Measures of Systemic Risk

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(joint work with Zachary Feinstein & Birgit Rudloff)

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Motivation

- Various financial crises have highlighted the paramount importance of systemic risk in the financial sector.
- The tremendous cost of systemic risk requires instruments for an efficient macroprudential regulation of financial institutions.
- **Goal of talk:**
  - **Novel approach to the measurement of systemic risk**
  - Systemic risk measures that are based on macroprudential objectives,
  - but enable at the same time systemic risk measurement on the level of firms.
Outline

(i) Measures of systemic risk
   • General definition and properties

(ii) Orthant risk measures
   • Conservative simplification that excludes externalities of capital levels

(iii) Numerical examples
   • Systemic risk aggregation as described in Chen, Iyengar & Moallemi (2013) and Kromer, Overbeck & Zilch (2015); network models as suggested by Eisenberg & Noe (2001) and Cifuentes, Shin & Ferrucci (2005), see also Awiszus & W. (2015)
Measures of Systemic Risk
The Basic Ingredients

Consider a one-period economy with $l$ entities.
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(i) **Structure of the underlying system**

- \( Y = (Y_k)_{k \in \mathbb{R}^l} \) non-decreasing random field
  - For each capital allocation \( k = (k_i)_{i=1,2,\ldots,l} \) the random variable \( Y_k \) captures the relevant stochastic outcome
  - The topological vector space of suitable random variables is denoted by \( \mathcal{X} \)
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(ii) **Objectives of a financial regulator**

- \( \mathcal{A} \subseteq \mathcal{X} \) set of random variables
  - Each element of \( \mathcal{A} \) is acceptable from the point of view of a regulatory authority
  - Mathematically: an acceptance set of a scalar monetary risk measure
Systemic Risk Measures – Definition

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Definition 1

Letting $P(\mathbb{R}_l; \mathbb{R}_+^l) := \{ B \subseteq \mathbb{R}_l \mid B = B + \mathbb{R}_+^l \}$ be the collection of upper sets with ordering cone $\mathbb{R}_+^l$, we call the function

$$R : \mathcal{Y} \times \mathbb{R}_l \rightarrow P(\mathbb{R}_l; \mathbb{R}_+^l)$$

a systemic risk measure, if for some acceptance set $\mathcal{A} \subseteq \mathcal{X}$ of a scalar monetary risk measure:

$$R(Y; k) = \{ m \in \mathbb{R}_l \mid Y_{k+m} \in \mathcal{A} \}.$$
Systemic Risk Measures – Properties

(i) **Cash-invariance:**

\[ R(Y; k) + m = R(Y; k - m) \]

(ii) **Monotonicity:**

\[ (\forall k \in \mathbb{R}^l : Y_k \geq Z_k) \Rightarrow (\forall k \in \mathbb{R}^l : R(Y; k) \supseteq R(Z; k)) \]

(iii) **Closed values:**

Suppose that \( \mathbb{R}^l \rightarrow \mathcal{X}, k \mapsto Y_k \) is continuous. Then \( R(Y; k) \) is a closed subset of \( \mathbb{R}^l \).
Systemic Risk Measures – Properties (2)

(i) **Convex values:**

Suppose that $\mathcal{A}$ is convex and that $\mathbb{R}^l \to \mathcal{X}, k \mapsto Y_k$ is concave. Then $R(Y; k)$ is a convex subset of $\mathbb{R}^l$ for all $k \in \mathbb{R}^l$, i.e. $R(Y; \cdot)$ has convex values.

(ii) **Diversification and quasi-convexity:**

The required notion of diversification is slightly more complicated for random fields.

An appropriate construction is described in Feinstein, Rudloff & W. (2015).
Examples of Random Fields
Examples of Non-Decreasing Random Fields

(i) **Aggregation mechanism insensitive to capital levels**
   - Setting of Chen, Iyengar & Moallemi (2013)
   - Based on axiomatic characterization of scalar systemic risk measures

(ii) **Aggregation mechanism sensitive to capital levels**
    - Extension that allows for feedback effects

(iii) **Financial networks with market clearing**
    - Essentially special case of example (ii)
Example (i): Aggregation mechanism insensitive to capital levels

- **Interconnected financial economy** of financial institutions
  
  \[ N = \{1, 2, \ldots, n\} \]

- \( X \in L^0(\mathbb{R}^n) \) future wealths of the agents in the financial sector

- \( \Lambda : \mathbb{R}^n \to \mathbb{R} \) increasing aggregation function

- Random output is
  
  \[ Y_k := \Lambda(X) + \sum_{i=1}^{n} k_i, \quad k \in \mathbb{R}^n \]

- In this case, \( n = l \).
Example (i): Aggregation mechanism insensitive to capital levels (cont.)

- If one assumes that there are \( l \) groups with identical capital levels, one could alternatively consider

\[
Y_k := \Lambda(X) + \sum_{i=1}^{n} g_i(k), \quad k \in \mathbb{R}^l
\]

with

\[
g(k) = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_l, \ldots, k_l),
\]

Group 1 \quad Group 2 \quad Group \( i \)

i.e. \( g : \mathbb{R}^l \rightarrow \mathbb{R}^n \) increasing.
Example (ii): Aggregation mechanism sensitive to capital levels

- Example (i) can be modified by setting
  \[ Y_k := \Lambda(X + k), \quad k \in \mathbb{R}^n. \]

- The aggregation mechanism is sensitive to capital levels. In particular, feedback from capital levels to the final outputs is captured.

- As in Example (i), we could have groups with equal capital levels which can be encoded by a function \( g : \mathbb{R}^l \to \mathbb{R}^n : \)
  \[ Y_k := \Lambda(X + g(k)), \quad k \in \mathbb{R}^l. \]
Example (iii): Financial networks


- Financial institutions $N = \{1, 2, \ldots, n\}$

- Society is additional node 0

- Nominal liability matrix: $(\bar{p}_{ij})_{i,j=0,1,2,\ldots,n}$, $\bar{p}_i = \sum_{j=0}^{n} \bar{p}_{ij}$

- Relative liabilities:

  $$ a_{ij} = \begin{cases} 
  \frac{\bar{p}_{ij}}{\bar{p}_i}, & \bar{p}_i > 0, \\
  0, & \bar{p}_i = 0. 
  \end{cases} $$

- Inverse demand function for illiquid asset: $f : \mathbb{R}_+ \to \mathbb{R}_{++}$
Example (iii): Financial networks (cont.)
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- **Liquid positions**: \( x \in \mathbb{R}^n \)
  - Obligations must be fulfilled via transfers of the liquid asset.

- **Illiquid positions**: \( s \in \mathbb{R}^n \)
  - If necessary, illiquid positions must be liquidated, but these are subject to price impact described by the inverse demand function.

- **Equilibrium** computed as unique fixed point:
  - Clearing vector: \( p(x; s) \in \mathbb{R}_+^{n+1} \)
  - Clearing price of illiquid asset: \( \pi(x; s) \in \mathbb{R}_+ \)
Example (iii): Financial networks (cont.)

\[ p_i(x; s) = \bar{p}_i \land \left( \sum_{j=0}^{n} p_j(x; s)a_{ji} + x_i + \pi(x; s)s_i \right), \quad i = 1, 2, \ldots, n \]

\[ \pi(x; s) = f \left[ \sum_{i=1}^{n} \left( \frac{1}{\pi(x; s)} \left[ \bar{p}_i - x_i - \sum_{j=0}^{n} a_{ji}p_j(x; s) \right]^{+} \land s_i \right) \right] \]

\[ p_0(x; s) = \bar{p}_0 \]

\[ \implies \]

\[ e_i(x; s) = \sum_{j \neq i} p_j(x; s)a_{ji} + x_i + \pi(x; s)s_i - \bar{p}_i, \quad i = 0, 1, 2, \ldots, n \]
Example (iii): Financial networks (cont.)

- Let $X, S \in L^0(\mathbb{R}^n)$ the random number of shares the agents hold at time $t = 1$ before market clearing.

- If we focus at the wealth of the society, then the relevant stochastic outcome is provided by the random field

$$Y_k := e_0(X + k; S), \quad k \in \mathbb{R}^n.$$ 

- Special case of example (ii) with aggregation function

$$\Lambda(\cdot) := e_0(\cdot; S).$$

- Again, one can consider groups with identical capital.
Orthant Risk Measures
A Pragmatic Approach

- **Systemic risk measurements** $R(Y; k)$ are sets of allocations of additional capital that lead to acceptable outcomes:
  - Financial firms cannot choose their capital independently of the other firms.
  - Risk measurements $R(Y; k)$ capture the essence of systemic risk, but are potentially difficult to communicate.

- A simple alternative consists in choosing a point $k^*$ in the boundary of $R(Y; k)$ and to require firms to hold capital inside
  \[
  k^* + \mathbb{R}_+
  \]
  - Construction is more conservative, without any externalities of the choices of capital levels, and easy to communicate.
A Pragmatic Approach (cont.)

Figure 1: Illustration of a minimal point $k^*$ of an upper set with the orthant $k^* + \mathbb{R}^2_+$ in blue.
Orthant Risk Measures – Definition

Definition 2 Let $\mathcal{P}(\mathbb{R}^l)$ be the power set of $\mathbb{R}^l$. A mapping $k^*: \mathcal{Y} \times \mathbb{R}^l \rightarrow \mathcal{P}(\mathbb{R}^l)$ is called an orthant risk measure associated with a systemic risk measure $R$, if the following properties are satisfied:

(i) Minimal values:
$$k^*(Y; k) \subseteq \text{Min} R(Y; k)$$

(ii) Convex values:
$$k^1, k^2 \in k^*(Y; k) \Rightarrow \alpha k^1 + (1 - \alpha) k^2 \in k^*(Y; k)$$

(iii) Cash-invariance:
$$k^*(Y; k) + m = k^*(Y; k - m)$$
Orthant Risk Measures – Characterization

Lemma 1 Let $R : \mathcal{Y} \times \mathbb{R}^l \to \mathcal{P}(\mathbb{R}^l; \mathbb{R}^l_+)$ be a systemic risk measure with convex values. For $w : \mathcal{Y} \to \mathbb{R}^l_+^+$ such that $w(Y) \in \text{recc} R(Y; 0)^+$, the set-valued mapping
\[
\hat{k}(Y; k) = \arg \min \left\{ \sum_{i=1}^l w(Y)_i m_i \mid m \in R(Y; k) \right\}
\] (1)
defines an orthant risk measure.

All orthant risk measures $k^*$ as defined above are included in orthant risk measures $\hat{k}$ of form (1), i.e. $k^*(Y; k) \subseteq \hat{k}(Y; k)$ for all $Y \in \mathcal{Y}$ and $k \in \mathbb{R}^l$.

- The lemma provides examples of orthant risk measures via a specific choice of the “regulatory price of capital” $w$. 

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Numerical Examples
Case Study A

- **Framework of Eisenberg & Noe (2001):**
  only *local interaction* in the network

- **Tiered graph:**
  - Connections are randomly generated, probabilities within tiers and between tiers are fixed
  - Size of obligations within tiers and between tiers along connections are fixed

- **2 Tiers/Groups:** few firms with large obligations, many firms with small obligations

- **Further ingredients:**
  random endowments, acceptance set defined by AV@R

- **Comparative statics:** varying the degrees of connectedness
Fixed Intra-Group Connections

![Graph showing the relationship between Group 1 and Group 2 capital requirements for different fixed intra-group connections.](image_url)
Fixed Inter-Group Connections
Fixed Inter-Group Connections

- $q_{I_1} = 60\%; q_{I_2} = 20\%; q_{I_3} = 20\%; q_{I_3} = 10\%$
- $q_{I_1} = 10\%; q_{I_2} = 20\%; q_{I_3} = 20\%; q_{I_3} = 10\%$
Case Study B

• Both local (network) and global (price impact) interaction

• Tiered graph:
  – Connections are randomly generated, probabilities within tiers and between tiers are fixed
  – Size of obligations within tiers and between tiers along connections are fixed

• 3 Tiers/Groups: few large, intermediate number of intermediate size, many small

• Further ingredients:
  random endowments, acceptance set defined by entropic risk measure

• Comparative statics: varying fraction in illiquid asset
Case Study B (2)
Conclusion
Conclusion

(i) Multi-variate approach to systemic risk
   • Integrates macroprudential objectives and systemic risk measurement on the level of the firms
   • Applicable to general financial system models

(ii) Pragmatic approach for implementation in practice
   • Conservative orthant risk measures derived from systemic risk measures via “regulatory price” of capital
   • Includes previous contributions as special cases

(iii) Implementation
   • Combination of Monte Carlo simulation and grid search
   • Successfully implemented in case studies
References


