $LRSQ(3,3)$: volatility of term structure factors containing unspanned components

- $\alpha = \alpha^*$ and range of $r_t$:

<table>
<thead>
<tr>
<th></th>
<th>$LRSQ(3,1)$</th>
<th>$LRSQ(3,2)$</th>
<th>$LRSQ(3,3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long ZCB yield</td>
<td>7.46%</td>
<td>6.88%</td>
<td>5.66%</td>
</tr>
<tr>
<td>$\alpha$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper bound on</td>
<td>20%</td>
<td>146%</td>
<td>72%</td>
</tr>
<tr>
<td>$r_t$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Empirical analysis
Level-dependence in factor volatilities

- Volatility of $Z_{it}$ with USV: $\sqrt{\sigma_i^2 Z_{it} + (\sigma_{i+3}^2 - \sigma_i^2) U_{it}}$
- Volatility of $Z_{it}$ without USV: $\sigma_i \sqrt{Z_{it}}$

![Figure 4: Level-dependence in volatility of the term structure factors. For each term structure factor, its instantaneous volatility is plotted against its level. The first, second, and third column correspond to the $LRSQ(3,1)$, $LRSQ(3,2)$, and $LRSQ(3,3)$ specification, respectively. The first, second, and third row correspond to $Z_{1,t}$, $Z_{2,t}$, and $Z_{3,t}$, respectively. Each plot contains 866 weekly observations from January 29, 1997 to August 28, 2013. The grey areas mark the possible range of factor volatilities for a given factor level.]

Empirical analysis
Volatility dynamics near the ZLB

Level-dependence in volatility, 3M/1Y swaption IV vs. 1Y swap rate

The figure shows the normal implied volatility of the 3-month option on the 1-year swap rate (in basis points) plotted against the level of the 1-year swap rate. The grey area marks the possible range of implied volatilities in case of the LRSQ(3,3) specification.
Regress weekly changes in the 3M swaption IV on weekly changes in the underlying swap rate

$$\Delta \sigma_{N,t} = \beta_0 + \beta_1 \Delta S_t + \epsilon_t$$

### Table 4: Level-dependence in volatility.

For each available swap maturity, the table reports results from regressing weekly changes in the 3-month normal implied volatility of the swap rate on weekly changes in the level of the swap rate (including a constant). Panel A shows the slope coefficients with t-statistics in parentheses, and Panel B shows the $R^2$s. Within each panel, the first row displays unconditional results, while the second to sixth rows display results conditional on the swap rate being in the intervals 0%-1%, 1%-2%, 2%-3%, 3%-4%, and 4%-5%, respectively. Each underlying time series consists of 866 weekly observations from January 29, 1997 to August 28, 2013. t-statistics are corrected for heteroscedasticity and serial correlation up to 12 lags using the method of Newey and West (1987). ∗, ∗∗, and ∗∗∗ denote significance at the 10%, 5%, and 1% level, respectively.
Figure 7: Level-dependence in volatility, $LRSQ(3,3)$

For each swap maturity, weekly changes in the 3-month normal implied volatility of the swap rate are regressed on weekly changes in the level of the swap rate (including a constant). Regressions are run unconditionally as well as conditional on the swap rate being in the intervals 0%-1%, 1%-2%, 2%-3%, 3%-4%, and 4%-5%, respectively. Panels A and C show the average (across swap maturities) slope coefficients and $R^2$s, respectively. Panels B and D show the average (across swap maturities) model-implied slope coefficients and $R^2$s, respectively. In each panel, the first bar corresponds to the unconditional regressions, while the second to sixth bars correspond to the conditional regressions. Model-implied values are obtained by running the regressions on data simulated from the $LRSQ(3,3)$ specification, where each time series consists of 2,600,000 weekly observations (50,000 years).
Conclusion

Key features of framework:
- Respects ZLB on interest rates
- Easily accommodates unspanned factors affecting volatility and risk premia
- Admits semi-analytical solutions to swaptions

Extensive empirical analysis:
- Parsimonious model specification has very good fit to interest rate swaps and swaptions since 1997
- Captures many features of term structure, volatility, and risk premia dynamics.