A Bayesian methodology for systemic risk assessment in financial networks

Luitgard A. M. Veraart

London School of Economics and Political Science

September 2015

Joint work with Axel Gandy (Imperial College London)

7th General AMaMeF and Swissquote Conference, EPFL

The problem

Consider interbank market as network:

- **Nodes** consist of \(n\) banks with indices in \(\mathcal{N} = \{1, \ldots, n\}\).
- **Edges** \(L_{ij}\) represent nominal interbank liability of bank \(i\) to bank \(j\).

Stress tests: Suppose some banks default on their liabilities. How do losses spread along the edges? What if edges are not observable?

A matrix \(L = (L_{ij}) \in \mathbb{R}^{n \times n}\) is a liabilities matrix if \(L_{ij} \geq 0\), \(L_{ii} = 0\) \(\forall i, j\)

Total nominal interbank liabilities of bank \(i\): \(r_i(L) := \sum_{j=1}^{m} L_{ij}\).

Total nominal interbank assets of bank \(i\): \(c_i(L) := \sum_{j=1}^{m} L_{ji}\).

In practice, \(L_{ij}\) not fully observable, but \(r_i(L)\), \(c_i(L)\) are.

How to fill in the missing data? Implications for stress testing?
Main contributions

- Development of Bayesian framework (Gibbs sampler) for sampling from distribution of liabilities matrix conditional on its row and column sums.

- Application to systemic risk assessment:
  - Can give probabilities for outcomes of stress tests.
  - Results show limitations of classical approach.

- Show that general monotonicity arguments relating severity of systemic risk to number of edges do not hold in general.

- Code is available as R-package (systemicrisk) on CRAN.
Existence of admissible liabilities matrix

Theorem (Existence of an admissible liabilities matrix)

Consider two vectors $a, l \in [0, \infty)^n$ satisfying $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} l_i$. Then there exists a matrix $L \in [0, \infty)^{n \times n}$ with

$$\text{diag}(L) = 0, \quad c(L) = a, \quad r(L) = l$$

if and only if

$$a_i \leq \sum_{\substack{j=1 \atop j \neq i}}^{n} l_j \quad \forall i \in \mathcal{N}.$$ 

Proof contains algorithm giving explicit construction.
The Bayesian framework

- Constructs adjacency matrix $A = (A_{ij})$; attaches liabilities $L_{ij}$.
- Model:
  
  For $i, j \in \mathcal{N}$:
  
  $A_{ij} \sim \text{Bernoulli}(p_{ij})$
  
  $L_{ij} | \{A_{ij} = 1\} \sim \text{Exponential}(\lambda_{ij})$
  
  $L_{ij} = 0$ if $A_{ij} = 0$.

- Parameters:
  
  - $p \in [0, 1]^{n \times n}$, $\text{diag}(p) = 0$; $p_{ij}$ probability of existence of directed edge from $i$ to $j$,
  
  - $\lambda \in \mathbb{R}^{n \times n}$, governs distribution of weights given that edge exists.
The Bayesian framework

- Constructs adjacency matrix $A = (A_{ij})$; attaches liabilities $L_{ij}$.

- Model:
  
  For $i, j \in \mathcal{N}$:
  
  $A_{ij} \sim \text{Bernoulli}(p_{ij})$
  
  $L_{ij} | \{A_{ij} = 1\} \sim \text{Exponential}(\lambda_{ij})$
  
  $L_{ij} = 0$ if $A_{ij} = 0$.

- Parameters:
  
  - $p \in [0, 1]^{n \times n}$, $\text{diag}(p) = 0$; $p_{ij}$ probability of existence of directed edge from $i$ to $j$,
  
  - $\lambda \in \mathbb{R}^{n \times n}$, governs distribution of weights given that edge exists.

- Observations: $c(L) = a \in \mathbb{R}^n$, $r(L) = l \in \mathbb{R}^n$. 

The Bayesian framework

- Constructs adjacency matrix $\mathcal{A} = (A_{ij})$; attaches liabilities $L_{ij}$.
- **Model:**

  For $i, j \in \mathcal{N}$:
  
  $A_{ij} \sim \text{Bernoulli}(p_{ij})$
  
  $L_{ij} \mid \{A_{ij} = 1\} \sim \text{Exponential}(\lambda_{ij})$
  
  $L_{ij} = 0$ if $A_{ij} = 0$.

- **Parameters:**
  
  - $p \in [0, 1]^{n \times n}$, $\text{diag}(p) = 0$; $p_{ij}$ probability of existence of directed edge from $i$ to $j$,
  
  - $\lambda \in \mathbb{R}^{n \times n}$, governs distribution of weights given that edge exists.

- **Observations:** $c(L) = a \in \mathbb{R}^n$, $r(L) = l \in \mathbb{R}^n$.

- **Main interest:** Distribution of $h(L) \mid a, l$. 
Gibbs sampling for $L|a, l$

- **Markov Chain Monte Carlo (MCMC):** Interested in sampling from a given distribution. Construct a Markov chain with this stationary distribution. Run chain. Chain converges to stationary distribution.

- Key idea of **Gibbs sampler**: a step of the chain updates one or several components of the entire parameter vector by sampling them from their joint conditional distribution given the remainder of the parameter vector.

- Here parameter vector is matrix $L$:
  - Initialise chain with matrix $L$ that satisfies $r(L) = l$, $c(L) = a$.
  - MCMC sampler produce a sequence of matrices $L^1, L^2, \ldots$.
  - Quantity of interest: $\mathbb{E}[h(L)|l, a] \approx \frac{1}{N} \sum_{i=1}^{N} h(L^{i\delta+b})$,
    - $N$ number of samples, $b$ burn-in period, $\delta \in \mathbb{N}$ thinning parameter.
Updating components of $L$

- Need to decide **which elements of $L$** need to be updated.

- Need to determine **how the new values will be chosen**, i.e., need to determine their **distribution conditional on remainder of elements of $L$**.
## Illustration of updating submatrices

<table>
<thead>
<tr>
<th>$L_{i1j1}$</th>
<th>$L_{i1j2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{i2j1}$</td>
<td>$L_{i2j2}$</td>
</tr>
<tr>
<td>$L_{i3j1}$</td>
<td>$L_{i3j2}$</td>
</tr>
<tr>
<td>$L_{i4j1}$</td>
<td>$L_{i4j2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L_{i1j1}$</th>
<th>$L_{i1j2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{i2j2}$</td>
<td>$L_{i2j3}$</td>
</tr>
<tr>
<td>$L_{i3j3}$</td>
<td>$L_{i4j4}$</td>
</tr>
</tbody>
</table>

The Bayesian framework
Balance sheets and fundamental defaults

- **Balance sheet of bank $i$:**

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>external assets</td>
<td>$a_i^{(e)}$</td>
</tr>
<tr>
<td>interbank assets</td>
<td>$a_i := c_i(L)$</td>
</tr>
<tr>
<td>external liabilities</td>
<td>$l_i^{(e)}$</td>
</tr>
<tr>
<td>interbank liabilities</td>
<td>$l_i := r_i(L)$</td>
</tr>
<tr>
<td>net worth</td>
<td>$w_i := w_i(L, a_i^{(e)}, l_i^{(e)}) := a_i^{(e)} + c_i(L) - l_i^{(e)} - r_i(L)$</td>
</tr>
</tbody>
</table>

- **Stress tests:** apply proportional shock $s \in [0, 1]^n$ to external assets; shocked external assets are $s_i a_i^{(e)} \forall i$.

- **Fundamental defaults:** $\{ i \mid w_i(L, s_i a_i^{(e)}, l_i^{(e)}) < 0 \}$

- **Fundamental defaults** can be checked from balance sheet aggregates without needing to know the whole matrix $L$!

- To check for **contagious defaults** we need to know $L$. 
Balance sheet data (in million Euros) from banks in the EBA 2011 stress test:

<table>
<thead>
<tr>
<th>Bank code</th>
<th>Bank</th>
<th>$a^{(e)} + a$</th>
<th>$a$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE017</td>
<td>DEUTSCHE BANK AG</td>
<td>1,905,630</td>
<td>47,102</td>
<td>30,361</td>
</tr>
<tr>
<td>DE018</td>
<td>COMMERZBANK AG</td>
<td>771,201</td>
<td>49,871</td>
<td>26,728</td>
</tr>
<tr>
<td>DE019</td>
<td>LANDESBank BADEN-WURTTEMBER</td>
<td>374,413</td>
<td>91,201</td>
<td>9,838</td>
</tr>
<tr>
<td>DE020</td>
<td>DZ BANK AG</td>
<td>323,578</td>
<td>100,099</td>
<td>7,299</td>
</tr>
<tr>
<td>DE021</td>
<td>BAYERISCHE LANDESBank</td>
<td>316,354</td>
<td>66,535</td>
<td>11,501</td>
</tr>
<tr>
<td>DE022</td>
<td>NORDDEUTSCHE LANDESBank -GZ-</td>
<td>228,586</td>
<td>54,921</td>
<td>3,974</td>
</tr>
<tr>
<td>DE023</td>
<td>HYPO REAL ESTATE HOLDING AG</td>
<td>328,119</td>
<td>7,956</td>
<td>5,539</td>
</tr>
<tr>
<td>DE024</td>
<td>WESTLB AG, DUSSELDORF</td>
<td>191,523</td>
<td>24,007</td>
<td>4,218</td>
</tr>
<tr>
<td>DE025</td>
<td>HSH NORDBank AG, HAMBURG</td>
<td>150,930</td>
<td>4,645</td>
<td>4,434</td>
</tr>
<tr>
<td>DE027</td>
<td>LANDESBank BERLIN AG</td>
<td>133,861</td>
<td>27,707</td>
<td>5,162</td>
</tr>
<tr>
<td>DE028</td>
<td>DEKABANK DEUTSCHE GIROZENTRALE</td>
<td>130,304</td>
<td>30,937</td>
<td>3,359</td>
</tr>
</tbody>
</table>
Stress testing

- We apply a **deterministic shock to external assets** of all 11 banks in the network by considering the shocked external assets $s_i a_i^{(e)}$ with $s_i = 0.97 \forall i \in \mathcal{N}$.

- Shock causes **fundamental default of 4 banks**: DE017, DE022, DE023, DE024.

- We apply the **clearing approach** by Eisenberg & Noe (2001) and [Rogers & V. (2013)] to determine which banks suffer **contagious defaults**.

- Gibbs sampler allows us to derive posteriori **default probabilities for remaining 7 banks**.
Default probabilities of banks as a function of $p$
Default probabilities for clearing with default costs

(a) Clearing with $\alpha = 1, \beta = 0$.7

Luitgard A. M. Veraart (LSE)
Mean out-degree of banks, i.e., $\mathbb{E}[\sum_j A_{ij} \mid a, l]$, for different $p^{ER}$ in the Erdős-Rényi network

<table>
<thead>
<tr>
<th>l</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE020</td>
<td>99936</td>
<td>3.50</td>
<td>4.40</td>
<td>5.40</td>
<td>6.20</td>
<td>6.90</td>
<td>7.60</td>
<td>8.30</td>
<td>9.00</td>
</tr>
<tr>
<td>DE019</td>
<td>91314</td>
<td>3.30</td>
<td>4.20</td>
<td>5.10</td>
<td>6.00</td>
<td>6.70</td>
<td>7.50</td>
<td>8.20</td>
<td>8.90</td>
</tr>
<tr>
<td>DE021</td>
<td>66494</td>
<td>2.90</td>
<td>3.70</td>
<td>4.70</td>
<td>5.50</td>
<td>6.40</td>
<td>7.20</td>
<td>8.00</td>
<td>8.80</td>
</tr>
<tr>
<td>DE022</td>
<td>54907</td>
<td>2.70</td>
<td>3.50</td>
<td>4.40</td>
<td>5.30</td>
<td>6.10</td>
<td>7.00</td>
<td>7.80</td>
<td>8.80</td>
</tr>
<tr>
<td>DE018</td>
<td>49864</td>
<td>2.60</td>
<td>3.40</td>
<td>4.30</td>
<td>5.10</td>
<td>6.00</td>
<td>6.90</td>
<td>7.80</td>
<td>8.70</td>
</tr>
<tr>
<td>DE017</td>
<td>46989</td>
<td>2.50</td>
<td>3.30</td>
<td>4.20</td>
<td>5.10</td>
<td>5.90</td>
<td>6.80</td>
<td>7.70</td>
<td>8.70</td>
</tr>
<tr>
<td>DE028</td>
<td>30963</td>
<td>2.20</td>
<td>2.80</td>
<td>3.60</td>
<td>4.50</td>
<td>5.40</td>
<td>6.30</td>
<td>7.30</td>
<td>8.40</td>
</tr>
<tr>
<td>DE027</td>
<td>27679</td>
<td>2.10</td>
<td>2.70</td>
<td>3.50</td>
<td>4.30</td>
<td>5.20</td>
<td>6.10</td>
<td>7.10</td>
<td>8.30</td>
</tr>
<tr>
<td>DE024</td>
<td>23971</td>
<td>1.90</td>
<td>2.60</td>
<td>3.30</td>
<td>4.10</td>
<td>5.00</td>
<td>5.90</td>
<td>7.00</td>
<td>8.20</td>
</tr>
<tr>
<td>DE023</td>
<td>8023</td>
<td>1.40</td>
<td>1.80</td>
<td>2.30</td>
<td>2.80</td>
<td>3.50</td>
<td>4.30</td>
<td>5.40</td>
<td>6.90</td>
</tr>
<tr>
<td>DE025</td>
<td>4841</td>
<td>1.20</td>
<td>1.50</td>
<td>1.90</td>
<td>2.40</td>
<td>2.90</td>
<td>3.60</td>
<td>4.60</td>
<td>6.10</td>
</tr>
</tbody>
</table>
Development of Bayesian framework (Gibbs sampler) for sampling from distribution of liabilities matrix conditional on its row and column sums.

Can be used for stress tests using empirical data.

Can be used as a simulation tool to analyse heterogeneous networks.

Can incorporate additional information such as expert views etc. on the network structure.

R package (systemicrisk) available from CRAN.
