Agency Conflicts and Short- vs Long-Termism in Corporate Policies *

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Abstract

We develop an analysis of short- and long-termism in corporate policies based on agency conflicts within firms. To do so, we build a dynamic agency model in which the agent controls both current earnings via short-term effort and firm growth via long-term effort. Under the optimal contract, agency conflicts can induce both over- and underinvestment in short- and long-term efforts, leading to short- or long-termism in corporate policies. The paper shows how firm characteristics shape the optimal contract and the horizon of corporate policies, thereby generating a number of novel empirical predictions on the optimality of short- vs. long-termism.

Keywords: Optimal short- and long-termism; Agency conflicts; Multi-tasking.

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Corporate short-termism has been the subject of considerable debate among academics, with much of the discussion focusing on whether it destroys value. A view commonly expressed in this literature is that stock market pressure leads management to take actions that can deliver immediate returns to shareholders at the expense of long-term value creation. Consistent with this view, Asker, Farre-Mensa, and Ljungqvist (2015) show that closely held firms tend to invest more than similar publicly listed companies. Bernstein (2015) finds that when firms go public their best inventors tend to leave and those who remain produce fewer patents. Gutierrez and Philippon (2017) find that the more companies are held by institutional investors, the less they tend to invest. These studies suggest that publicly held companies may focus on short-term goals at the expense of long-term investments. By contrast, Kaplan (2017) and Roe (2018) find that the presumed short-term orientation of managers does not show up in corporate profits, price-earnings ratios, or R&D. Gianetti and Yu (2018) find that following large permanent negative shocks, firms with more short-term institutional investors suffer smaller drops in sales, investment and employment and have better long-term performance than similar firms affected by the shocks.

While existing models provide a rich intuition as to why stock market pressure could lead firms or managers to engage in short-termism, they have not been entirely successful at explaining why short-termism does not always lead to poor outcomes. In this paper, we attempt to provide an answer to this question through the lens of agency theory. To do so, we develop a dynamic agency model in which the agent can exert unobservable effort to affect both current earnings and firm growth (i.e. future earnings). With this model, we show that the same firm can find it optimal at times to be short-termist—i.e. favor earnings—and at other times to be long-termist—i.e. favor growth. Our findings are generally consistent with the views expressed in the Economist, 1 that “long-termism and short-termism both have their virtues and vices—and these depend on context.”

We start our analysis by formulating a dynamic agency model in which an investor (the principal) hires a manager (the agent) to operate a firm. In this model, agency problems

1See “The Tyranny of the Long-Term,” The Economist, November 22, 2014.
arise because the manager can take hidden actions that affect both earnings and firm growth. As in He (2009) or Bolton, Wang, and Yang (2017), earnings are proportional to firm size, which is stochastic and governed by a (controlled) geometric Brownian Motion. In contrast with these models, earnings are also subject to moral hazard and to short-term shocks that do not necessarily affect (or correlate with) long-term prospects (i.e. shocks to firm value). The agent controls the drifts of both the earnings process and the firm size process through unobservable effort. As a result, the agent can exert effort for the short term to stimulate current earnings and/or for the long term to increase the rate at which the firm grows.

Exerting effort is costly, which requires the compensation contract to provide sufficient incentives to the agent. Under the optimal contract, the manager is thus punished (rewarded) if either cash-flow or firm growth is worse (better) than expected. Because the manager has limited liability, penalties accumulate until the termination of the contract, which occurs once the agent’s expected wealth falls to zero. Since termination is inefficient and generates deadweight costs, maintaining incentive compatibility is costly. Based on these tradeoffs, we derive an incentive compatible contract that maximizes the value that the principal derives from owning the firm. We then examine the implications of the optimal contract for long- and short-term effort provision and use the model to shed light on the relation between firm characteristics and the optimality of short- vs long-termism in corporate policies.

Our theory of short- and long-termism differs from existing contributions in three important respects. First, unlike most dynamic agency models, which generally focus either on short- or long-term agency conflicts, we consider a multi-tasking framework with both long- and short-term agency conflicts. Second, unlike most models on short-termism, we consider dynamics and do not assume that focusing either on the short or the long term is optimal. Third, we endogenize both long- and short-term efforts and relate them to firm characteristics. These unique features allow us to generate a rich set of testable predictions about firms’ optimal investment rates and the horizon of corporate policies.

We highlight our main findings. Considering first effort provision, we depart substantially from contributions with one-dimensional moral hazard, such as He (2009), DeMarzo and
Sannikov (2006) or DeMarzo, Fishman, He, and Wang (2012), in that first-best effort is not always implemented. In our model, the principal optimally balances the costs and benefits of incentivizing the manager over the short or the long term. This can lead the optimal contract to distort policies toward short- or long-term effort depending on firm characteristics and financial slack. Our results are thus in line with the views recently expressed in Harvard Business Review that companies “feel they need to strike the right “balance” between the short term and long term.” and in the Financial Times that “while there is little doubt that too much short-termism has negative effects, one should not assume that […] extreme long-termism is always for the best.”

Considering next the relation between firm performance and short-termism, our model predicts that firms with a high risk of liquidation—i.e. firms that perform worse and have little financial slack—should find it optimal to focus on the short term (i.e. current earnings) while firms with a low risk of liquidation—i.e. cash rich firms that perform well—should find it optimal to focus on the long term (i.e. asset growth). Interestingly, a recent study by Barton, Manyika, and Williamson (2017) finds using a data set of 615 large- and mid-cap US publicly listed companies from 2001 to 2015 that “the long-term focused companies surpassed their short-term focused peers on several important financial measures.” While our model does indeed predict that firm performance should be positively related to the corporate horizon, it in fact suggests the reverse causality.

More generally, we find that firm characteristics have first order effects on incentives and effort provision and, as a result, on the horizon of corporate policies. The model predicts for example that an increase in the rate of return on assets should favor short-termism while

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2In our model, investment behavior is related to incentives and compensation. Recent empirical studies by Edmans, Fang, and Huang (2018) and Ladika and Sautner (2018) show that short-term stock price concerns, such as vesting equity, can induce CEOs to take value-reducing actions, thereby suggesting that CEO incentives affect the horizon of corporate policies.

3See “Long-termism is just as bad as short-termism,” September 25, 2014.

4See “Cultish long-termism can hobble investors,” September 12, 2017.

5Interestingly, this causality issue is already discussed in The Economist, Schumpeter’s article “Corporate short-termism is a frustratingly slippery idea” who writes: “Do short-term firms become weak or do weak firms rationally adopt strategies that might be judged short term?” Similarly, Barton et al. (2017) write in their own study “one caveat: we’ve uncovered a correlation between managing for the long term and better financial performance; we haven’t shown that such management caused that superior performance.”
an increase in the growth rate of assets should favor long-termism. When either rate is sufficiently high, the optimal contract can end up always distorting the corporate horizon towards short-termism or long-termism. The model also predicts that a positive correlation between short-run shocks to earnings and long-run shocks to firm value, by increasing risk, leads to externalities between effort choices with higher correlation leading to decreased long-term effort. Another prediction of the model is that an increase in short-run volatility favors long-termism while an increase in long-run volatility favors short-termism (via their impact on the cost of incentive provision). Here again, when either volatility is sufficiently high, the optimal contract can end up always promoting short-termism or long-termism. Our results therefore suggest that the nature of the risks facing firm is key in determining the corporate horizon. Importantly, our finding that short-termism is more likely to be optimal when the volatility of the firm value process is high, the firm is financially weak, and the correlation between shocks to earnings and firm value is positive, i.e. when stock return volatility is high, is consistent with the evidence in Brochet, Loumioti, and Serafeim (2012, 2013). The other predictions of the model on the optimality of short- vs long-termism are new.

As in prior contributions, incentives are provided in the optimal contract by making the agent’s compensation contingent on firm performance. In prior dynamic contracting models, the optimal contract generates just enough incentives to motivate the agent to exert effort because incentive provision comes with the threat of termination and is therefore costly to implement. A distinctive feature of our model is that the optimal contract introduces (additional) exposure to permanent shocks that is not needed to incentivize effort. To understand this feature, note that a positive permanent shock makes liquidation more inefficient. As a result, the agent’s promised wealth under the optimal contract increases in response to a positive shock in order to reduce the likelihood of termination.

While our model shares this feature with the studies of Hoffmann and Pfeil (2010) and DeMarzo et al. (2012), the principal additionally needs to incentivize the manager to exert long-term effort in our framework with multi-tasking. This generates the distinct prediction that extra pay-for-performance is introduced and the manager’s wealth is fully exposed to
permanent shocks only when her stake in the firm is large enough. Notably, when her stake is low, the extra pay-for-performance effect is shut down and the incentive compatibility constraint is binding. In other words, positive permanent shocks lead to additional pay-for-performance and negative permanent shocks eventually eliminate this extra sensitivity to performance implied by the optimal contract. Our model therefore provides a rationale for the asymmetry of pay-for-performance observed in the executive compensation data (see e.g. Garvey and Milbourn (2006) and Francis, Iftekhar, Kose, and Zenu (2013)).

Another set of unique predictions from our model relates to the determinants of asymmetric pay-for-performance in executive compensation. We find for example that an increase in risk, i.e. in the volatility of earnings or in the volatility of firm value, should encourage the use of asymmetric pay-for-performance. We also find that an increase in the return on or growth rate of assets should lead to a greater use of asymmetric pay-for-performance. Our finding that high growth should favor asymmetric pay-for-performance is consistent with the evidence in Francis et al. (2013) that “asymmetric benchmarking of pay is more prevalent in high growth industries.” To the best of our knowledge, all the other predictions are novel.

Our paper relates to the literature on short-termism. Influential contributions in this literature include Stein (1989) or Bolton, Scheinkman, and Xiong (2006) in which stock market pressure leads managers to boost short-term earnings at the expense of long-term value. In related work, Thakor (2018) builds a model in which short-termism is efficient as it limits managerial rent extraction and leads to a better allocation of managers to projects. Narayanan (1985) develops a model in which short-term projects privately benefit managers by enhancing reputation and increasing wages. Von Thadden (1995) studies a dynamic model of financial contracting in which the fear of early project termination by outsiders leads to short-term biases of investment. Aghion and Stein (2008) analyze a multi-tasking model in which firms can improve sales growth or margins but also need to incorporate the stock market’s expectations in their policy choices. A key difference with these models is that in our setup there is no intrinsic conflict between short- and long-term effort.\(^6\)

\(^6\)Darrough (1987) shows that the equilibrium found by Narayanan vanishes if shareholders provide an appropriate incentive scheme. Jeon (1991) shows that the behavior identified by Stein does not prevail in
In related research, Zhu (2018) develops a model of persistent moral hazard in which the agent can choose between a short- and a long-term action and characterizes the contract that implements the long-term action. Hoffmann and Pfeil (2018) build a model in which the agent privately observe cash flows that he can divert and/or invest to increase the likelihood of adoption of future technologies. Marinovic and Varas (2017) develop a model in which managers can increase short-term performance at the expense of firm value. Hackbarth, Rivera, and Wong (2018) use a dynamic agency model to show that shareholder-debtholder conflicts may make short-termism optimal for shareholders. None of these models addresses the issue of short- vs long-termism in corporate policies. In addition, these models do not generate optimal long-termism or asymmetric pay-for-performance. Lastly, our modeling of cash flows with permanent and transitory shocks is similar to that in Décamps, Gryglewicz, Morellec, and Villeneuve (2017). Their model does not feature agency conflicts.

Our paper is more generally related to the growing literature on dynamic agency conflicts. Most contributions in this literature study agency conflicts over the short run, using a stationary environment characterized by identically and independently distributed cash flow shocks; see for example DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), Sannikov (2008), Malenko (2017), Miao and Rivera (2016), or Szydlowski (2016). In these models, the manager can affect current but not future firm performance. In contrast, He (2009) and He (2011) focus on agency conflicts over the long run by considering a framework in which the manager can affect firm growth. In these last two models, earnings are not subject to short-term moral hazard. Our model combines both strands of the literature in a unified framework and provides the first analysis of the effects of agency conflicts on short- vs long-termism in corporate policies.

Section 1 presents the model and its solution. Section 2 analyzes the implications of the model for optimal effort and the horizon of corporate policies. Section 3 analyzes optimal incentives and asymmetric pay-for-performance. Section 4 concludes. Technical developments are gathered in the Appendix.

steady state if stock prices take into account that the strategic behavior of the manager.
1 The Model

1.1 Assumptions

Throughout the paper, time is continuous and uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$ with the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions. We consider a principal-agent model in which the risk-neutral owner of a firm (the principal) hires a risk-neutral manager (the agent) to operate the firm’s assets. The owner of the firm has constant discount rate denoted by $r \geq 0$.

The firm employs physical capital for production and we denote by $K_t$ the level of the capital stock at time $t \geq 0$. Earnings are proportional to the capital stock $K_t$ (i.e. the firm employs an “$AK$” technology) and depend on the agent’s unobservable effort choices. Earnings are also subject to permanent (long-term) and transitory (short-term) shocks. Permanent shocks change the long-term prospects of the firm and influence cash flows permanently by affecting firm size. Notably, following He (2009) and Bolton, Wang, and Yang (2017), we consider that the firm’s capital stock (firm size) $\{K\} = \{K_t\}_{t \geq 0}$ evolves according to the controlled geometric Brownian motion process:

\[ dK_t = (\ell_0 \mu - \delta) K_t dt + \sigma_K K_t dZ^K_t, \quad (1) \]

where $\mu > 0$ is a constant, $\delta > 0$ is the rate of depreciation, $\sigma_K > 0$ is a constant volatility parameter, $\ell_0 \in [0, \ell_{max}]$ with $\ell_{max} < \frac{r + \delta}{\mu}$ is the manager’s long-term effort choice, and $\{Z^K\} = \{Z^K_t\}_{t \geq 0}$ is a standard Brownian motion. In addition to these permanent shocks, cash-flows are subject to short-term shocks that do not necessarily affect long-term prospects. Specifically, cash-flows $dX_t$ are proportional to $K_t$ but uncertain and governed by:

\[ dX_t = K_t dA_t = K_t \left( s_t \alpha dt + \sigma_X dZ^X_t \right), \quad (2) \]

\[ This specification for capital accumulation and revenue in which capital dynamics are governed by a controlled geometric Brownian motion has been used productively in asset pricing (e.g. Cox, Ingersoll, and Ross (1985) or Kogan (2004)), corporate finance (e.g. Abel and Eberly (2011) or Bolton, Wang, and Yang (2017)), or macroeconomics (e.g. Gertler and Kiyotaki (2010) or Brunnermeier and Sannikov (2014)). \]

where $\alpha$ and $\sigma_X$ are strictly positive constants, $s_t \in [0, s_{\text{max}}]$ is the manager’s short-term effort choice, and $\{Z^X\} = \{Z^X_t\}_{t \geq 0}$ is a standard Brownian motion. In the following, $\{Z^X\}$ is allowed to be correlated with $\{Z^K\}$ with correlation coefficient $\rho$, in that:\footnote{In general, the correlation coefficient $\rho$ between short-term and permanent cash flow shocks can be positive or negative. Considering for example the automobile industry, there is a general tendency for buyers of moving away from diesel cars towards electric cars. In the case of Volkswagen, this negative permanent demand shock on diesel cars has been compounded by the diesel gate, implying a positive correlation between short-run and long-run cash flow shocks. In the case of Tesla Motors, the positive long-run demand shock on electric cars has been dampened by negative shocks on the supply chain (notably for Model 3), implying a negative correlation between short-run and long-run cash flow shocks. Additional examples of a negative correlation include decisions to invest in R&D or to sell assets. When the firm sells assets today, it experiences a positive cash flow shock. However, it also decreases permanently future cash flows. Examples of positive correlation include price changes due to the exhaustion of existing supply of a commodity or improving technology for the production and discovery of a commodity. Chang, Dasgupta, Wong, and Yao (2014) estimate that for firms listed in the Compustat Industrial Annual files between 1971 and 2011, the correlation between short-term and permanent cash flow shocks is negative.}

$$\mathbb{E}[dZ^K_t dZ^X_t] = \rho dt, \text{ with } \rho \in [-1, 1].$$

While $\{K\}$ and $\{X\}$ are observable and contractible, the manager’s effort choices $\{s\}$ and $\{\ell\}$ are not. There are therefore two sources of agency conflicts as the principal cannot disentangle the manager’s actions from the Brownian shocks in equations (1) and (2).

As shown by equations (1) and (2), long-term effort improves firm growth while short-term effort improves current earnings.\footnote{Alternatively, short-term effort choice measures how effectively the manager is using the firm’s capital while long-term effort choice can be thought of as how much effort the manager puts in the selection/ adoption of investment projects. We thank Lukas Schmid for suggesting this interpretation.} Exerting effort however is costly for the manager. Notably, the cost of effort is convex in the effort choice and given by:

$$C(s_t, \ell_t) = \frac{1}{2} K_t \left( \lambda_s s_t^2 \alpha + \lambda_\ell \ell_t^2 \mu + 2\phi \alpha s_t \ell_t \right),$$

where $\phi$ is a constant and $\lambda_s$ and $\lambda_\ell$ are positive constants.\footnote{Our specification implies that the standard convex adjustment cost function for capital investment used in the neoclassical investment literature can be seen as a cost of incentivizing the manager. Our model can easily incorporate additional costs unrelated to managerial effort.} The model allows for environments in which short- and long-term efforts are substitutes ($\phi \geq 0$) or complements ($\phi \leq 0$). The above specification also implies that the cost of effort increases with firm size as administering a larger firm requires more effort; a similar assumption is made for example in He
(2009). An important difference between the two models is that while the return on invested capital is constant in He (2009), in that $$\frac{dX_t}{K_t} = \alpha dt$$, this is not the case in our model in which it is subject to short-term moral hazard and short-run shocks, in that $$\frac{dX_t}{K_t} = s_t \alpha dt + \sigma_X dZ_t^X$$.

As in DeMarzo and Sannikov (2006), Biais et al. (2007), or DeMarzo et al. (2012), the agent is more impatient than the principal and has a discount rate $$\gamma > r$$. As a result, the principal cannot indefinitely postpone payments to the agent. The agent possesses an outside option normalized to zero, has no wealth, and is protected by limited liability, which rules out negative wages.\(^{11}\) With no loss in generality, the manager cannot maintain a savings account.\(^ {12}\) Her employment starts at time $$t = 0$$ and is terminated at an endogenous stopping time $$\tau$$ at which point the firm is liquidated (this assumption is made for simplicity; our model only requires the firm to incur costs when terminating the contract of the manager). At the time of liquidation, the principal recovers a fraction $$R \geq 0$$ of assets, valued as $$RK_\tau$$. Liquidation is inefficient and generates deadweight losses. It is however necessary to incentivize the manager who is protected by limited liability.

To maximize firm value, the investor offers a contract to the agent at time $$t = 0$$ and commits to a compensation scheme $$\{C\}$$, recommended effort processes $$\{\hat{s}\}$$ and $$\{\hat{\ell}\}$$, and a termination time $$\tau$$. Because the agent has limited liability, the process $$\{C\}$$ is non-decreasing. We let $$\Pi \equiv (\{C\}, \{\hat{s}\}, \{\hat{\ell}\}, \tau)$$ represent the contract, where all elements are progressively measurable with respect to $$F$$, and assume certain regularity conditions that are gathered in the appendix. We will refer to these conditions collectively as the “usual regularity condition.” We call a contract incentive compatible if $$a_t = \hat{a}_t$$ for $$a \in \{s, \ell\}$$ and all $$\tau \geq t \geq 0$$. Since we focus without loss of generality on incentive compatible contracts, we will not formally distinguish actual and prescribed effort levels in the main text.

Before proceeding, note that the modeling of cash flows in (1) and (2) nests two popular frameworks as special cases. When $$\sigma_K = 0$$, we obtain the stationary environment of the

\(^{11}\)As in Albuquerque and Hopenhayn (2004) or Rampini and Viswanathan (2013), we could assume that the manager is able to appropriate a fraction of firm value so that the manager has reservation value $$\theta K_t$$, where $$\theta \geq 0$$ is a constant parameter. The entire analysis can be conducted by replacing 0 with $$\theta$$.

\(^{12}\)As in DeMarzo and Sannikov (2006), it is possible to show that savings cannot be part of the optimal contract. Indeed, because $$\gamma > r$$, it is cheaper for the principal to save and make direct payments to the agent. As a consequence, the manager consumes all payments she receives immediately.
dynamic agency models of DeMarzo and Sannikov (2006) and DeMarzo et al. (2012). Models analyzing the effects of financing frictions on firm decisions, such as Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011) or Hugonnier, Malamud, and Morellec (2015), also employ this cash-flow environment. Since there is no noise to hide the long-term effort choice, the long-term agency conflict is irrelevant in that case. By contrast, when $\sigma_X = 0$, there are no short-term shocks and we obtain the cash-flow environment used in the dynamic capital structure (e.g. Leland (1994) or Strebulaev (2007)) and real options literatures (e.g. Carlson, Fisher, and Giammarino (2006) or Morellec and Schürhoff (2011)) as well as in the dynamic agency models of He (2009, 2011). Since there is no noise to hide short-term effort choice, the short-term agency conflict is irrelevant in that case.

1.2 The Contracting Problem

Consider a contract $\Pi$, fix the effort processes $\{s\}$ and $\{\ell\}$, and define the agent’s expected payoff at time $t \geq 0$, i.e. her continuation value, as

$$ W_t(\Pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-\gamma(u-t)} dC_u - \int_t^\tau e^{-\gamma(u-t)} \frac{1}{2} K_u \left( \lambda_s s_u^2 \alpha + \lambda_\ell \ell_u^2 \mu + 2 \phi \alpha \mu s_u \ell_u \right) du \right]. $$

$W_t = W_t(\Pi)$ equals the promised value the agent gets if she follows the recommended path from time $t \geq 0$ onwards, net of the cost of implementing the recommended effort level. $W_0 = W_0(\Pi)$ is the agent’s expected payoff at inception at time $t = 0$.

The investor receives the firm cash flows and pays the compensation to the manager. As a result, given the contract $\Pi$, the investor’s expected payoff can be written as:

$$ P(W_0, K_0) = \mathbb{E} \left[ \int_0^\tau e^{-rt} dX_t + e^{-rt} RK_\tau - \int_0^\tau e^{-rt} dC_t \right]. $$

The objective of the principal is therefore to maximize the present value of the firm cash flows plus termination value net of the agent’s compensation, where we make the usual assumption that the principal possesses full bargaining power (see DeMarzo et al. (2012) for alternative specifications). Denote the set of incentive compatible contracts by $\mathbb{IC}$. The
investor’s optimization problem reads

$$\max_{\Pi \in \mathcal{C}} P(W_0, K_0) \text{ such that } W_0(\Pi) = W_0 \geq 0, \text{ and } P(W_0, K_0) \geq RK_0.$$  \hfill (3) \hfill

We denote the solution to this problem as $\Pi^* \equiv (C^*, s^*, \ell^*, \tau^*)$.

### 1.3 Benchmark Case: Effort Choice under First-Best

We start by deriving the value of the firm and the optimal effort levels absent agency conflicts. This is the case when there is no noise to hide the agent’s action, so that $\sigma_X = \sigma_K = 0$. Throughout the paper, we refer to this case as the first-best (FB) outcome. Given the stationarity of the firm’s optimization problem, the choice of $s$ and $\ell$ is time-invariant absent agency conflicts and the first-best firm value reads

$$P^{FB}(K_0) = \max_{(s, \ell) \in [0, s_{\text{max}}] \times [0, \ell_{\text{max}}]} \frac{K_0}{r + \delta - \mu \ell} \left[ \alpha s - \frac{1}{2} \left( \lambda_s \alpha s^2 + \lambda_\ell \mu \ell^2 + 2 \phi \alpha \mu s \ell \right) \right],$$

where the short- and long-term effort choice $\{s^{FB}, \ell^{FB}\}$ maximize the value of the firm. This leads to the following result:

**Proposition 1** (First-best firm value and effort choices). Assume the bounds $i_{\text{max}}$ for $i \in \{s, \ell\}$ are such that the first-best solution is interior. Then the following holds:

1. First-best short-term effort satisfies: $s^{FB} = \frac{1 - \phi \mu \ell}{\lambda_s}$.

2. First-best long-term effort satisfies: $\ell^{FB} = \frac{1}{\mu} \left[ r + \delta - \sqrt{(r + \delta)^2 - \frac{\mu \alpha \alpha (1 - 2(r + \delta) \phi)}{\lambda_s \lambda_\ell - \phi^2 \alpha \mu}} \right]$.

### 1.4 Model Solution

We now solve the model with agency conflicts over the short and long term, that is, assuming $\sigma_K \lambda_\ell > 0$ and $\sigma_X \lambda_s > 0$. In our model, $\lambda_s$ ($\lambda_\ell$) reflects the potential gains for the agent of deviating from the recommended effort choice over the short term (long term). The volatility parameters $\sigma_X$ and $\sigma_K$ reflect the difficulty for the principal of detecting shirking. When
cash-flow and/or firm size are volatile, it becomes difficult for the principal to determine whether adverse outcomes are due to bad luck or to the agent shirking.

Denote by $W_t$ the continuation payoff of the agent at time $t \geq 0$ under the optimal contract. The contract relies on cash payments $dC_t$ and changes in the value of future payments $dW_t$. The manager continues within the firm if and only if promised future transfers exceed the value of her outside option. To compensate the agent for her time preference and effort cost, incremental compensation $dC_t + dW_t$ must equal $(\gamma W_t + C(s_t, \ell_t))dt$ on average:

$$
\mathbb{E}_t[dC_t + dW_t] = \left[ \gamma W_t + \frac{1}{2} \tilde{K}_t \left( \lambda_s s_t^2 \alpha + \lambda_\ell \ell_t^2 \mu + 2 \phi \alpha \mu \ell \right) \right] dt.
$$

(4)

Equation (4) corresponds to the ‘Promise Keeping Condition’ in the discrete time formulation of DeMarzo and Fishman (2007). While this condition determines how much the agent should earn on average, her compensation must also be sufficiently sensitive to firm performance, as captured by $dX_t$ and $dK_t$, to maintain incentive compatibility. By punishing (rewarding) the manager if either asset growth or cash-flow is worse (better) than expected—i.e. falls short of (exceeds) its expectation—incentive compatibility is maintained. Using the martingale representation theorem, this sensitivity of the manager’s incremental payoff to earnings and productivity changes can be formalized as follows (see Appendix B):

$$
dW_t = \gamma W_t dt + \frac{1}{2} K_t \left( \lambda_s s_t^2 \alpha + \lambda_\ell \ell_t^2 \mu + 2 \phi \alpha \mu \ell \right) dt - dC_t
+ \beta^s_t (dX_t - \alpha s_t K_t dt) + \beta^\ell_t (dK_t - (\mu \ell_t - \delta) K_t dt),
$$

(5)

where the sensitivities $\beta^s_t$ and $\beta^\ell_t$ are used to satisfy incentive compatibility conditions with respect to short- and long-term effort, respectively. To understand why such a compensation scheme may lead the agent to exert effort, suppose that the agent decides to marginally deviate from the recommended choice $\hat{s}_t$ over the time interval $[t, t + dt)$. By doing so, she saves cost $\alpha K_t(\lambda_s \hat{s}_t + \phi \mu \ell)dt$. However, as a consequence of her behavior, realized earnings fall below their expectation and thus her compensation is reduced by $\alpha K_t \beta^s_t dt$. Incentivizing the manager to exert effort therefore requires that $\beta^s_t = \lambda_s \hat{s}_t + \phi \mu \ell$ if $\hat{s}_t$ is interior and
\[ \beta^s_t \geq \lambda_s \hat{s}_t + \phi \mu \ell \] if \( \hat{s}_t = s_{\text{max}} \). Similarly, \[ \beta^f_t = \lambda_f \hat{l}_t + \phi \alpha s \] if \( \hat{l}_t \) is interior and \[ \beta^f_t \geq \lambda_f \hat{l}_t + \phi \alpha s \] if \( \hat{l}_t = l_{\text{max}} \). Both incentive compatibility constraints require that the agent has enough skin in the game, reflected by sufficient exposure to firm performance.

The investor’s value function in an optimal contract, given by \( P(W, K) \), is the highest expected payoff the investor may obtain given \( K \) and \( W \). While there are two state variables in our model, the scale invariance of the firm’s environment allows us to write \( P(W, K) = K p(w) \) and reduce the problem to a single state variable: \( w \equiv \frac{W}{K} \), the scaled promised payments to the agent as in He (2009) or DeMarzo, Fishman, He, and Wang (2012).

To characterize the optimal compensation policy and its effects on the investor’s (scaled) value function \( p(w) \), note that it is always possible to compensate the agent with cash so that it costs at most $1 to increase \( w \) by $1 and \( p'(w) \geq -1 \). That is, the marginal cost of delaying payouts can never exceed the cost of an immediate transfer. As shown by equation (5), deferring compensation increases the growth rate of \( w \) and produces direct benefits by lowering the risk of liquidation when \( w \) is close to zero. As a result, the optimal contract will set \( dc \equiv \frac{dC}{K} \) to zero for low values of \( w \). However, due to the relative impatience of the agent \( (\gamma > r) \), postponing payments is costly. Since the benefits of delaying payouts decrease while the cost increases with \( w \), we may conjecture that \( p(w) \) is concave and that there exists a threshold \( \bar{w} \) above which it is optimal to directly pay the manager, i.e. such that

\[ p'(\bar{w}) = -1 \text{ and } dc = \max\{0, w - \bar{w}\}, \tag{6} \]

where the optimal payout boundary is determined by the smooth-pasting condition:

\[ p''(\bar{w}) = 0. \tag{7} \]

Lastly, when \( w \) falls to zero, the contract is terminated and the firm is liquidated so that

\[ p(0) = R. \tag{8} \]
When \( w \in [0, \bar{w}] \), the agent’s compensation is deferred and \( dc = 0 \). The Hamilton-Jacobi-Bellman equation for the principal’s problem is then given by (see Appendix B):

\[
(r + \delta)p(w) = \max_{s,t,\beta^s,\beta^\ell} \left\{ \alpha s + p'(w)w(\gamma + \delta - \mu \ell) + \frac{p'(w)}{2} \left( \lambda_s \mu \ell^2 + 2 \phi \alpha \mu s \ell \right) + \mu \ell p(w) + \frac{p''(w)}{2} \left[ (\beta^s \sigma_X)^2 + \sigma_K^2 (\beta^\ell - w)^2 + 2 \rho \sigma X \sigma_K \beta^s (\beta^\ell - w) \right] \right\}
\]

subject to the incentive compatibility constraints on \( \beta^s \) and \( \beta^\ell \). Evaluating the above HJB-equation at the payout boundary \( \bar{w} \) and using equations (6) and (7) yields

\[
(r + \delta - \mu \ell)p(\bar{w}) + \bar{w}(\gamma + \delta - \mu \ell) = \alpha s - \frac{1}{2} \left( \lambda_s s^2 \alpha + \lambda_\ell \ell^2 \mu + 2 \phi \alpha \mu s \ell \right).
\]

Postponing cash payments by increasing \( \bar{w} \) reduces the risk of termination. Delaying compensation is efficient until the investor’s and manager’s required returns \( (r + \delta - \mu \ell)p(\bar{w}) + \bar{w}(\gamma + \delta - \mu \ell) \) exhaust the available expected net cash-flow \( \alpha s - \frac{1}{2} \left( \lambda_s s^2 \alpha + \lambda_\ell \ell^2 \mu + 2 \phi \alpha \mu s \ell \right) \). Note that the required total rate of return (of both the investor and the manager) is reduced by the growth rate \( \mu \ell - \delta \) since both benefit from the expanding firm size, leading to effective discount rates \( \gamma + \delta - \mu \ell \) and \( r + \delta - \mu \ell \). Consequently, the prospect of higher future payments makes the manager and the investor loosely speaking more patient.

Due to the scale invariance, i.e. \( P(W, K_0) = p(w)K_0 \), the investor’s maximization problem at \( t = 0 \) can now be rewritten as

\[
\max_{w_0 \in [0, \bar{w}]} p(w_0) K_0
\]

with unique solution \( w_0 = w^* \) satisfying

\[
p'(w^*) = 0.
\]

As a consequence, the principal initially promises the agent utility \( w^* K_0 \) and expects a payoff \( P(K_0 w^*, K_0) = p(w^*)K_0 \). For convenience, we normalize \( K_0 \) to unity in the following and
refer to $p(w^*)$ as expected payoff instead of scaled expected payoff. The following Proposition summarizes our results about the optimal contract. Its proof is deferred to Appendix B.

**Proposition 2** (Firm value and optimal compensation under agency). Suppose that $\lambda_j \sigma_j > 0$ for $j \in \{X, K\}$ and $i \in \{s, \ell\}$ and let $\Pi^* \equiv (C^*, s^*, \ell^*, \tau^*)$ denote the optimal contract solving problem (3). The following holds true:

1. There exist $\mathbb{F}$-progressive processes $\{\beta^s\}$ and $\{\beta^\ell\}$ such that the agent’s continuation utility $W_t$ evolves according to (5). The optimal contract is incentive compatible in that $\beta^s = \lambda_s s(w) + \phi \mu \ell(w)$ for $s(w) < s_{\text{max}}$ and $\beta^s \geq \lambda_s s(w) + \phi \mu \ell(w)$ for $s(w) = s_{\text{max}}$ and $\beta^\ell = \lambda_\ell \ell(w) + \phi \alpha s(w)$ for $\ell(w) < \ell_{\text{max}}$ and $\beta^\ell \geq \lambda_\ell \ell(w) + \phi \alpha s(w)$ for $\ell(w) = \ell_{\text{max}}$.

2. The investor’s value function $P(W, K)$ is proportional to firm size and satisfies $P(W, K) = K p(w)$, where $p(w)$ is the unique solution to equation (9) subject to (6), (7), and (8) on $[0, \bar{w}]$. For $w > \bar{w}$ the scaled value function satisfies $p(w) = p(\bar{w}) - (w - \bar{w})$. Scaled cash payments $d_c = \frac{dC}{K}$ reflect $w$ back to $\bar{w}$.

3. The function $p(w)$ is strictly concave on $[0, \bar{w})$.

### 2 Model Analysis

This section examines the features of the optimal compensation policy and the implications of agency conflicts for long- and short-term effort choice. For clarity of exposition, we assume in the first subsection that the correlation $\rho$ between short- and long-run shocks and the externality parameter $\phi$ in the cost function are equal to zero. Section 2.2 analyzes the effects of non-zero correlation and cost externality on effort choice and incentive provision. Section 2.3 provides a quantitative analysis of the effects of short- and long-term agency conflicts and firm characteristics on investment and the horizon of corporate policies.
2.1 Incentive Provision and Effort Choice

We start by analyzing optimal effort choice $\{s(w), \ell(w)\}$ when $\rho = \phi = 0$. In our model with short- and long-term agency conflicts, the optimal effort choice $\{s(w), \ell(w)\}$ is determined by $s(w) = \min\{s_{\text{max}}, s^*(w)\}$ and $\ell(w) = \min\{\ell_{\text{max}}, \ell^*(w)\}$, where $s$ and $\ell$ can take values between $[0, s_{\text{max}}]$ and $[0, \ell_{\text{max}}]$. The value $s^*(w)$ is pinned down using the incentive-compatibility conditions and taking the first-order condition in equation (9). This leads to the following result:

**Proposition 3** (Optimal short-term effort). Optimal short-term effort is given by $s(w) = \min\{s_{\text{max}}, s^*(w)\}$. If short-term effort is interior, the marginal cost of effort equals its marginal benefit at the optimum and the optimal short-term effort is given by

$$s(w) = \alpha \frac{-p'(w)\lambda_s \alpha - p''(w)\lambda_s \sigma X^2}{\lambda_s \alpha - p''(w)\lambda_s \sigma X^2}.$$  \(11\)

When optimal short-term effort is interior, as determined by equation (11), the marginal benefit of short-term effort is simply the cash flow rate $\alpha$ and is at the first-best level. The direct (marginal) cost of effort for the manager is $\lambda_s \alpha$. Since the investor compensates the manager for this cost by increasing her continuation utility, the cost scales by $-p'(w)$ from the principal’s perspective. In the region where $p'(w) > 0$ (that is, for small $w$), this direct cost effect is negative as increasing $w$ benefits the investor by reducing the risk of inefficient liquidation. When the continuation payoff of the agent $w$ is large, the direct-cost effect is positive. At $w = \overline{w}$, where $p'(\overline{w}) = -1$, the effect is exactly at the first best-level. The second term in the denominator of equation (11), the volatility cost of effort, relates to the concavity of the investor’s value function. Because of this concavity, it is costly to increase effort as this requires additional incentives $\beta^s$, which increases the volatility of $w$. The effect is strongest when $p''(w)$ is the largest in absolute value and disappears at $w = \overline{w}$ where $p''(\overline{w}) = 0$. The top two panels of Figure 1 show how the direct cost of effort and the volatility cost of effort
Figure 1: The determinants of the optimal short- and long-term efforts. The solid curves represent the four effects in equations (11) and (12) in the typical case in which \( p'(0) > 0 \). The dashed lines represent the effects in the first-best problem. The direct-benefit effect in the choice of \( s \) (not presented) is constant in \( w \) and at the first-best level.

vary with the (scaled) promised payments to the agent \( w \).

Similarly, the first-order condition with respect to long-term effort in (9) yields:

**Proposition 4** (Optimal long-term effort). Optimal long-term effort is given by

\[
\ell(w) = \min \left\{ \ell_{\text{max}}, \ell^*(w) \right\}.
\]

If long-term effort is interior, the marginal cost of effort equals its marginal benefit at the optimum and the optimal long-term effort is given by

\[
\ell(w) = \frac{\mu (p(w) - p'(w)w)}{-p'(w)\lambda \mu} - \frac{p''(w)w\lambda K \sigma^2}{-p'(w)(\lambda K \sigma^2)^2},
\]

As shown by Propositions 3 and 4, there are two structural differences between equations
(11) for short-term effort $s(w)$ and (12) for long-term effort $\ell(w)$. A first difference is that optimal $\ell(w)$ has an additional benefit of effort compared to $s(w)$: the scaling effect. Since $p''(w) \leq 0$, this scaling effect unambiguously increases long-term effort. To understand the source of this effect, note that a positive permanent shock $dZ^K > 0$ has two opposing consequences. First, the agent is rewarded via the sensitivity $\beta^\ell$ and is promised higher future payments $W$. This increases $w = \frac{W}{K}$ by $\beta^\ell \sigma_K dZ^K$, which equals $\lambda_\ell \ell(\ell(w)) \sigma_K dZ^K$ when the incentive-compatibility constraint is binding and the long-term effort choice is interior. Second, firm size $K$ grows more than expected, thereby diluting the agent’s stake $w = \frac{W}{K}$ by $-w\sigma_K dZ^K$ and increasing the likelihood of inefficient termination. Because these two effects move $w$ in the opposite direction, the scaling/dilution effect reduces the volatility of the continuation payoff of the manager (and therefore the cost of incentivizing the manager), leading to an increase in optimal long-term effort.\footnote{We show below that when the incentive compatibility constraint is not binding because the manager’s continuation payoff is large (section 3), the principal optimally sets $\beta^\ell = w$ such that $(\beta^\ell - w) \sigma_K dZ^K = 0$. In such instances, the agent is incentivized for the long term without raising the threat of termination and incentive provision optimally involves asymmetric pay-for-performance.}

A second difference between optimal short- and long-term efforts is that the direct benefit of effort is constant in (11) while it is scaled by $p(w) - p'(w)w$ in (12). Two observations follow. First, $p(w) - p'(w)w$ is time-varying and increases with $w$. Second, $p(w) - p'(w)w$ is strictly less than $p^{FB}$, reducing the direct benefit of long-term effort compared to first best ($p^{FB}$ scales the benefit of effort in the first-best problem). The bottom two panels of Figure 1 plot the two effects specific to the long-term effort as functions of $w$.

Using the results in Propositions 3 and 4, it is possible to get a simple characterization of effort choices at the payout boundary. Indeed, when $w = \overline{w}$, the total marginal cost of short-term effort under moral hazard (the denominator of (11)) is exactly equal to the marginal cost of effort under the first-best. As the benefit of short-term effort is always at the first-best level, the optimal short-term effort under moral hazard is at the first-best level at $w = \overline{w}$,
in that \( s(\bar{w}) = \frac{1}{\lambda_s} = s^{FB} \). While short-term agency conflicts are locally resolved at \( w = \bar{w} \), this is not the case of long-term agency conflicts. At \( w = \bar{w} \) and with interior long-term effort, the scaling effect is absent since \( p''(\bar{w}) = 0 \). In this case, the weaker direct-benefit effect under agency conflicts generates underinvestment in long-term effort. To understand this result, note that the benefit of investing in long-term effort when \( w = \bar{w} \) is proportional to total firm value \( p(\bar{w}) + \bar{w} \) (since \( p'(\bar{w}) = -1 \)). This benefit is therefore strictly smaller than the benefit in the first-best firm, which is proportional to \( p^{FB} \). Moreover, at \( w = \bar{w} \), as in the case with the short-term effort, the marginal cost of long-term effort is equal to the marginal cost of effort under the first-best. This leads to under-investment in long-term effort in that \( \ell(\bar{w}) < \ell^{FB} \). The following Proposition summarizes these results:

**Proposition 5.** Assume that the bounds \((0, i_{\text{max}})\) for \( i \in \{s, \ell\} \) are such that first-best effort levels are interior. At the payout boundary, the firm always optimally invests in the short term in that \( s(\bar{w}) = s^{FB} \) and underinvests for the long term in that \( \ell(\bar{w}) < \ell^{FB} \).

Agency frictions generate additional costs and decrease benefits of effort and thus can lead to under-investment in short- and/or long-term effort for \( w < \bar{w} \). Interestingly, optimal over-investment can also occur for \( w < \bar{w} \). This can happen because the firm prioritizes long-term (resp. short-term) effort at the expense of short-term (resp. short-term) effort. This can also happen when there are externalities in effort choices. Section 2.3 presents a quantitative analysis of the occurrence of long- and short-termism under agency frictions.

### 2.2 Externalities in Effort Choices and Incentive Provision

Short- and long-term efforts in our model are indirectly related via the principal’s decision on how to optimally provide the manager’s continuation utility. Thus, the optimal levels of \( s(w) \) and \( \ell(w) \) in Propositions 3 and 4 depend on each other via the level and shape of principal’s value function \( p(w) \). We now consider direct externalities between effort choices and incentive provision that arise either from the interaction parameter \( \phi \) in the effort cost function or from correlation \( \rho \) between shocks. To identify the first-order channels through
which $\rho$ (resp. $\phi$) affects effort choice, we assume in the following that $\phi = 0$ (resp. $\rho = 0$). This simplification allows for a more intuitive interpretation of the main effects at work.

When $\phi = 0$ and $\rho \neq 0$, optimal short- and long-term efforts—if interior—are given by:

$$s(w) = \alpha + \frac{p''(w)\rho \sigma_X \sigma_K \lambda_s (\lambda_\ell \ell(w) - w)}{-p'(w)\lambda_s \alpha - p''(w)(\lambda_s \sigma_X)^2}$$  \hspace{1cm} (13)

and

$$\ell(w) = \mu (p(w) - p'(w)w) + \frac{p''(w)\rho \sigma_X \sigma_K \lambda_\ell \lambda_s s(w) - p''(w)w \lambda_\ell \sigma_\ell^2 K}{-p'(w)\lambda_\ell \mu - p''(w)(\lambda_\ell \sigma_K)^2}.$$ \hspace{1cm} (14)

Compared to equations (11) and (12), new terms appear that affect both optimal effort levels and incentives. As $s(w)$ depends on $\ell(w)$ and vice versa, there are direct externalities between efforts and incentives. With non-zero correlation, the agent’s exposure $\beta^s(w) = \lambda_s s(w)$ to transitory cash flow shocks poses an externality on the choice of $\beta^\ell(w) = \lambda_\ell \ell(w)$. Intuitively, when the two sources of risk are positively correlated, exposing the manager’s continuation payoff $W$ to both transitory and permanent shocks creates excess volatility and is therefore costly. As a result, leaving $\beta^s$ unchanged, a higher correlation between short- and long-run shocks $\rho$ makes it optimal for the principal to decrease $\beta^\ell$—and thus the volatility of $w$—to limit the risk of inefficient liquidation. With interior effort levels, the sensitivities $\beta^s$ and $\beta^\ell$ directly relate to $s(w)$ and $\ell(w)$ and there are therefore first-order externalities of short-term effort on long-term effort and vice versa. This externality of $s(w)$ on $\ell(w)$ is negative (positive) if $\rho > 0$ ($\rho < 0$) and is captured in equation (14) by

$$p''(w)\rho \sigma_X \sigma_K \lambda_\ell \beta^s = p''(w)\rho \sigma_X \sigma_K \lambda_\ell \lambda_s s(w).$$

The magnitude of the externality scales with the curvature of the value function $p''(w)$ and is therefore relatively weaker once the firm has accumulated sufficient financial slack (i.e. once $w$ is sufficiently high). Thus optimal short- and long-term efforts are most closely interrelated
when \( w \) is low and the likelihood of inefficient liquidation is high.

Equation (13) demonstrates that the choice of long-term effort \( \ell(w) \) also feeds back in the choice of short-term effort \( s(w) \). However, the correlation effect in the numerator of \( s(w) \) in (13) has an additional term and therefore consists of two separate components:

\[
p''(w)\rho\sigma_X\sigma_K\lambda_s(\lambda_\ell(w) - w) = p''(w)\rho\sigma_X\sigma_K\lambda_\ell(w) - p''(w)\rho\sigma_X\sigma_K\lambda_s w.
\]

This decomposition shows that when the correlation between shocks is non-zero, incentives for the short-run are also used to counteract the dilution in the manager’s stake arising upon positive permanent shocks \( dZ^K > 0 \). As discussed in section 2.1, the principal counteracts this dilution effect by tying the manager’s compensation to permanent shocks when there is no correlation. When the two sources of cash-flow risk are correlated, this can be done by changing either long-term or short-term incentives. This leads to lower short-term effort and incentives when \( \rho < 0 \) and to higher short-term effort and incentives when \( \rho > 0 \). The overall pattern is thus that negative \( \rho \) promotes long-term effort at the expense of short-term effort and positive \( \rho \) promotes short-term effort at the expense of long-term effort.

Next, we turn to the effects of cost externalities with non-zero \( \phi \) and zero correlation. In this case, the optimal effort levels are given by

\[
s(w) = \alpha + \frac{\phi p'(w)\alpha\mu\ell(w) + \sigma^2_X\mu\lambda_\ell(w) + \sigma^2_K\lambda_\ell(w) - w)}{-p'(w)\lambda_s\alpha - p''(w)\lambda_s\sigma_X^2 - p''(w)(\phi\alpha\sigma_K)^2}
\]  

(15)
and

\[
\ell(w) = \frac{\mu (p(w) - p'(w)w) - p''(w)w\lambda \ell \sigma^2_K}{-p'(w)\lambda \ell \mu - p''(w)(\lambda \ell \sigma_K)^2 - p''(w)(\phi \mu \sigma_X)^2} + \frac{\phi p'(w)\alpha \mu s(w)}{-p'(w)\lambda \ell \mu - p''(w)(\lambda \ell \sigma_K)^2 - p''(w)(\phi \mu \sigma_X)^2}.
\]

The first effect in the numerators of (15) and (16) simply reflects the marginal cost externality scaled by \(-p'(w)\) as the investor compensates the manager for this cost by increasing her continuation utility. To interpret the ‘Incentive externality’ effect, which shows up in the numerator and the denominator of (15) and (16), suppose that the principal wants to marginally increase short-term effort \(s(w)\) and that \(\phi > 0\). Evidently, this requires a higher exposure to transitory shocks \(\beta^s = \lambda s(w) + \phi \mu \ell(w)\). Because providing additional short-run incentives is ceteris paribus more costly when \(\beta^s\) is large, a higher value of \(\phi\) exacerbates the volatility cost of effort the principal incurs. Likewise, raising short-term effort \(s(w)\) mandates also higher long-term incentives to implement a given level \(\ell(w) = \beta^\ell(w) - \phi \alpha s(w)\). This requires to raise the sensitivity of the agent’s stake \(w\) to capital shocks, which again is more costly for large values of \(\ell(w)\) and \(\beta^\ell(w)\). The effects outlined above are obviously reversed for \(\phi < 0\). Put differently, providing high powered incentives for the short-run alleviates (resp. exacerbates) agency conflicts over the long-term and vice versa, which leads to positive (resp. negative) incentive externalities if \(\phi < 0\) (resp. \(\phi > 0\)).

### 2.3 Short- vs long-termism in corporate policies

In sections 2.1 and 2.2, we showed that both short-term and long-term efforts can exceed the first-best levels leading to \textit{absolute} short- and long-termism. This section analyzes the patterns of under- and over-investment in effort relative to the first-best levels. We will investigate both absolute short- or long-termism, by comparing \(i(w)\) with \(i^{FB}\) for \(i \in \{s, \ell\}\), and relative short- or long-termism, by comparing \(\frac{\ell(w)}{s(w)}\) with \(\frac{i^{FB}}{s^{FB}}\) with ratios of these two.
quantities above (resp. below) 1 indicating long-termism (resp. short-termism). Since the optimal effort levels with agency frictions depend on the promised payments \(w\) to the agent (and the first-best levels do not), so will the investment patterns.

In our quantitative analysis, we use the following input parameter values. We set the discount rate parameters to \(r = 4.6\%\) and \(\gamma = 5\%\) and the depreciation rate to \(\delta = 12.5\%\), in line with DeMarzo et al. (2012). For convenience, we normalize first best effort levels to one, i.e. \(s^{FB} = \ell^{FB} = 1\), and set \(\mu = 13.2\%\) so that first best investment \(\mu\ell^{FB}\) is as in Bolton et al. (2017). This implies the cost parameter values \(\lambda_s = 1\) and \(\lambda_\ell = 1.19\). The volatility parameter of the long term shock is set to \(\sigma_K = 15\%\), in line with Kogan (2004), while the volatility parameter of the short-term shock is set to \(\sigma_X = 25\%\), in line with DeMarzo et al. (2012). The drift parameter for the profitability/productivity process is set to \(\alpha = 25\%\), implying that the (expected) return on assets is 4.7\% at first best, in line with the estimates in Morellec, Nikolov, and Schürhoff (2012) and Graham, Leary, and Roberts (2015). Finally, we assume the recovery value \(R = 20\%\) and an externality parameter \(\phi = 0\) in the cost function. In our analysis, we also investigate the effects of varying these parameters around their base case values.\(^{14}\)

We start by examining the effects of agency conflicts on the horizon of corporate policies. We do so by comparing the relative investment in efforts to the first best using the ratio \(R(w) \equiv \frac{\ell(w)/s(w)}{\ell^{FB}/s^{FB}}\). Figure 2 plots \(R(w)\) as a function of \(w\) for different values of the model parameters. The figure first shows that policies are distorted predominantly for low and intermediate values of \(w\). Second, it demonstrates that close to termination—i.e. when the firm is financially weak and has little financial slack as captured by \(w\)—the optimal contract generally distorts policies towards short-termism. For intermediate values of \(w\), the optimal contract generally distorts policies towards long-termism.

Figure 2 also demonstrates the central role of the risk parameters for the corporate horizon. For example, it shows that an increase in short-run volatility \(\sigma_X\) tends to favor

\(^{14}\)One important difference between our model and prior work is that effort is generally not at first best and varies between 0 and \(i_{\text{max}}\) for \(i = s, \ell\). This is why we base our calibration of the return on assets and of the growth rate of capital on the first best case.
Figure 2: Relative short-and long-termism. The parameters for our base case environment are \(\alpha = 0.25\), \(\mu = 0.132\), \(\sigma_X = 0.25\), \(\sigma_K = 0.15\), \(\lambda_s = 1\), \(\lambda_l = 1.19\), \(r = 0.046\), \(\gamma = 0.05\), \(\delta = 0.125\), \(\phi = 0\), \(s_{\text{max}} = 2\), \(\ell_{\text{max}} = 1.25\), and \(R = 0.2\). A value of the ratio below 1 indicates that policies are distorted towards short-termism. A value of the ratio above 1 indicates that policies are distorted towards long-termism.
Table 1: Relative short- and long-termism. The parameter values are such that $s^{FB} = \ell^{FB} = 1$. The parameters are $\alpha = 0.25$, $\mu = 0.132$, $\sigma_X = 0.25$, $\sigma_K = 0.2$, $\lambda_s = 1$, $\lambda_\ell = 1.19$, $r = 0.046$, $\gamma = 0.05$, $\delta = 0.125$, $\phi = 0$, $s_{\text{max}} = 2$, $\ell_{\text{max}} = 1.25$, and $R = 0.2$.

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</tbody>
</table>

long-termism while an increase in long-run volatility $\sigma_K$ tends to favor long-termism. When short-run risk is high (e.g. $\sigma_X = 0.3$), policies are always distorted towards long-termism. When short-run risk is low (e.g. $\sigma_X = 0.2$), policies are always distorted towards short-termism. Symmetrically, when long-run risk is high (e.g. $\sigma_K = 0.2$), policies are always distorted towards short-termism. When long-run risk is low (e.g. $\sigma_K = 0.13$), policies are always distorted towards long-termism. The figure also illustrates the importance of the correlation between shocks. A negative correlation leads to more long-termism in corporate policies, consistent with the discussion in section 2.2.

To better understand the quantitative impact of the risk parameters on the corporate horizon, Table 1 reports the ratio $R(w)$ for different parameter combinations when $w = 0$ and $w = w^*$ (we focus on these two values as Figure 2 shows that this is where the distortions in corporate policies are the largest). The table shows that the effects of the parameters can be quantitatively very large, leading to extreme long-termism (with values of the ratio $R(w)$ above 20) or to extreme short-termism (with values of the ratio $R(w)$ equal to zero).

Consistent with economic intuition, the principal puts more emphasis on long-run effort
(i.e., asset growth) when it is less costly to incentivize (i.e., when the volatility $\sigma_K$ of permanent shocks is low) and short-run effort is more costly to incentivize (i.e., when the volatility $\sigma_X$ of transitory shocks is low). Figure 2 and Table 1 also show that optimal long-termism primarily arises when $\rho$ is negative and the manager’s continuation payoff is close to $w^*$ solving equation (10). Our results therefore suggest that the nature of the risks facing firm is key in determining the corporate horizon. Notably, our analysis suggests that even in high growth industries firms may not want to invest in long-term assets (or stimulate long-run effort) if they face shocks that are mostly of permanent nature, in that $\sigma_K$ is high and/or $\sigma_X$ is low. Therefore, our model offers a potential explanation for the puzzling empirical evidence that in recent years capital is not allocated to the industries with the best growth opportunities (as recently shown by Lee, Shin, and Stulz (2018)) and that firms tend to underinvest relative to their growth opportunities (Gutierrez and Philippon (2017)).

Another result illustrated by Figure 2 and Table 1 is that short-termism arises when $\sigma_X$ is low, $\sigma_K$ is high, $\rho$ is positive, and $w$ is close to zero. Optimal short-termism also arises when the firm has little slack and short- and long-run shocks are positively correlated. As discussed earlier, a positive correlation between shocks increases the risk of liquidation. When the likelihood of liquidation is high, the benefits of long-term growth are limited. By contrast, stimulating short-term effort increases the cash flow rate of the firm and reduces the risk of termination. Short-termism is therefore optimal in such instances.

The above analysis implies that in our model short-termism arises exactly when the principal’s return on the firm is very volatile. Figure 3 plots the volatility of the return on the principal’s stake $\Sigma_p^*(w)$ (derived in Appendix G) as a function of the manager’s continuation payoff $w$ for three different values of the correlation coefficient $\rho$. Because the principal’s stake can be seen as the firm’s equity, our results are consistent with the evidence in Brochet, Loumioti, and Serafeim (2012) and Brochet, Loumioti, and Serafeim (2013), who show that corporate short termism is associated with higher stock-return volatility.

To assess firm policies “close to termination”, we analyze effort choice “just before” termination, given by $\lim_{w \downarrow 0} k(w)$ for $k \in \{s, \ell\}$. For convenience, we informally write $k(0) = \lim_{w \downarrow 0} k(w)$, even though effort at $w = 0$ equals zero, in that in general $0 = k(0) \neq \lim_{w \downarrow 0} k(w)$ and $k(\cdot)$ is not right-continuous at $w = 0$.  

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Figure 3: Return volatility $\Sigma_p^*(w)$ for different values of $\rho$ in our base case environment. The parameters are $\alpha = 0.25$, $\mu = 0.132$, $\sigma_X = 0.25$, $\sigma_K = 0.2$, $\lambda_s = 1$, $\lambda_\ell = 1.19$, $r = 0.046$, $\gamma = 0.05$, $\delta = 0.125$, $\phi = 0$, $s_{\text{max}} = 2$, $\ell_{\text{max}} = 1.25$, and $R = 0.2$.

Figure 2 also shows that a decrease in the expected return on invested capital $\alpha$ leads to more long-termism as the returns on short-term effort are lower. When $\alpha$ is sufficiently low, the contract almost always distort policies towards long-termism. Symmetrically, the contract almost always distort policies towards short-termism when $\alpha$ is sufficiently high (i.e. $\alpha > 0.27$ in our base case). An increase in the growth rate $\mu$ increases the returns on long-term effort and distort policies towards long-termism. As expected, an increase in the cost $\lambda_s$ (resp. $\lambda_\ell$) of short-term (resp. long-term) effort favors long-termism (resp. short-termism). A decrease in the externality parameter $\phi$ in the cost function favors long-termism.

To get a more complete picture of the effects of the various parameters of the model on effort choices, Figure 4 plots short- and long-term efforts $s(w)$ and $\ell(w)$ as functions of the continuation payoff of the manager $w$ for different values of the correlation coefficient $\rho$ between short- and long-run shocks, the volatility coefficients $\sigma_K$ and $\sigma_X$, the expected profitability $\alpha$, the growth rate $\mu$, the depreciation rate $\delta$, and the externality parameter in the cost function $\phi$. A first striking feature of the plots is that, with the exception of the "short-run risk parameters ($\rho$ and $\sigma_X$)" short-term effort essentially only responds to changes in the manager’s continuation payoff $w$, which can be seen as a measure of financial slack (as in DeMarzo et al. (2012)). The other parameters have virtually no effect on $s(w)$. By contrast, long-term effort $\ell(w)$ is very sensitive to the various parameters of the model.
Figure 4: Absolute short-and long-termism. The parameters for our base case environment are $\alpha = 0.25$, $\mu = 0.132$, $\sigma_X = 0.25$, $\sigma_K = 0.15$, $\lambda_s = 1$, $\lambda_{\ell} = 1.19$, $r = 0.046$, $\gamma = 0.05$, $\delta = 0.125$, $\phi = 0$, $s_{\text{max}} = 2$, $\ell_{\text{max}} = 1.25$, and $R = 0.2$. 
Notably, and as expected, long-term effort increases with the efficiency of the firm long-run technology as measured by $\mu$. It also increases with the duration of assets as measured by $\delta$. Also, a decrease in the externality parameter $\phi$ in the cost function makes long-term effort cheaper, leading to an increase in $\ell(w)$.

Figure 4 also shows that both short- and long-term effort are highly sensitive to the risk parameters with an increase in long-run risk $\sigma_K$ (resp. short-run risk $\sigma_X$) leading to a decrease in long-term effort $\ell(w)$ (resp. short-term effort $s(w)$). A negative correlation between permanent and transitory shocks combined with a low volatility of permanent shocks and a high volatility of transitory shocks leads to absolute long-termism. Symmetrically, a positive correlation between shocks combined with a high volatility of permanent shocks and a low volatility of transitory shocks leads to absolute short-termism (not shown).

Overall, our richer model thus yields the new insight that agency issues may lead to higher managerial effort and to optimal short-termism or long-termism in corporate policies.

3 Incentives for the short and long run

3.1 Asymmetric pay in executive compensation

In this section, we turn to analyze incentive provision. For clarity of exposition, we assume again that the correlation $\rho$ between short- and long-run shocks and the externality parameter $\phi$ in the cost function are equal to zero. When the level of effort is interior, the incentive-compatibility constraint directly maps effort to incentives since $\beta^s = \lambda_s s(w)$ and $\beta^\ell = \lambda_\ell \ell(w)$. Higher effort means higher incentives and the discussion of effort and incentives cannot be separated. This section therefore focuses on corner levels of effort, i.e. situations in which $s(w) = s_{\text{max}}$ and $\ell(w) = \ell_{\text{max}}$. The upper bounds on the effort levels can be related to the maximum time the manager can spend on the job. The upper bound on long-term effort, i.e. $\ell_{\text{max}} < \frac{r + \delta}{\mu}$, also naturally arises in our model as a necessary condition to obtain finite firm values. Corner levels of efforts obtain in the model whenever the costs of efforts are low (low $\lambda_s \sigma_X$ or low $\lambda_\ell \sigma_K$) or the maximum effort level is low (low $s_{\text{max}}$ or low $\ell_{\text{max}}$).
Corner levels of efforts are the only relevant cases in a model with binary effort choices (i.e. \( s \in \{0, s_{\text{max}}\} \) or \( \ell \in \{0, \ell_{\text{max}}\} \), as in He (2009), or in a model with effort cost functions that are linear in effort levels, as in Biais et al. (2007) or DeMarzo et al. (2012).

The objective of the principal when choosing the manager’s exposure to firm performance is to maximize the value derived from the firm, given a promised payment \( w \) to the manager. To do so, the principal equivalently minimizes the agent’s exposure to shocks, while maintaining incentive compatibility (see equation (9)). An application of Itô’s formula implies that the dynamics of scaled promised payments are given by:

\[
dw = \left( (\gamma + \delta - \mu\ell)w + \sigma^2_K(w - \beta^\ell) + \frac{\lambda_s s^2 \alpha + \lambda \ell^2 \mu}{2} \right) dt + \beta^s \sigma_X dX + (\beta^\ell - w) \sigma_K dK.
\]

Minimizing risk exposure amounts to minimizing the instantaneous variance of the scaled promised payments:

\[
\Sigma(w) = (\beta^s \sigma_X)^2 + (\beta^\ell - w)^2 \sigma_K^2 \quad \text{subject to } \beta^s \geq \lambda_s s_{\text{max}} \text{ and } \beta^\ell \geq \lambda \ell_{\text{max}}.
\]

This leads to the following result:

**Proposition 6** (Asymmetric benchmarking in executive compensation). Assume that the bounds \((0, i_{\text{max}})\) for \( i \in \{s, \ell\} \) are such that the optimal effort levels are at the corner. Then, optimal incentives are given by \( \beta^s = \lambda_s s_{\text{max}} \) and \( \beta^\ell = \lambda \ell_{\text{max}} + \max\{0, w - \lambda \ell_{\text{max}}\} \).

The finding that the incentive compatibility constraint \( \beta^s \geq \lambda_s s_{\text{max}} \) in Proposition 6 is tight is standard and intuitive. The principal needs to expose the agent to firm performance but this is costly because this increases the risk of inefficient liquidation. Thus, the principal optimally exposes the agent to as little short-run risk as possible.

The finding that the incentive compatibility constraint \( \beta^\ell \geq \lambda \ell_{\text{max}} \) is not necessarily tight stems from the fact that the principal optimally wants to expose the manager’s continuation payoff to long-run, permanent shocks. Indeed, as noted in Hoffmann and Pfeil (2010) and DeMarzo et al. (2012), a positive permanent shock makes liquidation more inefficient. As a result, under the optimal contract, the agent’s promised wealth increases in response to a
The IC constraint is always binding:

\[ 0 \leq w \leq \lambda \ell \ell_{\text{max}} \]

- \( \beta^t = \lambda \ell \ell_{\text{max}} \) is binding;
- \( \Sigma(w) \geq (\beta^s \sigma_X)^2 \)

The IC constraint only binding for low \( w \):

- \( \beta^t = \lambda \ell \ell_{\text{max}} \) is binding;
- \( \Sigma(w) \geq (\beta^s \sigma_X)^2 \)

\[ \beta^t = w > \lambda \ell \ell_{\text{max}}; \]
- All risk stems from transitory shocks:
- \( \Sigma(w) = (\beta^s \sigma_X)^2 \).

Figure 5: Binding and non-binding incentive-compatibility constraint for the long-run effort in the case of \( \rho = 0 \).

positive shock in order to reduce the likelihood of inefficient liquidation. In our model, and as noted above, a positive permanent shock \( dZ^K > 0 \) has two effects. First, the agent is rewarded for good performance and is promised higher future payments \( W \), which increases the stake \( w \) by \( \beta^t \sigma_K dZ^K \). Second, firm size \( K \) grows more than expected, thereby reducing the agent’s stake in the firm by \( -w \sigma_K dZ^K \). If \( w > \beta^t \), the scaling/dilution effect outweighs the absolute reward due to good performance and the relative stake \( w \) decreases following a positive permanent shock, thereby increasing the likelihood of inefficient termination. To eliminate this negative effect, and thus to make \( w \) less volatile, the investor can increase the sensitivity of future payments to changes in firm size to \( \beta^t = w > \lambda \ell \ell(w) \). In such instances, we have \( (\beta^t - w) \sigma_K dZ^K = 0 \) and the effects from scaling/dilution and performance based compensation exactly cancel out. As a result, the agent is incentivized for the long run without raising the threat of termination. This requires exposing the manager to long-run shocks beyond the level needed to incentivize long-term effort.

Importantly, Hoffmann and Pfeil (2010) and DeMarzo et al. (2012) also find if firm value
is exposed to permanent shocks, it is optimal to tie managerial wealth to these shocks. In contrast with these papers, the principal needs to incentivize the manager to exert long-run effort in our framework with multi-tasking. This generates the distinct prediction that extra pay-for-performance is introduced and the manager’s wealth is fully exposed to permanent shocks only when her stake in the firm is large enough. By contrast, when \( w \) is low, the extra pay-for-performance effect is shut down, the incentive compatibility constraint is binding, and the contract implies that \( \beta^\ell = \lambda^\ell \ell \). The resulting pattern of pay-for-performance is thus asymmetric in that excessive exposure to permanent shocks is the largest at the highest \( w = \bar{w} \) and vanishes as \( w \) falls below \( \lambda^\ell \ell_{\text{max}} \). In other words, positive shocks lead to additional pay-for-performance and negative shocks eventually eliminate this extra sensitivity to performance implied by the optimal contract. This is consistent with evidence on the asymmetry of pay-for-performance in executive compensation (see for example Garvey and Milbourn (2006) and Francis, Iftekhar, Kose, and Zenu (2013)).

Accordingly, two different scenarios can occur in our model. First, when \( \lambda^\ell \ell_{\text{max}} > \bar{w} \), incentive-compatibility constraints are always tight and there is no asymmetric pay-for-performance, as in related models by DeMarzo and Sannikov (2006), DeMarzo et al. (2012), or He (2009). This case is depicted in the top panel if Figure 5. Second, a novelty of our model is that \( \bar{w} \) can exceed \( \lambda^\ell \ell_{\text{max}} \) and thus we can have \( \beta^\ell = w > \lambda^\ell \ell_{\text{max}} \) for all \( w \in (\lambda^\ell \ell_{\text{max}}, \bar{w}] \). Hence, the incentive-compatibility constraint corresponding to the long-run will be loose in some states and incentive provision will involve extra pay-for-performance, as illustrated in the bottom panel of Figure 5. Whenever \( w > \lambda^\ell \ell_{\text{max}} \), all risk the agent is exposed to stems exclusively from transitory cash-flow shocks (since \( \rho = 0 \)) and incentive provision for the long run becomes effectively costless.

Remarkably, the effects described above critically depends on the presence of both short-run and long-run moral hazard, as we show in the following Proposition:

**Proposition 7** (Optimal incentives with one-dimensional moral hazard). If long-term effort is observable, long-run incentives satisfy \( \beta^\ell = w \) for all \( w \in [0, \bar{w}] \). If short-term effort is observable, long-run incentives satisfy \( \beta^\ell > w \) for all \( w \in [0, \bar{w}] \).
To close this section note that our findings differ from those in He (2009). In contrast to his work, all risk from permanent cash flow shocks can be eliminated in our model even under the assumption that the agent is more impatient than the principal $\gamma > r$. In He (2009), this can only happen in the extreme case of equally patient agent and principal, where the resulting agency conflict does not affect the firm value anymore after sufficiently strong past performance. This implies that the firm eventually becomes riskless and the agent works forever. As a result, the first-best outcome can be achieved. By contrast, when the firm is exposed to both permanent and transitory shocks, only long-run agency conflicts may be temporarily harmless. Indeed, sufficiently adverse cash-flow shocks may lower $w$, drive it below $\lambda_{t\ell}$, and even trigger liquidation, implying that first-best will never be reached.

3.2 Which Firms Use Asymmetric Pay-for-Performance?

As discussed above, when the firm has accumulated sufficient financial slack, in that $w > \lambda_{t\ell_{\text{max}}}$, the manager receives additional exposure to permanent shocks, beyond what is required to provide sufficient incentives. We now analyze which firms are more likely to make use of the asymmetric and convex pattern in executive compensation. To do so, we simulate firms in our model and examine how the probability $\mathcal{P}[\beta_{t\ell} > \lambda_{t\ell_{\text{max}}}]$ that the firm uses asymmetric pay changes with the model parameters. To ensure the model admits a stationary distribution, we assume that the manager is replaced upon termination by paying a (e.g. search) cost $k$, so that $p(0) = p(w^*) - k$ and the firm is never liquidated.

Figure 6 reveals that firms facing higher risks, i.e. firms with a high earnings volatility $\sigma_X$ or with high firm value volatility $\sigma_K$, are more likely to use asymmetric pay-for-performance. Indeed, an increase in risk makes it more likely that the contract of the manager will be terminated, thereby generating deadweight costs. To reduce the expected value of these costs, the principal finds it optimal to increase the payout threshold $\overline{w}$. This in turn increases the size of the region $[\lambda_{t\ell_{\text{max}}}, \overline{w}]$ over which providing extra incentives to management is optimal and therefore the likelihood that the firm will use asymmetric pay-for-performance.

Figure 6 additionally shows that firms with a high return on invested capital $\alpha$ and better
growth perspectives $\mu$ should make greater use of asymmetric pay-for-performance. This is again intuitive. Indeed, since an increase in either $\alpha$ or $\mu$ boosts overall profitability, the firm optimally delays payouts to the management further by raising $\overline{w}$ and exposing the manager’s compensation package more and more to long-run performance in an attempt to reduce the risk of termination. Our finding that firms with a high growth potential $\mu$ should be more prone to use asymmetric pay-for-performance is consistent with the evidence in Francis et al. (2013) “that asymmetric benchmarking of pay is more prevalent in high growth industries.” To the best of our knowledge, all the other predictions are novel.

4 Conclusion

We develop a dynamic agency model in which the agent controls current earnings via short-term effort and firm growth via long-term effort. In this multi-tasking model, the principal optimally balances the costs and benefits of incentivizing the manager over the short- or the long-term, leading to both overinvestment, i.e. effort above the first-best level, and underinvestment, i.e. effort below the first-best level. As shown in the paper, this leads to optimal short-termism when financial slack is low and the likelihood of inefficient termination.
is high and to optimal long-termism when it is low.

The model additionally predicts that that a decrease in the return on invested capital should favor long-termism while an increase in the growth rate of assets should favor long-termism. It also predicts that the nature of the risks facing firm is key in determining the corporate horizon. We show for example that the correlation between between shocks to earnings and to firm value, by increasing risk, leads to externalities between effort choices with higher correlation leading to a shortening of the corporate horizon. Another prediction of the model is that an increase in earnings volatility tends to favor long-termism while an increase in firm value volatility tends to favor short-termism. Importantly, our finding that short-termism is more likely to be optimal when the volatility of the firm value process is high and the correlation between shocks to earnings and firm value is positive, i.e. when stock return volatility is high, is consistent with the available evidence. The other predictions of the model are novel and provide grounds for further empirical work on short- vs long-termism in corporate policies.

Incentives are provided in the optimal contract by making the agent’s compensation contingent on firm performance. Because the firm is subject to long-run, permanent shocks, it is optimal to introduce exposure to long-run volatility that is not needed to incentivize effort in the contract. In our model with multi-tasking, however, the principal needs to incentivize the manager to exert long-run effort. This generates the distinct prediction that extra pay-for-performance is introduced and the manager’s wealth is fully exposed to permanent shocks only when her stake in the firm is large enough. Notably, when her stake is low, the extra pay-for-performance effect is shut down and the incentive compatibility constraint is binding. In other words, positive permanent shocks lead to additional pay-for-performance and negative permanent shocks eventually eliminate this extra sensitivity to performance implied by the optimal contract. Our model therefore provides a rationale for the asymmetry of pay-for-performance observed in the executive compensation data.
Appendix

Without loss in generality, we consider throughout the whole Appendix that the depreciation rate of capital $\delta$ equals zero. To ensure the problem is well-behaved, we impose the following regularity conditions:

a) Square integrability of the payout process $\{C\}$:
$$E \left[ \left( \int_0^r e^{-\gamma s} dC_s \right)^2 \right] < \infty.$$

b) The processes $\{s\}$ and $\{\ell\}$ are of bounded variation.

c) The sensitivities $\{\beta_s\}$ and $\{\beta_\ell\}$ are almost surely bounded, so that there exists $M > 0$ with $P(\beta^K_t < M) = 1$ for each $t \geq 0$ and $K \in \{s, \ell\}$. We make this assumption for purely technical reasons and can choose $M < \infty$ arbitrarily large (enough), such that this constraint never binds in optimum.

A Proof of Proposition 1

Proof. The first best effort levels $(s^{FB}, \ell^{FB})$ maximize
$$\hat{p}(s, \ell) = \frac{1}{r + \delta - \mu \ell} \left[ \alpha s - \frac{1}{2} \left( \lambda_s \alpha s^2 + \lambda_\ell \mu \ell^2 + 2\phi \alpha \mu s \ell \right) \right].$$

The first order condition with respect to $s$, that is $\frac{\partial \hat{p}(s, \ell)}{\partial s} = 0$ implies $s^{FB} = \frac{1 - \phi \mu \ell}{\lambda_s}$ provided that effort choice is interior and it is evident that $\frac{\partial^2 \hat{p}(s, \ell)}{\partial s^2} < 0$. After imposing this first best choice, the first order condition with respect to $\ell$ reads
$$K_0 \left[ \alpha \mu (2(r + \delta)\phi - 1) + 2(r + \delta) \mu (\alpha \mu \phi^2 - \lambda_s) \ell + \mu^2 (\lambda_s \lambda_\ell - \phi \alpha \mu) \ell^2 \right] = 0,$$

which can be rewritten as
$$\Phi(\ell) \equiv \alpha \mu (2(r + \delta)\phi - 1) + 2(r + \delta) \mu (\alpha \mu \phi^2 - \lambda_s) \ell + \mu^2 (\lambda_s \lambda_\ell - \phi \alpha \mu) \ell^2 = 0.$$

Due to the second order condition of the maximization, it must be that $\frac{\partial^2 \hat{p}(s, \ell)}{\partial \ell^2} \leq 0$, which implies $\Phi'(\ell^{FB}) \leq 0$. Solving yields the desired expression for $s^{FB}$ and $\ell^{FB}$. \qed

B Proof of Proposition 2

A.1 Auxiliary Results

We first show, that each effort path ($\{s\}, \{\ell\}$) induces a probability measure under certain conditions. To begin with, fix a probability measure $P^0$, such that
$$dX_t = \sigma_X K_t d\bar{W}_t^X \quad \text{and} \quad dK_t = \sigma_K K_t d\bar{W}_t^K$$
with correlated standard Brownian motions \( \{ \tilde{W}^X \}, \{ \tilde{W}^K \} \) under this measure, both progressive with respect to \( \mathcal{F} \). The measure \( \mathcal{P}^0 \) corresponds to perpetual zero effort. Define \( \tilde{W}_t \equiv (\tilde{W}^X_t, \tilde{W}^K_t)' \) and let the (unconditional) covariance matrix of \( \tilde{W}_t \) under \( \mathcal{P}^0 \) be
\[
\mathbb{V}^0(\tilde{W}_t) = \mathbb{E}^0(\tilde{W}_t \tilde{W}_t') = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \cdot t \equiv C t.
\]
Here, \( \mathbb{V}^0(\cdot) \) denotes the variance operator with respect to the measure \( \mathcal{P}^0 \). Let us employ a Cholesky decomposition to write \( M^{-1}(M^{-1})' = C \) or equivalently \( M'M = C^{-1} \) for an invertible, deterministic matrix \( M \). Observe that
\[
\mathbb{V}^0(M\tilde{W}_t) = ME^0(\tilde{W}_t \tilde{W}_t') M' = MCM' \cdot t = M(M'M)^{-1}M' \cdot t = I \cdot t,
\]
where \( I \in \mathbb{R}^{2 \times 2} \) denotes the identity matrix. Because the two components of \( \tilde{W}_t \) are jointly normal and uncorrelated, they are also independent, in that the process \( \{ \tilde{W}_T \} \equiv \{ M\tilde{W} \} \) follows a bidimensional standard Brownian motion. We can now apply Girsanov’s theorem to \( \{ \tilde{W}_T \} \) where all components, by definition, are mutually independent.

As a first step, we define
\[
\Theta_t = \Theta_t(s, \ell) = \left( \frac{\alpha S_t}{\sigma_X}, \frac{\mu \ell_t}{\sigma_K} \right)' \quad \text{and} \quad \tilde{\Theta}_t = \tilde{\Theta}_t(s, \ell) \equiv M\Theta_t(s, \ell).
\]
Further, let
\[
\Gamma'_t = \Gamma'_t(s, \ell) \equiv \exp \left( \int_0^t \tilde{\Theta}_u \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t ||\tilde{\Theta}_u||^2 du \right),
\]
where \( || \cdot || \) denotes the Euclidean norm in \( \mathbb{R}^2 \) and
\[
\int_0^t \tilde{\Theta}_u \cdot d\tilde{W}_u = \int_0^t \sum_{i=1,2} \tilde{\Theta}_{u,i} d\tilde{W}_{u,i} = \sum_{i=1,2} \int_0^t \tilde{\Theta}_{u,i} d\tilde{W}_{u,i}.
\]
Throughout the paper, we will assume that the processes \( \{ s \}, \{ \ell \} \) are such that the so-called ‘Novikov condition’ is satisfied, in that
\[
\mathbb{E}^0 \left[ \exp \left( \frac{1}{2} \int_0^T ||\tilde{\Theta}_t||^2(s, \ell)dt \right) \right] < \infty.
\]
Then, \( \{ \Gamma' \} \) follows a martingale under \( \mathcal{P}^0 \) rather than just a local martingale. Due to \( \mathbb{E}^0 \Gamma'_0 = \mathbb{E}^0 \Gamma'_0 = 1 \), the process \( \{ \Gamma' \} \) is a progressive density process and defines the probability measure \( \mathcal{P}^{s,\ell} \) via the Radon-Nikodym derivative
\[
\left( \frac{d\mathcal{P}^{s,\ell}}{d\mathcal{P}^0} \right) |_{\mathcal{F}_t} = \Gamma'_t.
\]
\[\text{For a matrix-valued random variable } Y : \Omega \to \mathbb{R}^{n \times k} \text{ we denote the transposed random variable by } Y' : \Omega \to \mathbb{R}^{k \times n}.\]
By Girsanov’s theorem

\[ \{Z_t^T = \widetilde{W}_t^T - \int_0^t \tilde{\Theta}_u du : t \geq 0\} \]

follows a bidimensional, standard Brownian motion under the measure \( \mathcal{P}^{s,\ell} \). The linearity of the (Riemann-) integral implies

\[ M\left(\begin{pmatrix} Z_t^X \\ Z_t^K \end{pmatrix}\right) \equiv Z_t^T = M\left(\widetilde{W}_t - \int_0^t \Theta_u du\right) = M\left(\begin{pmatrix} \widetilde{W}_t^X \\ \widetilde{W}_t^K \end{pmatrix}\right) = M\left(\begin{pmatrix} \int_0^t \Theta_{u,1} du \\ \int_0^t \Theta_{u,2} du \end{pmatrix}\right). \]

Therefore, for each \( t \geq 0 \)

\[ dZ_t^X = \frac{dX_t - K_t \alpha_s dt}{K_t \sigma_X} \quad \text{and} \quad dZ_t^K = \frac{dK_t - K_t \mu_\ell dt}{K_t \sigma_K} \]

are the increments of a standard Brownian motion under \( \mathcal{P}^{s,\ell} \) with instantaneous correlation \( \rho dt \). In the following, we say the measure \( \mathcal{P}^{s,\ell} \) is induced by the processes \( \{s\}, \{\ell\} \). Note that all probability measures of the family \( \{\mathcal{P}^{s,\ell}\}_{\{s\},\{\ell\}} \) are mutually equivalent.

**B.2 Proof of Proposition 2.1**

**Proof.** Consider an incentive compatible contract \( \Pi \equiv (\{C\}, \{\hat{s}\}, \{\hat{\ell}\}, \tau) \). Further, assume in the following without loss of generality that \( \mathbb{F} \) is the filtration generated by \( \{X\}, \{K\} \), in that \( \mathcal{F}_t = \sigma(X_s, K_s : 0 \leq s \leq t) \). Then, the agent’s continuation utility at time \( t \) (under the principal’s information) is defined by

\[ W_t(\Pi) \equiv \mathbb{E}^{\hat{s},\hat{\ell}}_{t}\left[ \int_0^\tau e^{-\gamma(z-t)} dC_z - \int_0^\tau e^{-\gamma(z-t)} K_z C(\hat{s}_z, \hat{\ell}_z) dz \right], \]

where \( \mathbb{E}^{\hat{s},\hat{\ell}}_{t}(\cdot) \) denotes the conditional expectation given \( \mathcal{F}_t \), taken under the probability measure \( \mathcal{P}^{\hat{s},\hat{\ell}} \) induced by \( \{\hat{s}\} \) and \( \{\hat{\ell}\} \). Define for \( t \leq \tau \):

\[ \Gamma_t(\Pi) \equiv \mathbb{E}^{\hat{s},\hat{\ell}}_{t}[W_0(\Pi)] = \int_0^t e^{-\gamma z} dC_z - \int_0^t e^{-\gamma z} K_z C(\hat{s}_z, \hat{\ell}_z) dz + e^{-\gamma t} W_t(\Pi). \quad \text{(A1)} \]

By construction, \( \{\Gamma_t(\Pi) : 0 \leq t \leq \tau\} \) is a square-integrable martingale under \( \mathcal{P}^{\hat{s},\hat{\ell}} \), progressive with respect to \( \mathbb{F} \).

Next, observe that any sigma-algebra is invariant under an injective transformation of its generator. In particular, let \( M \in \mathbb{R}^{2 \times 2} \) an invertible, deterministic matrix with \( \det(M) \neq 1 \) and note that

\[ \mathcal{F}_t = \sigma(X_s, K_s : s \leq t) = \sigma(Z^1_s, Z^2_s : s \leq t) = \sigma(Z_s : s \leq t) = \sigma(M \cdot Z_s : s \leq t) \]
with $Z_t \equiv (Z^1_t, Z^2_t)'$. Here,

$$dZ^1_t \equiv \frac{dX_t - K_t a^s_t dt}{K_t \sigma_X} \quad \text{and} \quad dZ^2_t \equiv \frac{dK_t - K_t \dot{\mu}_t dt}{K_t \sigma_K}$$

are the increments of a standard Brownian motion under the probability measure $\mathcal{P}^{s, \ell}$. Note that $dZ^1_t = dZ^X_t$, $dZ^2_t = dZ^K_t$, whenever $a_t = \dot{a}_t$ for all $a \in \{s, \ell\}$.

As in the previous section, let the covariance matrix $\mathbf{V}(Z_t) = \mathbf{C}t$ and employ a Cholesky decomposition $\mathbf{M}' \mathbf{M} = \mathbf{C}^{-1}$. We have already shown that $\{Z^T_t \equiv \mathbf{M}Z_t : 0 \leq t \leq \tau\}$ follows a bidimensional, standard Brownian-motion under $\mathcal{P}^{s, \ell}$, where both components are mutually independent. By the martingale representation theorem (see e.g. Shreve (2004)), there exists a bidimensional process $\{b_t\}_{t \geq 0}$, progressively measurable with respect to $\mathcal{F}$, such that

$$d\Gamma_t(\Pi) = e^{-\gamma t} b'_t \cdot dZ^T_t = e^{-\gamma t} b'_t \cdot \mathbf{M}^{-1} \cdot dZ^T_t = e^{-\gamma t} K_t \left( \beta^s_t \sigma_X dZ^1_t + \beta^l_t \sigma_K dZ^2_t \right),$$

where we exploit the linearity of the Itô integral - i.e. $d(\mathbf{MZ}^T_t) = \mathbf{M} dZ^T_t$ - and set $(\beta^s_t \sigma_X, \beta^l_t \sigma_K) \equiv b'_t \mathbf{M}/K_t$ for all $t$. Taking the look of the stochastic differential both sides of (A1) with respect to $t$, one can verify that

$$d\Gamma_t(\Pi) = e^{-\gamma t} K_t \left( \beta^s_t \sigma_X dZ^1_t + \beta^l_t \sigma_K dZ^2_t \right) = e^{-\gamma t} \left[ dC_t - K_t \mathcal{C}(\hat{s}_t, \hat{\ell}_t) dt \right] - \gamma e^{-\gamma t} W_t(\Pi) dt + e^{-\gamma t} dW_t(\Pi)$$

and thus equation (5) holds after rearranging. Indeed, since the right hand side of (5) satisfies a Lipschitz-condition under the usual regularity conditions (i.e. square integrability of $\{C\}$ and $\{\hat{s}\}, \{\hat{\ell}\}$ of bounded variation), $\{W\}$ is the unique strong solution to the stochastic differential equation (5).

Next, we provide necessary and sufficient conditions for the contract $\Pi$ to be indeed incentive compatible. For this purpose, let the ‘recommended’ effort processes $\{\hat{s}\}$ and $\{\hat{\ell}\}$ and the expected payoff of the agent at time $t$ be $W_t$, when following the recommended strategy from time $t$ onwards. Further, let $\{s\}$ and $\{l\}$ represent the ‘actual’ effort processes, which may in principle differ from $\{\hat{s}\}$ and $\{\hat{\ell}\}$. We have

$$W_t \equiv \mathbb{E}^s,\ell_t \left[ \int_t^\tau e^{-\gamma(z-t)} dC_z - \int_t^\tau e^{-\gamma(z-t)} K_z \mathcal{C}(\hat{s}_z, \hat{\ell}_z) dz \right].$$

Recall that $\mathbb{E}^s,\ell_t$ denotes the expectation, conditional on the filtration $\mathcal{F}_t$, taken under the probability measure $\mathcal{P}^{s, \ell}$. As shown above, the process $\{W\}$ solves the stochastic differential equation:

$$dW_t = \gamma W_t dt + K_t \mathcal{C}(\hat{s}_t, \hat{\ell}_t) dt + \beta^s_t (dX_t - K_t a \dot{s}_t dt) + \beta^l_t (dK_t - K_t \dot{\mu}_t dt) - dC_t.$$
We can rewrite this stochastic differential equation as

\[ dW_t + dC_t = \gamma W_t dt + K_t \mathcal{C}(s_t, \ell_t) dt + \mathcal{K}^{s,t} \mathcal{E}(s_t, \ell_t) dt + K_t \beta^s_t [\alpha(s_t - \hat{s}_t) dt + \sigma_X dZ^X_t] + K_t \beta^\ell_t [\mu(\ell_t - \hat{\ell}_t) dt + \sigma_K dZ^K_t] \]

with

\[ dZ^X_t \equiv \frac{dX_t - K_t \alpha s_t dt}{K_t \sigma_X} \quad \text{and} \quad dZ^K_t \equiv \frac{dK_t - K_t \mu \ell_t dt}{K_t \sigma_K}. \]

Girsanov’s theorem implies now that \( dZ^X_t \equiv \frac{dX_t - K_t \alpha s_t dt}{K_t \sigma_X} \) and \( dZ^K_t \equiv \frac{dK_t - K_t \mu \ell_t dt}{K_t \sigma_K} \) are the increments of a standard Brownian motion under the measure \( P^{s,\ell} \) induced by \( \{s\}, \{\ell\} \).

Next, define the auxiliary gain process

\[ g_t = g_t(\{s\}, \{\ell\}) \equiv \int_0^t e^{-\gamma z} dC_z - \int_0^t e^{-\gamma z} K_z \mathcal{C}(s_z, \ell_z) dZ^X_z + e^{-\gamma t} W_t \]

and recall that \( W_T = 0 \). Now, note that the agent’s ‘actual’ expected payoff under the strategy \( \{s\}, \{\ell\} \) reads

\[ W'_0 = \max_{\{s\}, \{\ell\}} \mathbb{E}^{s,\ell} \left[ \int_0^\tau e^{-\gamma z} dC_z - \int_0^\tau e^{-\gamma z} K_z \mathcal{C}(s_z, \ell_z) dZ^X_z \right] = \max_{\{s\}, \{\ell\}} \mathbb{E}^{s,\ell} [g_T(\{s\}, \{\ell\})]. \]

We obtain

\[ e^{\gamma t} d\hat{g}_t = K_t \left[ \mathcal{C}(\hat{s}_t, \hat{\ell}_t) - \mathcal{C}(s_t, \ell_t) \right] dt \]

\[ + K_t \left[ \alpha \beta^s_t (s_t - \hat{s}_t) + \mu \beta^\ell_t (\ell_t - \hat{\ell}_t) \right] dt + K_t \left[ \beta^s_t \sigma_X dZ^X_t + \beta^\ell_t \sigma_K dZ^K_t \right] \]

\[ \equiv \mu_t^g dt + K_t \left[ \beta^s_t \sigma_X dZ^X_t + \beta^\ell_t \sigma_K dZ^K_t \right]. \]

It is now easy to see that, when choosing \( s_t = \hat{s}_t, \ell_t = \hat{\ell}_t \), the agent can always ensure that \( \mu_t^g = 0 \), in which case \( \{g_t\}_{t \geq 0} \) follows a martingale under \( P^{s,\ell} \). Hence,

\[ W'_0 = \max_{\{s\}, \{\ell\}} \mathbb{E}^{s,\ell} [g_T(\{s\}, \{\ell\})] \geq \mathbb{E}^{\hat{s},\ell} [g_T(\{\hat{s}\}, \{\hat{\ell}\})] = W_0. \]

The inequality is strict if and only if there exist processes \( \{s\}, \{\ell\} \) and a stopping time \( \tau' \) with \( P^{s,\ell}(\tau' < \tau) > 0 \) such that \( \mu_{\tau'}^g > 0 \). This arises because then there also exists a set \( \mathcal{A} \subseteq [0, \tau) \times \Omega \) with

\[ \mu_t^G(\omega) > 0 \quad \text{for all} \quad (t, \omega) \in \mathcal{A} \quad \text{and} \quad \mathcal{L} \otimes P^{s,\ell}(\mathcal{A}) > 0, \]

where \( \mathcal{L} \) is the Lebesgue-measure on the Lebesgue sigma-algebra in \( \mathbb{R} \). Because \( P^{s,\ell}(\tau < \infty) \) for all admissible \( \{s\}, \{\ell\} \) it follows that \( e^{-\gamma t} \mu_t^G(\omega) > 0 \) for all \( (t, \omega) \in \mathcal{A} \). Whence,

\[ W'_0 \geq \int_{\mathcal{A}} e^{-\gamma z} \mu_z^G(\omega) d(\mathcal{L}(z) \otimes P^{s,\ell}(\omega)) + W_0 > W_0. \]
In case \( W'_t > W_0 \), either \( s_\omega(\omega) \neq \hat{s}_\omega(\omega) \) or \( \ell_\omega(\omega) \neq \hat{\ell}_\omega(\omega) \) on the set \( \mathcal{A} \), which has positive measure, so that \( \Pi \) is not incentive compatible.

Hence, for \( \Pi \) to be incentive compatible, it must for all \( t \geq 0 \) (almost surely) hold that

\[
\max_{s_t, \ell_t} \left\{ \alpha \beta_t^s (s_t - \hat{s}_t) + \mu \beta_t^\ell (\ell_t - \hat{\ell}_t) + [\mathcal{C}(s_t, \ell_t) - \mathcal{C}(s_t, \ell_t)] \right\} = 0
\]

or equivalently

\[
(s_t, \ell_t) = \arg \max_{s_t, \ell_t} \left\{ \alpha \beta_t^s (s_t - \hat{s}_t) + \mu \beta_t^\ell (\ell_t - \hat{\ell}_t) + [\mathcal{C}(s_t, \ell_t) - \mathcal{C}(s_t, \ell_t)] \right\}
\]

for given \( \beta_t^s, \beta_t^\ell \). After going through the maximization, we obtain that this is satisfied if \( \hat{s}_t \lambda + \phi \mu \hat{\ell}_t = \beta_t^s \) and \( \hat{\ell}_t \lambda + \phi \alpha \hat{s}_t = \beta_t^\ell \), in case \((s_t, \ell_t) \in (0, s_{\text{max}}) \times (0, \ell_{\text{max}})\). If \( \{\hat{a}_t, \hat{b}_t\} = \{s_t, \ell_t\} \) and \( \hat{a}_t \in \{s_t, \ell_t\} \) is not interior, in that \( \hat{a}_t = a_{\text{max}} \) for \( a \in \{s, \ell\} \), then \( a_t = \hat{a}_t \) solves the above maximization problem if and only if \( \beta_t^s \geq \lambda a_t + \phi \chi \hat{b}_t \), where \( \chi = \mu \) if \( a = s \) and \( \chi = \alpha \) if \( a = \ell \). The result follows.

\[\square\]

B.3 Proof of Proposition 2.2

In this section, we proceed as follows. First, we represent \( P(W, K) \) as a twice continuously differentiable solution of a HJB-equation and then show that there exists a function \( \Pi \) is not incentive compatible.

Hence, for \( \Pi \) to be incentive compatible, it must for all \( t \geq 0 \) (almost surely) hold that

\[
\max_{s_t, \ell_t} \left\{ \alpha \beta_t^s (s_t - \hat{s}_t) + \mu \beta_t^\ell (\ell_t - \hat{\ell}_t) + [\mathcal{C}(s_t, \ell_t) - \mathcal{C}(s_t, \ell_t)] \right\} = 0
\]

or equivalently

\[
(s_t, \ell_t) = \arg \max_{s_t, \ell_t} \left\{ \alpha \beta_t^s (s_t - \hat{s}_t) + \mu \beta_t^\ell (\ell_t - \hat{\ell}_t) + [\mathcal{C}(s_t, \ell_t) - \mathcal{C}(s_t, \ell_t)] \right\}
\]

for given \( \beta_t^s, \beta_t^\ell \). After going through the maximization, we obtain that this is satisfied if \( \hat{s}_t \lambda + \phi \mu \hat{\ell}_t = \beta_t^s \) and \( \hat{\ell}_t \lambda + \phi \alpha \hat{s}_t = \beta_t^\ell \), in case \((s_t, \ell_t) \in (0, s_{\text{max}}) \times (0, \ell_{\text{max}})\). If \( \{\hat{a}_t, \hat{b}_t\} = \{s_t, \ell_t\} \) and \( \hat{a}_t \in \{s_t, \ell_t\} \) is not interior, in that \( \hat{a}_t = a_{\text{max}} \) for \( a \in \{s, \ell\} \), then \( a_t = \hat{a}_t \) solves the above maximization problem if and only if \( \beta_t^s \geq \lambda a_t + \phi \chi \hat{b}_t \), where \( \chi = \mu \) if \( a = s \) and \( \chi = \alpha \) if \( a = \ell \). The result follows.

\[\square\]

A.3.1 Scaling of the value function

Given the optimal control and stopping problem (3), suppose that the principal's value function \( P(W, K) \) satisfies the HJB-equation

\[
rP = \max_{s, \ell, \beta^s, \beta^\ell} \left\{ \alpha s W + \gamma W + K C(s, \ell) + P_K \mu \ell K + \frac{1}{2} \left( P_{WW} \left( \beta^s \sigma_X K \right)^2 \right. \right.
\]

\[
+ \left( \beta^\ell \sigma_K K \right)^2 + 2 \rho \sigma_X \sigma_K K^2 \beta^\ell \beta^s \right) + P_{KK} \left( \sigma_K K \right)^2 + 2 P_{WK} \left[ (\sigma_K K)^2 \beta^\ell + \rho \sigma_X \sigma_K K^2 \beta^s \right) \right\}
\]

in some region \( \mathcal{S} \subset \mathbb{R}^2 \), subject to the boundary conditions

\[
P(0, K) = RK, P(W, 0) = 0, P_W(W, K) = -1, P_{WW}(W, K) = 0.
\]
Here, \( \overline{W} \equiv \overline{W}(K) = \overline{w}K \) parametrizes the boundary of \( S \), on which \( W, K > 0 \). Taking the guess \( P(W, K) = p(W/K)K \) for some function \( p \in C^2 \), we obtain
\[
P_W = p'(w), P_K = p(w) - wp'(w), P_{WW} = -w/Kp''(w), P_{WK} = p''(w)/K, p_{KK} = w^2/Kp''(w),
\]
which implies the HJB-equation (9) and its boundary conditions.

In the following, we will assume that (9) admits an unique, twice continuously differentiable solution \( p(\cdot) \) on \( [0, \overline{w}] \). A formal existence proof is beyond the scope of the paper and therefore omitted.\(^{17}\)

We first rewrite the principal’s problem (3) in a convenient manner. Let
\[
\Psi_t = (\rho \sigma_K t, \sigma_K t)' \quad \text{and} \quad \tilde{\Psi}_t = \mathbf{M} \Psi_t,
\]
where \( \mathbf{M}' \mathbf{M} = \mathbf{C}^{-1} \) and \( \mathbf{C} t \) is the covariance matrix of \( (Z^X_t, Z^K_t) \). Next, define the equivalent, auxiliary probability measure \( \tilde{\mathcal{P}} \) according to the Radon-Nikodym derivative
\[
\left( \frac{d\tilde{\mathcal{P}}}{d\mathcal{P}} \right)|_{\mathcal{F}_t} \equiv \exp \left\{ \int_0^t \tilde{\Psi}_u du - \frac{1}{2} \int_0^t ||\tilde{\Psi}_u||^2 du \right\}.
\]
By arguments similar to the ones in Appendix B.1, Girsanov’s theorem implies that
\[
\tilde{Z}^X_t = Z^X_t - \rho \sigma_K t \quad \text{and} \quad \tilde{Z}^K_t = Z^K_t - \sigma_K t
\]
are both standard Brownian motions with correlation \( \rho t \) under \( \tilde{\mathcal{P}} \). An application of Itô’s Lemma consequently yields that the scaled continuation value \( \{w\} \) evolves according to
\[
dw_t + dc_t = \left[ (\gamma - \mu \ell_t)w_t + \mathcal{C}(s_t, \ell_t) \right] dt + \beta_t \sigma_x d\tilde{Z}^X_t + (\beta_t^t - w_t) \sigma_K d\tilde{Z}^K_t
\]
under \( \tilde{\mathcal{P}} \). Finally, we are able to rewrite the principal’s problem (3) as
\[
\max_{\{c, s, \ell, w\}} \quad \tilde{\mathbb{E}} \left[ \left. \int_0^\tau e^{-rt+\mu \int_0^t \ell_s^s dZ^K_s + \alpha s_t dt} - \int_0^\tau e^{-rt+\mu \int_0^t \ell_s^s dZ^K_s + \alpha s_t dt} R \right| w_0 = w^* \right],
\]
where the expectation \( \tilde{\mathbb{E}}[\cdot] \) is taken under the equivalent, auxiliary measure \( \tilde{\mathcal{P}} \). Here, \( dc_t \equiv dC_t/K_t = \max\{w_t - \overline{w}, 0\} \). The stated integral expression is implied by following Lemma.

**Lemma 1.** Suppose \( \{w\} \) is the unique, strong solution to the stochastic differential equation
\[
dw_t = \delta_t dt + \Delta_t w_t dt - dc_t + (\beta_t^t - w_t) \sigma_K d\tilde{Z}^K_t + \beta_t^t \sigma_x d\tilde{Z}^X_t
\]
for \( t \leq \tau \), standard Brownian motions \( \{Z^X\}, \{Z^K\} \) with correlation \( \rho \) and progressive processes \( \{\delta\}, \{\Delta\}, \{\beta^t\}, \{\beta^t\} \) of bounded variation.\(^{18}\) Assume that \( dw_t = 0 \) for \( t > \tau \) where

\(^{17}\)Indeed, the possible discontinuities of the functions \( s(\cdot), \ell(\cdot) \) cause technical complications. If \( s_{\max}, \ell_{\max} \) are sufficiently large, this problem is not present anymore. Then, the existence and uniqueness of the solution follow from the Picard-Lindelöf theorem, since the required Lipschitz condition is evidently satisfied.

\(^{18}\)We call a process \( \{Y\} \) ‘of bounded variation’ if it can be written as the difference of two almost surely increasing processes. Similarly, a function \( F \in \mathbb{R}^{[a,b]} \) is called ‘of bounded variation’ if it can be written as
\[ \tau = \min\{t \geq 0 : w_t = 0\}. \] Furthermore, \( dc_t = \max\{w_t - \overline{w}, 0\} \) with threshold \( \overline{w} > 0 \). Let now \( g : [0, \overline{w}] \to \mathbb{R} \) of bounded variation. Then the twice continuously differentiable function \( f : [0, \overline{w}] \to \mathbb{R} \) (i.e. \( f \in C^2 \)) solves the differential equation

\[ r_t f(w_t) = g(w_t) + f'(w_t)\beta_t \Delta w_t + f''(w_t) \left[ \sigma_K^2 (\beta_t^t - w_t)^2 + (\beta_t^s \sigma_X)^2 + 2 \rho \sigma_X \sigma_K \beta_t^s (\beta_t^t - w_t) \right] \quad (A2) \]

with boundary conditions \( f(0) = R, f'(\overline{w}) = -1 \) if and only if

\[
 f(w) = E \left[ \int_0^\tau e^{-\int_0^t r_u du} g(w_u) dt - \int_0^\tau e^{-\int_0^t r_u du} dc_t + e^{-\int_0^t r_u du} R \Big| w_0 = w \right]
\]

for a progressive discount rate process \( \{r\} \) of bounded variation.

**Proof.** Suppose \( f(\cdot) \) solves (A2). Define

\[
 h_t \equiv \int_0^t e^{-\int_0^s r_u du} g(w_s) dz - \int_0^t e^{-\int_0^s r_u du} dz - \int_0^t e^{-\int_0^s r_u du} f(w_t).
\]

Applying Itô’s Lemma, we obtain

\[
 e^{\int_0^t r_u du} dh_t = \left\{ g(w_t) - r_t f(w_t) + \frac{f''(w_t)}{2} \left[ \sigma_K^2 (\beta_t^t - w_t)^2 + (\beta_t^s \sigma_X)^2 + 2 \rho \sigma_X \sigma_K \beta_t^s (\beta_t^t - w_t) \right] \right. \\
 + f'(w_t) (\delta_t + \Delta_t w_t) \right\} dt - \left\{ (1 + f'(w_t)) dc_t + f'(w_t) [dZ_t X \beta_t^s \sigma_X + dZ_t K (\beta_t^t - w_t) \sigma_K] \right\}.
\]

The first term in curly brackets equals zero because \( f(\cdot) \) solves (A2). Since \( f'(\overline{w}) = -1 \) and \( dc_t = 0 \) for all \( w_t \leq \overline{w} \), the second term in square brackets equals also zero and therefore \( \{h\} \) follows a martingale up to time \( \tau \). As a result, we have:

\[
 f(w_0) = f(w) = h_0 = E[h_\tau] = E \left[ \int_0^\tau e^{-\int_0^t r_u du} g(w_t) dt - \int_0^\tau e^{-\int_0^t r_u du} dc_t + e^{-\int_0^t r_u du} R \Big| w_0 = w \right].
\]

The result follows. \( \square \)

**B.3.2 Verification**

**Proof.** Next, we verify the optimality of the contract \( \Pi^* \) among all contracts \( \Pi \) satisfying incentive compatibility. To do so, we show that the principal obtains under any contract \( \Pi \in \Pi \) at most (scaled) payoff \( \tilde{G}(\Pi)/K \leq p(w^*) \), with equality if and only if \( \Pi = \Pi^* \). Here, \( p(\cdot) \) solves the HJB-equation (9) with corresponding payout threshold \( \overline{w} \) and \( w_0 = w^* \).

Consider any incentive-compatible contract \( \Pi \equiv (\{C\}, \{s\}, \{\ell\}, \tau) \). For any \( t \leq \tau \), define its auxiliary gain process \( G \) as

\[
 G_t(\Pi) = \int_0^t e^{-ru} dX_u - \int_0^t e^{-ru} dC_u + e^{-rt} P(W_t, K_t),
\]

where the agent’s continuation payoff evolves according to (5). Recall that \( w_t = \frac{w_t}{K_t} \) and the difference of two increasing functions on the interval \([a, b]\).
\(P(W_t, K_t) = K_t p(w_t)\). Itô’s lemma implies that for \(t \leq \tau\):

\[
e^{rt} \frac{dG_t(\Pi)}{K_t} = \left[ - (r - \mu \ell_t) p(w_t) + \alpha s_t + p'(w_t) \left( w_t (\gamma - \mu \ell_t) + C(s_t, \ell_t) \right) + \frac{p''(w_t)}{2} \left[ \left( \beta^s_t \sigma_X \right)^2 + \sigma_K^2 \left( \beta^\ell_t - w_t \right)^2 + 2 \rho \sigma_X \sigma_K \beta^s_t (\beta^\ell_t - w_t) \right] \right] dt - (1 + p'(w_t)) dc_t \\
+ \sigma_K \left( p(w_t) + p'(w_t) (\beta^\ell_t - w_t) \right) dZ^K_t + \sigma_X (1 + \beta^s_t p'(w_t)) dZ^X_t.
\]

Under the optimal effort and incentives, the first term in square bracket stays at zero always. Other effort and incentive policies will make this term negative (owing to the concavity of \(p\)). The second term is non positive since \(p'(w_t) \geq -1\), but equal to zero under the optimal contract. Therefore, for the auxiliary gain process, we have

\[
dG_t(\Pi) = \mu_G(t) dt + e^{-rt} K_t \left[ \sigma_K \left( p(w_t) + p'(w_t) (\beta^\ell_t - w_t) \right) dZ^K_t + \sigma_X (1 + \beta^s_t p'(w_t)) dZ^X_t \right],
\]

where \(\mu_G(t) \leq 0\). Due to our assumption of bounded sensitivities \(\{\beta^s\}, \{\beta^\ell\}\), it follows that

\[
\mathbb{E} \left( \int_0^t e^{-ru} \left( p(w_u) + p'(w_u) (\beta_u^\ell - w_u) \right) dZ^K_u \right) = \mathbb{E} \left( \int_0^t e^{-ru} \left( 1 + \beta_u^s p'(w_u) \right) dZ^X_u \right) = 0,
\]

which implies that \(\{G_t\}_{t \geq 0}\) follows a supermartingale. Furthermore, under \(\Pi\), investors’ expected payoff is

\[
\tilde{G}(\Pi) \equiv \mathbb{E} \left[ \int_0^\tau e^{-ru} dX_u - \int_0^\tau e^{-ru} dC_u + e^{-rt} RK_\tau \right],
\]

As a result, we have that

\[
\tilde{G}(\Pi) = \mathbb{E} \left[ G_\tau(\Pi) \right] \\
= \mathbb{E} \left[ G_{\tau \wedge t}(\Pi) + 1_{\{t \leq \tau\}} \left( \int_t^\tau e^{-rs} (dX_s - dC_s) + e^{-rt} RK_\tau - e^{-rt} P(W_t, K_t) \right) \right] \\
= \mathbb{E} \left[ G_{\tau \wedge t}(\Pi) \right] + e^{-rt} \mathbb{E} \left[ 1_{\{t \leq \tau\}} \mathbb{E}_t \left( \int_t^\tau e^{-r(s-t)} (dX_s - dC_s) + e^{-r(\tau-t)} RK_\tau - P(W_t, K_t) \right) \right] \\
\leq G_0 + e^{-rt} \mathbb{E} \left[ P^{FB}(K_t) - W_t - P(W_t, K_t) \right] \\
\leq G_0 + e^{-rt} \frac{P^{FB}(K_0)}{K_t} \mathbb{E} \left[ K_t \right],
\]

where \(p^{FB} \equiv \frac{P^{FB}(K_0)}{K_t}\) is the (scaled) first best value. The inequalities follow from the supermartingale property of \(G_t\), the fact that the value of the firm with agency is below first best, and the fact that \(p^{FB} - w - p(w) \leq p^{FB} - R\). Since \(\mu_{\ell_{\text{max}}} < r\), it follows that \(\lim_{t \to \infty} e^{-rt} \mathbb{E} \left[ K_t \right] = 0\). Therefore, letting \(t \to \infty\) yields \(\tilde{G}(\Pi) \leq G_0 \equiv P(W_0, K_0) = p(w_0)K_0\) for all incentive compatible contracts. For the optimal contract \(\Pi^*\), the investors’ payoff
\( \tilde{G}(\Pi') \) achieves \( P(W_0, K_0) = p(w_0)K_0 \) since the above weak inequality holds in equality when \( t \to \infty \). This completes the argument.

\[ \square \]

### B.4 Proof of Proposition 2.3

#### B.4.1 Auxiliary Results

In this section, we prove the following auxiliary Lemma, which is key for establishing the concavity of the value function.

**Lemma 2.** Let \( p(\cdot) \) the unique, twice continuously differentiable solution to the HJB-equation (9) on the interval \([0, \bar{w}]\) subject to the boundary conditions \( p(0) = R, \ p'(\bar{w}) = -1 \) and \( p''(\bar{w}) = 0 \). Further, assume the processes \( \{\hat{s}\}, \{\hat{\ell}\} \) are of bounded variation. Then it follows for any \( w_1 \in (0, \bar{w}] \) with \( p''(w_1) = 0 \) that \( p'(w_1) < 0 \) and that the policy functions \( s(\cdot), \ell(\cdot) \) are continuous in a neighbourhood of \( w_1 \).

**Proof.** We start with an important observation. Because the processes \( \{\hat{s}\}, \{\hat{\ell}\} \) are hypothesis of bounded variation, they can be written as the difference of two almost surely increasing processes, such that \( \hat{a}_t = \hat{a}^1_t - \hat{a}^2_t \) for all \( t \geq 0, a \in \{s, \ell\} \) and \( \hat{a}^2_t \) increases almost surely. By Froda’s theorem,\(^{19}\) each of the processes \( \{\hat{\ell}\} \) has no essential discontinuity and at most countably many jump-discontinuities with probability one. Since \( \{w\} \) follows a Brownian semimartingale, this implies that any point of discontinuity of \( a(\cdot) \) can neither be an essential discontinuity nor can the set of discontinuity points of \( a(\cdot) \) be dense in \([0, \bar{w}]\) for all \( a \in \{s, \ell\} \).

We first prove that \( p'(w_1) < 0 \). Let us suppose to the contrary \( p'(w_1) \geq 0 \), hence \( w_1 < \bar{w} \). Note that for any \( \delta > 0 \) exists \( z \in (w_1 - \delta, w_1 + \delta) \) such that \( s(\cdot), \ell(\cdot) \) are continuous in a neighbourhood of \( z \), because discontinuity points do not form a dense set. Since \( p'(\cdot), p''(\cdot) \) are continuous, for any \( \varepsilon > 0 \) we can choose \( \delta > 0 \) and \( z \in (w_1 - \delta, w_1 + \delta) \) such that \( \min\{p'(z), p''(z)\} < -\varepsilon \). The HJB-equation (9) and the fact, that \( \ell(z) = \ell_{FB} \) is not necessarily optimal, imply

\[
(r - \mu \ell_{FB}) p(z) \geq \max_{s \in [0, s_{\max}]} \left\{ \alpha s + p'(z)(\gamma - \mu \ell_{FB})z + p'(z)C(s, \ell_{FB}) + p''(z)\Sigma(z) \right\} \\
\geq \max_{s \in [0, s_{\max}]} \left\{ \alpha s - \varepsilon [(\gamma - \mu \ell_{FB})z + C(s, \ell_{FB}) + \Sigma(z)] \right\}.
\]

It is now evident that there exists \( \varepsilon > 0 \) such that \( s = s(z) = s_{\max} \geq s_{FB} \) and

\[
\alpha s - \varepsilon [(\gamma - \mu \ell_{FB})z + C(s, \ell_{FB}) + \Sigma(z)] > \alpha s_{FB} - C(s_{FB}, \ell_{FB}) = (r - \mu \ell_{FB}) p_{FB}.
\]

Hence, there exists \( z \in [0, \bar{w}] \) such that \( p(z) > p_{FB} \), a contradiction.

Next, let us prove that \( \ell(\cdot) \) must be continuous in a neighbourhood of \( w_1 \) and assume to the contrary that there is no neighbourhood of \( w_1 \), on which \( \ell(\cdot) \) is continuous. Since the set

\(^{19}\)Froda’s theorem states that each real valued, monotone function has at most countably many points of discontinuity. It is clear that such a function cannot have an essential discontinuity, i.e. a point of oscillation.
of discontinuities of \( \ell(\cdot) \) must be discrete (not dense), it is immediate that

\[
\ell_- \equiv \lim_{w \uparrow w_1} \ell(w) \neq \lim_{w \downarrow w_1} \ell(w) \equiv \ell_+,
\]
i.e. \( \ell(\cdot) \) has a jump discontinuity at \( w_1 \) itself. Without loss of generality, we will assume that \( \ell_- < \ell_+ \) and \( w_1 < \underline{w} \).

Note that for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( z \in (w_1, w_1 + \delta) \) it holds that \( |\ell(z) - \ell_+| < \varepsilon \). The optimality of \( \ell(z) \) requires that \( \frac{\partial p(z)}{\partial \ell}|_{\ell=\ell(z)} \geq 0 \) with equality if \( \ell(z) \) is interior. Due to the continuity of \( p''(\cdot) \), the limit \( \varepsilon \to 0 \) yields \( \Gamma_\ell(w_1) \geq 0 \) for

\[
\Gamma_\ell(w) = p(w) - p'(w)w + p'(w)C_\ell(s, \ell_+) \quad \text{with} \quad C_\ell(s, \ell_+) = \frac{\partial C(s, \ell)}{\partial \ell}|_{\ell=\ell_+}
\]

In addition, for all \( \varepsilon > 0 \) it must be that there exists \( \delta > 0 \) such that for all \( x \in (w_1 - \delta, w_1) \) it holds that \( |\ell(x) - \ell_-| < \varepsilon \). Hence, for \( \varepsilon > 0 \) sufficiently small, \( \ell(x) < \ell_{\max} \) and therefore \( \frac{\partial p(z)}{\partial \ell}|_{\ell=\ell(x)} = 0 \), which implies together with the continuity of \( p''(\cdot) \) that \( \Gamma_\ell(w_1) = 0 \) for

\[
\hat{\Gamma}_\ell(w) = p(w) - p'(w)w + p'(w)C_\ell(s, \ell_-).
\]

Next, observe that

\[
0 \leq \Gamma_\ell(w_1) - \hat{\Gamma}_\ell(w_1) = \lambda \ell p'(w_1)(\ell_+ - \ell_-).
\]

Because we have already shown \( p'(w_1) < 0 \), it follows that \( \ell_- \geq \ell_+ \), a contradiction.

Finally, assume that there is no neighbourhood of \( w_1 \), on which \( s(\cdot) \) is continuous. Since the set of discontinuity points of \( s(\cdot) \) is discrete, this is equivalent to \( s_- \equiv \lim_{w \uparrow w_1} s(w) \neq \lim_{w \downarrow w_1} s(w) \equiv s_+ \). Without loss of generality, suppose \( s_+ > s_- \). Then, for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( z \in (w_1, w_1 + \delta) \) it holds that \( |s(z) - s_-| < \varepsilon \). Optimality requires \( \frac{\partial p(z)}{\partial s}|_{s=s(z)} \geq 0 \). Taking the limit \( \varepsilon \to 0 \), we obtain \( \Gamma_s(w_1) \geq 0 \) for \( \Gamma_s(w) = \alpha s(w) + p'(w)C_s(s_+, \ell) \). Similarly, \( \hat{\Gamma}_s(w_1) = 0 \) for \( \hat{\Gamma}_s(w) = s(w) + p'(w)C_s(s_-, \ell) \). Hence,

\[
0 \leq \Gamma_s(w_1) - \hat{\Gamma}_s(w_1) = \lambda s p'(w_1)(s_+ - s_-).
\]

Due to \( p'(w_1) < 0 \) it follows that \( s_- \geq s_+ \), a contradiction. \( \square \)

**B.4.2 Concavity of the value function**

**Proof.** Since \( p''(\cdot) \) is continuous on \([0, \underline{w}]\) and \( \{s\}, \{\ell\} \) are of bounded variation, it follows that the mappings \( s(\cdot), \ell(\cdot) \) are continuous on \([0, \underline{w}]\) up to a discrete set with (Lebesgue-) measure zero. On the set, where \( s(\cdot), \ell(\cdot) \) are continuous, the envelope theorem implies now that \( p'''(\cdot) \) exists and is given by

\[
p'''(w) = \frac{(r - \gamma)p'(w) - p''(w)(w(\gamma - \mu \ell) + C(s, \ell) - \sigma^2_K (\beta^\ell - w) - \rho \sigma_X \sigma_K \beta s) + \frac{1}{2} \left( (\beta^s \sigma_X)^2 + \sigma^2_K (\beta^\ell - w)^2 + 2 \rho \sigma_X \sigma_K \beta (\beta^\ell - w) \right)}{\left( \beta^s \sigma_X \right)^2 + \sigma^2_K (\beta^\ell - w)^2 + 2 \rho \sigma_X \sigma_K \beta (\beta^\ell - w)}.
\]

\( ^{20} \)Since \( p(\cdot) \) is extended linearly to the right of \( \underline{w} \), discontinuity to the right of \( \underline{w} \) is not an issue.
We have to show that $p''(w) < 0$ for all $0 \leq w < \overline{w}$.

By Lemma 2 we know that $s(\cdot), \ell(\cdot)$ are continuous in a neighbourhood of $\overline{w}$. Then, we observe that $p''(\overline{w}) \propto -r > 0$ due to $\beta^s \geq \lambda_s > 0$ and thus $p''(\cdot) > 0$ in a neighbourhood of $\overline{w}$. Hence, $p''(w) < 0$ on an interval $(\overline{w} - \varepsilon, \overline{w})$ with appropriate $\varepsilon > 0$.

Next, suppose there exists $w_0 \in [0, \overline{w}]$ with $p''(w_0) > 0$ and define $w_1 \equiv \sup\{w \in [0, \overline{w}) : p''(w) \geq 0\}$. By the previous step and continuity it follows that $p''(w_1) = 0$ and $w_1 < \overline{w}$. We obtain now from Lemma 2 that $s(\cdot), \ell(\cdot)$ are continuous in a neighbourhood of $w_1$ and that $p'(w_1) < 0$. However, this implies $p''(w_1) > 0$ and therefore $p''(\cdot) > 0$ in a neighbourhood of $w_1$. Thus, there exists $w' = w_1$ with $p''(w') > 0$, a contradiction to the definition of $w_1$. This completes the proof. \hfill \square

C Proofs of Propositions 3 and 4

Proof. This follows directly from the maximization of $p(w)$ over $s \in [0, s_{\text{max}}]$ and $\ell \in [0, \ell_{\text{max}}]$ for a given $w$, as indicated by the HJB-equation (9). Interior levels $s(w), \ell(w)$ must solve the respective first order conditions of maximization, that is $\frac{\partial p(w)}{\partial s}|_{s = s(w)} = 0$ and $\frac{\partial p(w)}{\partial \ell}|_{\ell = \ell(w)} = 0$. After rearranging the FOC’s of the maximization, one arrives at the desired expressions. \hfill \square

D Proof of Proposition 5

Proof. At the payout boundary $\overline{w}$, we have that

$$ rp(\overline{w}) = \max_{s,\ell} \left\{ \mu \ell p(\overline{w}) + \alpha s - \overline{w}(\gamma - \mu \ell) - \frac{1}{2} \left( \lambda_s s^2 \alpha + \lambda_\ell \ell^2 \mu \right) \right\}. $$

Taking the first order condition of maximization with respect to $s$, we obtain that $s(\overline{w}) = \frac{1}{\lambda_s}$, if interior, and $s = s_{\text{max}}$ otherwise, i.e. $s(\overline{w}) = s^{FB}$. Similarly, the first order condition with respect to $\ell$ reads

$$ 0 = p(\overline{w}) + \overline{w} - \lambda_\ell \ell < p^{FB} - \lambda_\ell \ell, $$

where $p^{FB} = P^{FB}/K_0$. Note that $\ell(\overline{w})$ solves this first order condition, as we assume it to be interior. Because $\ell^{FB}$ solves $p^{FB} - \lambda_\ell \ell = 0$, it follows that $\ell(\overline{w}) < \ell^{FB}$. \hfill \square

E Proof of Proposition 6

In this section, we prove the natural generalization of Proposition 6 for $\rho \neq 0$. We start with the following Lemma.

Lemma 3. Let $w \in (0, \overline{w}]$ such that in optimum $\ell(w) = \ell = \ell_{\text{max}}$ and $s(w) = s \in [0, s_{\text{max}}]$. Assume that parameters satisfy $-\rho \sigma_K \lambda \ell_{\text{max}} < \sigma_X \lambda s_{\text{max}}$ for $\rho \in (-1, 1)$. Then

$$ \beta^\ell \equiv \beta^\ell(w) = \max \{ \lambda_\ell \ell_{\text{max}}, w - \rho \frac{\sigma_X}{\sigma_K} \lambda_s s \} \quad \text{and} \quad \beta^s \equiv \beta^s(w) = \lambda_s s. $$
Hence, the short-run IC-condition is always tight under the conditions stated.

Proof. Given the optimal choice \( \ell(w) = \ell_{\text{max}} \), \( s(w) = s \), the tuple \( (\beta^s(w), \beta^\ell(w)) \) must satisfy

\[
(\beta^s(w), \beta^\ell(w)) = \arg \min_{\beta^s, \beta^\ell} \left[ (\beta^s \sigma_X)^2 + \sigma_{K}^2 (\beta^\ell - w)^2 + 2 \rho \sigma_X \sigma_K \beta^s (\beta^\ell - w) \right]
\]

subject to \( \beta^\ell \geq \lambda \ell_{\text{max}} \) and \( \beta^s \geq \lambda s \),

where the last inequality is tight, unless \( s = s_{\text{max}} \). Using standard arguments, one obtains:

\[
\beta^\ell \equiv \beta^\ell(w) = \max \{ \lambda \ell_{\text{max}}, w - \rho \frac{\sigma_X}{\sigma_K} \beta^s \};
\]

\[
\beta^s \equiv \beta^s(w) = \max \{ \lambda s_{\text{max}}, \rho \frac{\sigma_K}{\sigma_X} (w - \beta^\ell) \} \text{ if } s = s_{\text{max}} \text{ and } \beta^s = \lambda s \text{ otherwise}.
\]

The claim is trivial if \( s < s_{\text{max}} \) or \( \rho = 0 \).

Let us suppose \( s = s_{\text{max}}, \rho \neq 0 \) and \( \beta^s > \lambda s \). Hence, \( \beta^s = \rho \frac{\sigma_K}{\sigma_X} (w - \beta^\ell) \). If now \( \beta^\ell > \lambda \ell \), then \( \beta^\ell = w - \rho \sigma_X / \sigma_K \beta^s \). This implies \( \rho \sigma_K / \sigma_X (w - \beta^\ell) = \rho^2 \beta^s < \beta^s \) and hence \( \beta^s = \lambda s_{\text{max}} \), a contradiction.

Next, suppose \( \rho < 0 \) and \( \beta^\ell = \lambda \ell_{\text{max}} \). Hence, \( w > \lambda \ell_{\text{max}} \). Since \( \beta^\ell = \lambda \ell_{\text{max}} \) it follows that \( \lambda \ell_{\text{max}} > w - \rho \sigma_X / \sigma_K \beta^s \) and - using \( \beta^s = \rho \frac{\sigma_K}{\sigma_X} (w - \beta^\ell) \) - one obtains \( \lambda \ell_{\text{max}} > w - \rho^2 (w - \lambda \ell_{\text{max}}) \). Hence, \( \lambda \ell_{\text{max}} > w \), a contradiction.

Finally, assume \( s = s_{\text{max}}, \rho < 0 \) and \( \beta^\ell = \lambda \ell_{\text{max}} \). Hence, \( \lambda \ell_{\text{max}} > w \) and \( \rho \sigma_K / \sigma_X (w - \lambda \ell) > \lambda s_{\text{max}} \), which implies \( w - \lambda \ell_{\text{max}} < \lambda s_{\text{max}} \sigma_X / (\sigma_K \rho) \). Therefore, \( -\rho \sigma_K \lambda \ell_{\text{max}} > \sigma_X \lambda s_{\text{max}} \), which contradicts the hypothesis.

The claim of Proposition 6 follows then already from Lemma 3.

Next, we state Lemma 4, which analyzes the non-binding IC-condition in a more general setting.

**Lemma 4.** Assume the usual regularity conditions, \( -\rho \sigma_K \lambda \ell_{\text{max}} < \sigma_X \lambda s_{\text{max}} \) and \( R, p(\overline{w}) > 0 \). Further, assume that \( \overline{w} > \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma_K}{\sigma_X} \) and that \( \ell(w) = \ell_{\text{max}} = \ell^{\text{FB}} \) in a left-neighbourhood of \( \overline{w} \). Then the following holds true:

i) There exists \( w' \geq \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma_K}{\sigma_X} \lambda s(w') \), such that for all \( w \in [w', \overline{w}] \) long-run effort satisfies \( \ell(w) = \ell_{\text{max}} \). Further, the IC-condition \( \beta^\ell(w) \geq \lambda \ell_{\text{max}} \) is not tight on \( [w', \overline{w}] \), in that \( \beta^\ell(w) = w - \rho^+ \frac{\sigma_K}{\sigma_X} \lambda s > \lambda \ell_{\text{max}} \).

ii) Suppose \( s(w) = s_{\text{max}} \) for all \( w \geq \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma_K}{\sigma_X} \lambda s_{\text{max}} \) as well as \( \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma_K}{\sigma_X} \lambda s_{\text{max}} \geq w^* \). Then \( \beta^\ell(w) = w - \rho^+ \frac{\sigma_K}{\sigma_X} \lambda s_{\text{max}} > \lambda \ell_{\text{max}} \), and therefore \( \ell(w) = \ell_{\text{max}} \) for all \( w > \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma_K}{\sigma_X} \lambda s_{\text{max}} \).

iii) If \( \rho = 0 \), then \( \beta^\ell(w) = w \) and \( \ell(w) = \ell_{\text{max}} \) for all \( w \geq \max \{ \lambda \ell_{\text{max}}, w^* \} \).

**Proof.** i) Note that Lemma 2 implies that \( s(\cdot) \) must be continuous in a (left-) neighbourhood of \( \overline{w} \). Hence, the existence of the desired \( w' \geq \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma_K}{\sigma_X} \lambda s(w') \) follows from
the assumption
\[
\bar{w} > \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma X}{\sigma K} \lambda_s s(\bar{w}) = \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma X}{\sigma_K}
\]
and Lemma 3.

ii) Let us suppose to the contrary that there exists \( w' \geq \lambda \ell_{\text{max}} + \rho^+ \frac{\sigma_X}{\sigma_K} \lambda_s s_{\text{max}} \) with \( \ell(w') < \ell_{\text{max}} \). The optimality of \( \ell(w') \) implies
\[
p(w') + p'(w')(\lambda \ell_{\text{max}} - w') \leq 0
\]
and therefore \( p'(w') > 0 \), \( w' > \lambda \ell_{\text{max}} \) due to \( p(w) \geq \max\{p(0), p(\bar{w})\} > 0 \). This contradicts the concavity of \( p(\cdot) \) and the assumption that \( w' \geq w^* \). Again Lemma 3 yields the claim.

iii) If there existed \( w \geq \max\{\lambda \ell_{\text{max}}, w^*\} \) such that \( \ell(w) < \ell_{\text{max}} \), it would readily follow that \( p'(w) > 0 \) if \( w > w^* \) or \( p(w) = 0 \) if \( w = w^* \), a contradiction. Hence, the result follows by Lemma 3.

\[\square\]

**F Solution with One-dimensional Moral Hazard**

In this section, we solve the model when moral hazard is one-dimensional. Further, we assume that the cost of short- (long-) run effort is directly paid by the principal and \( \phi = 0 \), when short- (long-) run effort is observable, in order to connect to the models in He (2009) and DeMarzo et al. (2012) respectively. The specification of the cost does not impact short- (long-) run incentives, when short- (long-) run effort is observable. 21

**F.1 Moral Hazard Only over the Short-run**

Assume that the process \{\ell\} is observable for the principal and thus contractible, yet there are still both types of cash-flow shocks present. Formally, the (public) information filtration is given by

\[
\mathbb{F}_t^\ell \equiv \{\mathcal{F}_t^\ell : t \geq 0\} \quad \text{with} \quad \mathcal{F}_t^\ell = \sigma(X_s, K_s, \ell_s : 0 \leq s \leq t) = \mathcal{F}_t \wedge \sigma(\ell_s : 0 \leq s \leq t).
\]

21 As will become clear later, even without this assumption the proofs of Lemma 5 - because long-run effort is fixed there by hypothesis - and Lemma 6 - which is even proved for the baseline model - remain valid. From there it readily follows that the proof of Proposition 7 is not affected by this assumption either.
By standard arguments, one obtains the HJB-equation
\[
rp(w) = \max_{s,\ell,\beta^s,\beta^\ell} \left\{ \alpha s + \mu \ell p(w) - \lambda s \alpha \ell^2 + p'(w)w(\gamma - \mu \ell) + \frac{p'(w)}{2} \lambda_s \ell^2 \right. \\
+ \frac{p''(w)}{2} \left[ (\beta^s \sigma_X)^2 + \sigma_K^2 (\beta^\ell - w)^2 + 2 \rho \sigma_X \sigma_K \beta^s (\beta^\ell - w) \right] \},
\]
subject to \( p(0) = R, p'(w) = -1 \) and \( p''(w) = 0 \). In contrast to the baseline model, the maximization is only subject to the IC-condition \( \beta^s \geq \lambda_s s \), where the inequality is tight if \( s < s_{\text{max}} \). Without loss of generality, we focus on interior levels, i.e. assume for all \( w \) that \( s(w) \in (0, s_{\text{max}}) \).

The unconstrained maximization over \( \beta^\ell \) yields now the optimal value \( \beta^\ell = \beta^\ell(w) = w - \frac{\sigma_X}{\sigma_K} \beta^s \). Whence,
\[
\Sigma(w) \equiv \frac{\nabla (dw)}{dt} = \frac{[dw, dw]}{dt} = (\beta^s \sigma_X)^2 + \sigma_K^2 (\beta^\ell - w)^2 + 2 \rho \sigma_X \sigma_K \beta^s (\beta^\ell - w) = (\beta^s \sigma_X)^2 (1 - \rho^2).
\]
The optimal values of \( s, \ell \) are now given by
\[
s(w) = s = \frac{\alpha}{-p'(w) \lambda_s \alpha - p''(w) \lambda_s \sigma_X^2 (1 - \rho^2)} \quad \text{and} \quad \ell(w) = \ell = \frac{p(w) - wp'(w)}{\lambda_\ell}.
\]
Because \( p'(w) \geq -1 \), it follows that \( p(w) - wp'(w) \leq p(w) + w < p^{FB} \), which implies that \( \ell(w) \leq \ell^{FB} \), where the inequality is tight if \( \ell^{FB} < \ell_{\text{max}} \).

Because \( s^{FB} = 1/\lambda_s \), it follows that \( s(w) < s^{FB} \). Furthermore, a necessary and sufficient condition for \( s(w) < s^{FB} \) for all \( w \in [0, \bar{w}] \) is given by \( p'(w) \alpha + p''(w) \sigma_X^2 (1 - \rho^2) \leq -1 \) for \( w \in [0, \bar{w}] \). Since correlation \( \rho \) is ‘exploited’ to dampen the magnitude of transitory shocks \( \{Z^X\} \), the solution of the model with parameters \( (\sigma_X, \sigma_K, \rho) \) and observable \( \{\ell\} \) is isomorphic to the solution of the model with parameters \( (\sigma_X \sqrt{1 - \rho^2}, 0, 0) \), in that the value functions will be identical.

**F.2 Moral Hazard Only over the Long-run**

Assume that the process \( \{s\} \) is observable by the principal and thus contractible. Formally, the (public) information filtration is given by
\[
\mathbb{F}^s = \{ \mathcal{F}^s_t : t \geq 0 \} \quad \text{with} \quad \mathcal{F}^s_t = \sigma(X_k, K_k, s_k : 0 \leq k \leq t) = \mathcal{F}_t \vee \sigma(s_k : 0 \leq k \leq t).
\]

One obtains the HJB-equation
\[
rp(w) = \max_{s,\ell,\beta^s,\beta^\ell} \left\{ \alpha s + \mu \ell p(w) - \lambda s \alpha \ell^2 + p'(w)w(\gamma - \mu \ell) + \frac{p'(w)}{2} \lambda_s \ell^2 \right. \\
+ \frac{p''(w)}{2} \left[ (\beta^s \sigma_X)^2 + \sigma_K^2 (\beta^\ell - w)^2 + 2 \rho \sigma_X \sigma_K \beta^s (\beta^\ell - w) \right] \},
\]
subject to \( p(0) = R, \ p'(\bar{w}) = -1 \) and \( p''(\bar{w}) = 0 \). The unconstrained maximization over \( \beta^s \) gives \( \beta^s = \rho_{\sigma_X}^s (w - \beta^\ell) \). Whence,

\[
\Sigma(w) \equiv \frac{V(dw)}{dt} = \frac{dw, dw}{dt} = \left( \beta^s \sigma_X \right)^2 + \sigma_K^2 (\beta^\ell - w)^2 + 2 \rho \sigma_X \sigma_K \beta^s (\beta^\ell - w) = \sigma_K^2 (\beta^\ell - w)^2 (1 - \rho^2).
\]

The optimal values of \( s, \ell \), if interior, are now given by

\[
s(w) = s = \frac{1}{\lambda_s} \quad \text{and} \quad \ell(w) = \frac{\mu(p(w) - p'(w)w) - p''(w)w \lambda \ell \sigma_K^2 (1 - \rho^2)}{-p'(w) \lambda \mu - p''(w) \lambda \ell \sigma_K^2 (1 - \rho^2)}.
\]

Note that the solution in this model is in general well behaved, in that there are no points \( w \), for which \( \beta^\ell(w) = w \) and the second-order ODE collapses to one of the first-order. This is guaranteed by Proposition 7, to be proven in Appendix F. Since correlation \( \rho \) is ‘exploited’ to dampen the magnitude of transitory shocks \( \{Z^K\} \), the solution of the model with parameters \( (\sigma_K, \sigma_X, \rho) \) and observable \( \{s\} \) is isomorphic to the solution of the model with parameters \( (\sigma_K \sqrt{1 - \rho^2}, 0, 0) \), in that the value functions will be identical.

**G Proof of Proposition 7**

The claim regarding observable long-run effort \( \{\ell\} \) follows directly from Appendix E. We therefore only prove the claim regarding observable short-run effort \( \{s\} \). We split the proof in two parts.

**G.1 Part 1 - Auxiliary Results**

In this section, we establish two auxiliary results, that will be used in the second part of the proof of Proposition 7. It suffices to prove the claims for \( \phi = 0 \), since both \( s \) and \( \ell \) remain constant in the key part of the argument.

**Lemma 5.** Assume the usual regularity conditions and suppose that \( \{s\} \) is publicly observable (e.g. \( \sigma_X = 0 \)). Then short-run effort \( s(w) \) is contractible and constant over time. If in addition \( \ell(w) = \ell_{\text{max}} \) in a left neighbourhood of \( \bar{w} \), then it must be that \( \bar{w} < \lambda_{\ell} \ell_{\text{max}} \).

**Proof.** Without loss of generality, we normalize \( s(w) = s \equiv 1 \) for all \( w \), i.e. set \( \alpha = \lambda_s = 1 \). Further, in light of Appendix E it is obvious, that it suffices to prove the claim for \( \sigma_X = 0 \).

i) Define \( \ell(\bar{w}) = \ell = \ell_{\text{max}} \) and by assumption \( \ell(w) = \ell \) in an (open) left neighbourhood of \( \bar{w} \).

Let us first show that \( \lambda_{\ell} \ell \neq \bar{w} \). Suppose to the contrary \( \lambda_{\ell} \ell = \bar{w} \). Then

\[
p(\bar{w}) = \frac{1}{r - \mu \ell} \left( 1 - \frac{1}{2} \left( \lambda_s + \lambda_{\ell} \ell^2 \mu \right) - \bar{w} (\gamma - \mu \ell) \right).
\]

Let \( \varepsilon > 0 \) and consider the Taylor-expansion of \( p(\bar{w} - \varepsilon) \) around \( p(\bar{w}) \), given by \( p(\bar{w} - \varepsilon) = p(\bar{w}) + \varepsilon + o(\varepsilon^2) \). Further, define \( \ell_{\varepsilon} \equiv \ell(\bar{w} - \varepsilon) \) and note that in optimum
\[ \beta^\ell (\bar{w} - \varepsilon) = \lambda_\ell \ell. \] Hence,

\[
(r - \mu \ell_\varepsilon)p(\bar{w} - \varepsilon) = 1 - \frac{1}{2}\lambda_\varepsilon - p'(\bar{w} - \varepsilon)\left(\frac{1}{2}\lambda_\ell \ell_\varepsilon \mu + (\gamma - \mu \ell_\varepsilon)(\bar{w} - \varepsilon)\right) + \frac{\sigma^2_\ell (\lambda_\ell \ell_\varepsilon - \bar{w} + \varepsilon)^2}{2} p''(\bar{w} - \varepsilon) \
= 1 - \frac{1}{2}\lambda_\varepsilon + \left(1 + o(\varepsilon^2)\right)\left(\frac{1}{2}\lambda_\ell \ell_\varepsilon \mu + (\gamma - \mu \ell_\varepsilon)(\bar{w} - \varepsilon)\right) + \frac{\sigma^2_\ell (\lambda_\ell \ell_\varepsilon - \bar{w} + \varepsilon)^2}{2} p''(\bar{w} - \varepsilon),
\]

where we used that \( p'(\bar{w} - \varepsilon) = p'(\bar{w}) - \varepsilon p''(\bar{w}) + o(\varepsilon^2). \)

Combining the above yields

\[
p(\bar{w} - \varepsilon)\mu(\ell_\varepsilon - \ell) = \varepsilon(r - \mu \ell) + (\gamma - \mu \ell_\varepsilon)(\bar{w} - \varepsilon) - \bar{w}(\gamma - \mu \ell) + \frac{1}{2}\mu \lambda_\ell (\ell_\varepsilon^2 - \ell^2) - \frac{\sigma^2_\ell (\lambda_\ell \ell_\varepsilon - \bar{w} + \varepsilon)^2}{2} p''(\bar{w} - \varepsilon) + o(\varepsilon^2) + o(\varepsilon^3).
\]

Next, note that \( \ell = \ell_\varepsilon + \varepsilon \ell'(\bar{w} - \varepsilon) + o(\varepsilon^2), \) in case \( \ell(\cdot) \) is differentiable, which is guaranteed for \( \varepsilon > 0 \) sufficiently small. This yields

\[
\mu p(\bar{w} - \varepsilon)(-\varepsilon \ell'(\bar{w} - \varepsilon)) = \varepsilon(r - \gamma) - \bar{w} \mu \varepsilon \ell'(\bar{w} - \varepsilon) + o(\varepsilon^2) \quad \iff \quad o(\varepsilon) - \mu(p(\bar{w} - \varepsilon) + \bar{w}) \ell'(\bar{w} - \varepsilon) = r - \gamma.
\]

However, it follows that for \( \varepsilon > 0 \) sufficiently close to zero \( \ell'(\bar{w} - \varepsilon) = 0, \) a contradiction.

**ii)** Next, suppose \( \bar{w} > \lambda_\ell \ell_{\max}. \) Thus, there exists \( w' \geq \lambda_\ell \ell_{\max} \) with \( \bar{w} > w' \) such that \( \ell \equiv \ell(w) = \ell_{\max} \) for all \( w \geq w'. \) In optimum, \( \beta^\ell(w') = w'. \) Then,

\[
p(w') < p(\bar{w}) - (w' - \bar{w}) = \frac{1}{r - \mu \ell}\left(0.5 - \frac{\lambda_\ell \ell_\varepsilon^2 \mu}{2} - \bar{w}(\gamma - r) - w'(r - \mu \ell)\right) < \frac{1}{r - \mu \ell}\left(0.5 - \frac{\lambda_\ell \ell_\varepsilon^2 \mu}{2} - w'(\gamma - r) - w'(r - \mu \ell)\right) = \frac{1}{r - \mu \ell}\left(0.5 - \frac{\lambda_\ell \ell_\varepsilon^2 \mu}{2} - w'(\gamma - \mu \ell)\right),
\]

where the first inequality is due to strict concavity and the second one due to \( w' < \bar{w}. \) However, as the 'firm' becomes 'riskless' at \( w', \) we get that

\[
p(w') \geq \frac{1}{r - \mu \ell}\left(0.5 - \frac{\lambda_\ell \ell_\varepsilon^2 \mu}{2} - w'(\gamma - \mu \ell)\right).
\]

The inequality stems from the fact that constant, scaled payouts at rate \( w'(\gamma - \mu \ell) + \frac{\lambda_\ell \ell_\varepsilon^2 \mu}{2} \) and this way keeping \( w_t = w' \) constant for all future times \( t \) is always an option but not necessarily optimal. This yields the desired contradiction.
While the previous result is only valid under the assumption that short-run effort \( \{s\} \) is observable, we state now a related claim for interior levels and \( \phi = 0 \), which does not hinge on the observability of \( \{s\} \).

**Lemma 6.** Assume the usual regularity conditions and \( \min\{R, p(\overline{w})\} > 0 \). If in addition \( \rho \leq 0 \) and \( \ell(w) < \ell_{\text{max}} \) for all \( w \in [0, \overline{w}] \), it holds that \( \beta^\ell(w) > w \) or equivalently \( \ell(w) > w/\lambda_\ell \) for all \( w \in [0, \overline{w}] \).

**Proof.** Let us suppose to the contrary that there exists \( w_0 \in [0, \overline{w}] \), such that \( \beta^\ell(w_0) \leq w_0 \). By assumption, \( \ell(w_0) < \ell_{\text{max}} \) and by (14) (and \( \rho \leq 0 \)) it is evident, that \( \ell(w_0) > 0 \). Hence, \( \ell_0 = \ell(w_0) \) solves \( \frac{\partial p(w_0)}{\partial \ell} = 0 \), given \( \beta^s \). We may without loss of generality assume that \( \beta^s \geq 0 \). Hence,

\[
\mu [p(w_0) - w_0p'(w_0)] + \mu \lambda_\ell \ell_0 p'(w_0) + p''(w_0) \left[ \lambda_\ell (\lambda_\ell \ell_0 - w_0) + \rho \sigma_X \sigma_K \lambda_\ell \beta^s \right] = 0.
\]

Because \( \beta^\ell(w_0) = \lambda_\ell \ell_0 \leq w_0 \) and \( p''(w_0) \leq 0 \), it is immediate that

\[
p(w_0) + p'(w_0)(\lambda_\ell \ell_0 - w_0) \leq 0,
\]

which implies - due to \( p(w_0) \geq \min\{R, p(\overline{w})\} > 0 \) - that \( p'(w_0) > 0 \) as well as \( \lambda_\ell \ell_0 < w_0 \). Hence, \( w_0 < w^* \). Next, observe that at \( w = 0 \), it must be \( \ell(0) > 0 \) due to \( p(0) = R > 0 \). Because long-run effort is interior for all \([0, \overline{w}]\), the mapping \( \ell(\cdot) \) is continuous on the same interval (due to \( p \in C^2 \)). Hence, the function \( \chi : w \mapsto \lambda_\ell \ell(w) = w_0 \), defined on \([0, \overline{w}]\), is also continuous and must therefore have a root \( w_1 \) on \((0, w_0)\). However, the optimality of \( \ell(w_1) = w_1/\lambda_\ell \) implies \( p(w_1) \leq 0 \), which contradicts \( p(w_1) \geq \min\{R, p(\overline{w})\} > 0 \). \( \square \)

**G.2 Part 2**

We finalize now the proof of Proposition 7. Recall that the claim is stated for \( \phi = 0 \), but it would be straightforward to extend the result for general \( \phi \).

**Proof.** Without loss of generality, we normalize in the following \( \sigma_X \) to zero.

i) As a first step, we show that if there exists \( w' \in [0, \overline{w}] \) with \( \beta^\ell(w') = w' \), then it must be that \( \ell(w) = \ell_{\text{max}} \) for all \( w \in [w', \overline{w}] \) and \( \overline{w} > \lambda_\ell \ell_{\text{max}} \).

Hence, assume there is \( w' \in [0, \overline{w}] \), such that \( \beta^\ell(w') = w' \). Then, by Lemma 6 it follows that \( \ell(w') = \ell_{\text{max}} \). Hence, \( \lambda_\ell \ell_{\text{max}} \leq w' = \beta^\ell(w') \) as well as \( \overline{w} > \lambda_\ell \ell_{\text{max}} \). Observe that for \( \ell(w') = \ell_{\text{max}} \) to be optimal it must hold that

\[
p(w') + (\lambda_\ell \ell_{\text{max}} - w')p'(w') \geq 0.
\]

However, by strict concavity of \( p(\cdot) \) on \([w', \overline{w}]\), it follows for all \( w \in (w', \overline{w}] \) that

\[
\frac{\partial p(w)}{\partial \ell} \bigg|_{\ell = \ell_{\text{max}}} = p(w) + p'(w)(\lambda_\ell \ell_{\text{max}} - w) > p(w') + (\lambda_\ell \ell_{\text{max}} - w')p'(w') \geq 0.
\]

This readily implies that \( \ell(w) = \ell_{\text{max}} \) for all \( w \in [w', \overline{w}] \).
ii) Next, let us prove that $\beta^\ell(w') = w'$ cannot be for $w' < \overline{w}$. This can be proven by replicating part ii) of the proof of Lemma 5, because we have already shown that $\beta^\ell(w') = w'$ implies $\ell(\cdot) \equiv \ell_{\text{max}}$ on $[w', \overline{w}]$ and $\overline{w} > \lambda\ell_{\text{max}}$.

iii) To complete the proof, we establish that $\overline{w} \neq \beta^\ell(\overline{w})$. Hence, suppose to the contrary that $\overline{w} = \beta^\ell(\overline{w})$. Then, Lemma 6 implies that $\ell(w') \equiv \ell_{\text{max}}$ and therefore $\overline{w} \geq \lambda\ell_{\text{max}}$.

First, let us rule out $w = \lambda\ell_{\text{max}}$. If this were the case, we are able to establish - replicating the argumentation from part i) of the proof of Lemma 5 - that it must be $\lim_{w \uparrow \overline{w}} \left( o(\varepsilon) - \mu(p(\overline{w} - \varepsilon) + \overline{w}) \ell'(\overline{w} - \varepsilon) \right) = r - \gamma$.

If it were $\ell(w) = \ell_{\text{max}}$ in a left neighbourhood of $w_{\text{max}}$, the claim follows by Lemma 5. Thus, we may assume this is not the case.

Then, there exists for any $\delta > 0$ a value $w' \in (\overline{w} - \delta, \overline{w})$, such that $\ell(w') < \ell_{\text{max}}$. It follows that $\ell(w) < \ell_{\text{max}}$ on an interval $(\overline{w} - \delta, \overline{w})$, because $\ell(w') = \ell_{\text{max}}$ for $w' < \overline{w}$ implies $\ell(w) = \ell_{\text{max}}$ on $[w', \overline{w}]$ by part i) of the proof.

Further, by Lemma 6, it must be that $\ell(w) > w/\lambda$ in a left neighbourhood of and excluding $\overline{w}$, i.e. the interval $(\overline{w} - \delta, \overline{w})$. It follows that $p''(\cdot)$ exists on $(\overline{w} - \delta, \overline{w})$, with $\lim_{w \uparrow \overline{w}} p''(w) = \infty$. But then also

$$\lim_{w \uparrow \overline{w}} \frac{\partial \ell'(x)}{\partial x} \bigg|_{x=w} = \lim_{w \uparrow \overline{w}} \ell'(w) = \lim_{w \uparrow \overline{w}} \frac{\sigma_K^2 p(w) p''(w)}{\mu} = \infty.$$ 

However, this implies that

$$o(\varepsilon) - \mu(p(\overline{w} - \varepsilon) + \overline{w}) \ell'(\overline{w} - \varepsilon) = r - \gamma.$$ 

cannot hold for all $\varepsilon > 0$ due to $p(\overline{w}) + \overline{w} > 0$, a contradiction.

Second, we show that $\overline{w} > \lambda\ell_{\text{max}}$ cannot occur. Suppose now to the contrary $\overline{w} > \lambda\ell_{\text{max}}$. By part ii) of the proof, there cannot be $w' \in [\lambda\ell_{\text{max}}, \overline{w})$ such that $\ell(w') = \ell_{\text{max}}$. Hence, it follows that $\ell(w) < \ell_{\text{max}}$ for all $w \in [\lambda\ell_{\text{max}}, \overline{w})$. However, this implies that

$$p(w) + p'(w)(\lambda\ell_{\text{max}} - w) < 0$$

holds for all $w \in [\lambda\ell_{\text{max}}, \overline{w})$. Whence, $p'(w) > 0$ for all $w \in [\lambda\ell_{\text{max}}, \overline{w})$, a contradiction to $p'(\overline{w}) = -1$.

Combining Proposition 7 and Lemma 5, we even obtain that $\beta^\ell(w) > w$, whenever $\rho \leq 0$.

### H Return Volatility

The principal’s instantaneous return $dR_t$ at time $t$ is given by

$$dR_t = dX_t + dP(W_t, K_t),$$

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that is by the change in cash-flow and contract value. By Itô’s Lemma:
\[
dX_t + dP(W_t, K_t) = rP(W_t, K_t)dt + \sigma_X K_t dZ_t^X \\
+ P_W(W_t, K_t) \left( \beta_t^e \sigma_K K_t dZ_t^K + \beta_t^s \sigma_X K_t dZ_t^X \right) + P_K(W_t, K_t) \sigma_K K_t dZ_t^K,
\]
where we used that
\[
rP(W_t, K_t)dt = E \left[ dX_t + dP(W_t, K_t) \right] \iff (r - \mu_t) p(w_t) = \alpha s_t + \frac{\tilde{E} \left[ dp(w_t) \right]}{dt}
\]
holds in optimum by the HJB-equation (9). Note that the expectation \(\tilde{E} [\cdot]\) is taken under the auxiliary, equivalent probability measure \(\tilde{P}\), defined in Appendix A.3.1. Next, observe that \(P(W_t, K_t) = K_t p(w_t)\) and therefore \(P_W(W_t, K_t) = p'(w_t)\) as well as \(P_K(W_t, K_t) = p(w_t) - w_t p'(w_t)\). Consequently, straightforward calculations yield
\[
\frac{dR_t}{P(W_t, K_t)} = rdt + \frac{1 + \beta_t^s p'(w_t)}{p(w_t)} \sigma_X dZ_t^X + \frac{p(w_t) + (\beta_t^e - w_t) p'(w_t)}{p(w_t)} \sigma_K dZ_t^K \\
= rdt + \Sigma_X^* (w) dZ_t^X + \Sigma_K^* (w) dZ_t^K,
\]
where
\[
\Sigma_X^*(w) = \frac{1 + \beta_t^s p'(w)}{p(w)} \sigma_X \text{ and } \Sigma_K^*(w) = \sigma_K + \frac{p'(w) (\beta_t^e - w)}{p(w)} \sigma_K.
\]
The squared volatility term is given by
\[
(\Sigma_p^*)^2 \equiv \frac{\nabla (dR_t)}{P(W_t, K_t)^2 dt} = \frac{[dR_t, dR_t]}{P(W_t, K_t)^2 dt}.
\]
References


