Linear-Rational Term Structure Models

DAMIR FILIPOVIĆ, MARTIN LARSSON, and ANDERS B. TROLLE∗

ABSTRACT

We introduce the class of linear-rational term structure models in which the state price density is modeled such that bond prices become linear-rational functions of the factors. This class is highly tractable with several distinct advantages: (i) ensures non-negative interest rates, (ii) easily accommodates unspanned factors affecting volatility and risk premiums, and (iii) admits semi-analytical solutions to swaptions. A parsimonious model specification within the linear-rational class has a very good fit to both interest rate swaps and swaptions since 1997 and captures many features of term structure, volatility, and risk premium dynamics—including when interest rates are close to the zero lower bound.

The current environment with very low interest rates creates difficulties for many existing term structure models, most notably Gaussian or conditionally Gaussian models that invariably place large probabilities on negative future interest rates. Models that respect the zero lower bound (ZLB) on interest rates exist but often have limited ability to accommodate unspanned factors affecting volatility and risk premiums or to price many types of interest rate derivatives. In light of these limitations, the purpose of this paper is twofold:

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First, we introduce a new class of term structure models, the linear-rational, which is highly tractable and (i) ensures nonnegative interest rates, (ii) easily accommodates unspanned factors affecting volatility and risk premiums, and (iii) admits semi-analytical solutions to swaptions—an important class of interest rate derivatives that underlie the pricing and hedging of mortgage-backed securities, callable agency securities, life insurance products, and a wide variety of structured products. Second, we perform an extensive empirical analysis of a set of parsimonious model specifications within the linear-rational class.

The first contribution of the paper is to introduce the class of linear-rational term structure models. A sufficient condition for the absence of arbitrage opportunities in a model of a financial market is the existence of a state price density; that is, a positive adapted process \( \zeta_t \) such that the price \( \Pi(t, T) \) at time \( t \) of any time-\( T \) cash flow \( C_T \) is given by\(^1\)

\[
\Pi(t, T) = \frac{1}{\zeta_t} E_t[\zeta_T C_T].
\] (1)

Following Constantinides (1992), our approach to modeling the term structure is to directly specify the state price density. Specifically, we assume a multivariate factor process \( Z_t \), which has a linear drift, and a state price density, which is a linear function of \( Z_t \). In this case, bond prices and the short rate become linear-rational functions—that is, ratios of linear functions—of \( Z_t \), which is why we refer to the framework as linear-rational. We show that one can easily ensure that the short rate stays nonnegative.\(^2\)

We distinguish between factors that are spanned by the term structure and those that are unspanned, and we provide conditions such that all of the factors in \( Z_t \) are spanned. A key feature of the framework is that the term structure depends only on the drift of \( Z_t \). This leaves freedom to specify exogenous factors feeding into the martingale part of \( Z_t \). Such factors give rise to unspanned stochastic volatility (USV) and can be recovered from bond derivatives prices. We further distinguish between USV factors that directly affect the instantaneous bond return covariances and those that affect expected future bond return covariances.

Within the linear-rational framework we show how to construct a model in which \( Z_t \) is \( m \)-dimensional and there are \( n \leq m \) USV factors. The joint factor process is affine, and swaptions can be priced semi-analytically. This model is termed the linear-rational square-root (LRSQ) model. We also discuss an extension of the state price density specification that allows for much richer risk premium dynamics than the baseline model. It also allows for the introduction

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\(^1\) Throughout, we assume there is a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) on which all random quantities are defined, and \( E_t[\cdot] \) denotes \( \mathcal{F}_t \)-conditional expectation.

\(^2\) While zero is a natural lower bound on nominal interest rates, any lower bound is accommodated by the framework. In the United States, the Federal Reserve kept the federal funds rate in a range between 0 and 25 basis points from December 2008 to December 2015, and other money market rates mostly remained nonnegative during this period. However, in the Eurozone (as well as in Denmark, Sweden, and Switzerland), the ZLB assumption has recently been challenged as both policy rates and money market rates have moved into negative territory.
of unspanned risk premium factors, although this is not a focus of our empirical analysis.

The second contribution of the paper is an extensive empirical analysis of the LRSQ model. We use a panel data set consisting of both swaps and swaptions from January 1997 to August 2013. At a weekly frequency, we observe a term structure of swap rates with maturities from 1 year to 10 years as well as a surface of at-the-money implied volatilities of swaptions with swap maturities from 1 year to 10 years and option expiries from 3 months to 5 years. The estimation approach is quasi-maximum likelihood in conjunction with the Kalman filter. The term structure is assumed to be driven by three factors, and we vary the number of USV factors between one and three. A robust feature across all specifications is that parameters align such that under the risk-neutral measure and after a normalization of the factor process, the short rate mean-reverts to a factor that affects the intermediate part of the term structure (a curvature factor), which in turn mean-reverts toward a factor that affects the long end of the term structure (a slope factor). The preferred specification has three USV factors and simultaneously fits both swaps and swaptions well. This result continues to hold for the part of the sample period in which short-term rates are very close to the ZLB.

Using long samples of simulated data, we investigate the ability of the model to capture the dynamics of the term structure, volatility, and swap risk premiums. First, the model captures important features of term structure dynamics near the ZLB. Consistent with the data, the model generates extended periods of very low short rates. Furthermore, when the short rate is close to zero, the model generates highly asymmetric distributions of future short rates, with the most likely values of future short rates being significantly lower than the mean values. Related to this, the model also replicates how the first principal component of the term structure changes from a level factor during normal times to more of a slope factor during times of near-zero short rates.

Second, the model captures important features of volatility dynamics near the ZLB. Previous research shows that a large fraction of variation in volatility is effectively unrelated to variation in the term structure. We provide an important qualification to this result: volatility becomes compressed and gradually more level-dependent as interest rates approach the ZLB. This is illustrated by Figure 1, which shows the 3-month implied volatility of the 1-year swap rate plotted against the level of the 1-year swap rate. More formally, for each swap maturity, we regress weekly changes in the 3-month implied volatility of the swap rate on weekly changes in the level of the swap rate. Conditional on swap rates being close to zero, the regression coefficients are positive, large in magnitude, and very highly statistically significant, and the $R^2$s are around 0.50. However, as the level of swap rates increases, the relation between volatility and swap rate changes becomes progressively weaker, and volatility exhibits very little level-dependence at moderate levels of swap rates. Capturing these dynamics—strong level-dependence of volatility near the ZLB and predominantly USV at higher interest rate levels—poses a significant challenge for existing dynamic term structure models. Our model successfully meets this
challenge because it simultaneously respects the ZLB on interest rates and incorporates USV.

Third, the model captures several characteristics of risk premiums in swap contracts. We consider realized excess returns on zero-coupon bonds bootstrapped from the swap term structure and show that in the data the unconditional mean and volatility of excess returns increase with bond maturity, but in such a way that the unconditional Sharpe ratio decreases with bond maturity. We also find that implied volatility is a robust predictor of excess returns, while the predictive power of the slope of the term structure is relatively weak in our sample. The model largely captures unconditional risk premiums

\footnote{This result differs from a large literature on the predictability of excess bond returns in the Treasury market. The reason is likely some combination of our more recent sample period, our use of forward-looking implied volatilities, and structural differences between the Treasury and swap markets. As we note later, a key property of many equilibrium term structure models is a positive risk-return trade-off in the bond market, which is consistent with our results.}
and, as the dimension of USV increases, has a reasonable fit to conditional risk premiums.4

The linear-rational framework is related to the linearity-generating (LG) framework studied in Gabaix (2009) in which bond prices are linear functions of a set of factors.5 A specific LG term structure model is analyzed by Carr, Gabaix, and Wu (2009). However, the factor process in their model is time-inhomogeneous and nonstationary, while the one in the LRSQ model is time-homogeneous and stationary. Also, the volatility structure is very different in the two models, and while bond prices are perfectly correlated in the Carr, Gabaix, and Wu (2009) model, they exhibit a truly multifactor structure in the LRSQ model.

The exponential-affine framework—see, for example, Duffie and Kan (1996) and Dai and Singleton (2000)—is arguably the dominant one in the term structure literature. In this framework, one can ensure nonnegative interest rates (which requires all factors to be of the square-root type) or accommodate USV (which requires at least one conditionally Gaussian factor, see Joslin (2014)), but not both. Furthermore, no exponential-affine model admits semi-analytical solutions to swaptions.6 In contrast, the linear-rational framework accommodates all three features.7 The linear-rational framework also has the advantage that the term structure factors can be analytically inverted from coupon bond prices and swap rates, which are directly observable in the market. In contrast, in the exponential-affine framework, this can be done only from zero-coupon bond prices. Table I contrasts the two frameworks.8

We compare the LRSQ model with the exponential-affine model that relies on a multifactor square-root process. Because of the limitations of the latter model, we abstract from USV and swaption pricing to focus exclusively on the pricing of swaps. The two estimated models have a similar qualitative structure for the risk-neutral drift of the short rate, although the LRSQ model incorporates certain nonlinearities. Also, the factor loadings have similar shapes. Finally, the pricing performance of the two models is virtually identical despite the LRSQ model having a more parsimonious description of the term structure.

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4 The historical mean excess returns and Sharpe ratios are inflated by the downward trend in interest rates over the sample period. Indeed, the model-implied values are lower.
5 More generally, the linear-rational framework is related to the frameworks in Rogers (1997) and Flesaker and Hughston (1996).
6 Various approximation schemes for pricing swaptions have been proposed in the literature; see, for example, Singleton and Umantsev (2002) and the references therein.
7 Alternative frameworks that ensure nonnegative interest rates include the shadow-rate framework of Black (1995)—see, for example, Kim and Singleton (2012) for a recent application—and the exponential-quadratic framework of Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2002). Neither of these frameworks accommodates USV or admits semi-analytical solutions to swaptions.
8 Cieslak and Povala (2016) estimate a non-USV affine model using information on yield volatility. Their estimated volatility factors are effectively unspanned by the term structure. We prefer to impose USV because it results in a more parsimonious model and does not adversely affect the ability of the model to fit term structure dynamics.
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Table I
Comparison of Exponential-Affine and Linear-Rational Frameworks
The table contrasts the two frameworks along key dimensions. ZCB stands for zero-coupon bond, and LR stands for linear-rational. In the exponential-affine framework, respecting the zero lower bound (ZLB) on interest rates is only possible if all factors are of the square-root type, and accommodating unspanned stochastic volatility (USV) is only possible if at least one factor is conditionally Gaussian.

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<th>Exponential-affine</th>
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<td>ZCB price</td>
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<td>Analytical factor inversion</td>
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The paper is structured as follows. Section I lays out the linear-rational framework. Section II describes the LRSQ model. Section III extends the state price density specification. The empirical analysis is in Section IV, and Section V concludes. All proofs are given in the Appendix, and an Internet Appendix contains supplementary results.9

I. The Linear-Rational Framework

In this section, we introduce the linear-rational framework and present explicit formulas for zero-coupon bond prices and the short rate. We next distinguish between term structure factors and USV factors. We further describe interest rate swaptions and derive a swaption pricing formula. Finally, we describe a normalization of the term structure factors that may be helpful for model interpretation, and we relate the linear-rational framework to existing models.

A. Term Structure Specification

A linear-rational term structure model consists of two components: a multivariate factor process $Z_t$ with a linear drift and state space $E \subset \mathbb{R}^m$, and a state price density $\xi_t$ given as a linear function of $Z_t$. Specifically, we assume that $Z_t$ has dynamics of the form

$$dZ_t = \kappa (\theta - Z_t) dt + dM_t$$

9 The Internet Appendix is available in the online version of the article on the Journal of Finance website.
for some $\kappa \in \mathbb{R}^{m \times m}$ and $\theta \in \mathbb{R}^m$, and for some $m$-dimensional martingale $M_t$. The state price density is assumed to be given by
\[ \zeta_t = e^{-\alpha t} (\phi + \psi^\top Z_t) \] (3)
for some $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^m$ such that $\phi + \psi^\top x > 0$ for all $x \in E$, and for some $\alpha \in \mathbb{R}$. As we discuss below, the role of the parameter $\alpha$ is to ensure that the short rate stays nonnegative. We extend the state price density specification (3), without affecting the pricing functions, in Section III.

The linear drift of $Z_t$ implies that conditional expectations are of the linear form (see Lemma A3 in the Appendix)
\[ \mathbb{E}_t [Z_T] = \theta + e^{-\kappa (T-t)}(Z_t - \theta), \quad t \leq T. \] (4)

An immediate consequence is that zero-coupon bond prices and the short rate become linear-rational functions of $Z_t$, which is why we refer to this framework as linear-rational. Indeed, the basic pricing formula (1) with $C_T = 1$ shows that zero-coupon bond prices are given by
\[ P(t, T) = F(T-t, Z_t), \]
where
\[ F(\tau, z) = e^{-\alpha \tau} \frac{\phi + \psi^\top \theta + \psi^\top e^{-\tau \kappa} (z - \theta)}{\phi + \psi^\top z}. \] (5)

The short rate is obtained via the formula
\[ r_t = -\partial_T \log P(t, T)|_{T=t} \]
and is given by
\[ r_t = \alpha - \frac{\psi^\top \kappa (\theta - Z_t)}{\phi + \psi^\top Z_t}. \] (6)

The latter expression clarifies the role of the parameter $\alpha$: provided that the short rate is bounded from below, we may guarantee that it stays nonnegative by choosing $\alpha$ large enough. This leads to an intrinsic choice of $\alpha$ as the smallest value that yields a nonnegative short rate. In other words, we define
\[ \alpha^* = \sup_{z \in E} \frac{\psi^\top \kappa (\theta - z)}{\phi + \psi^\top z} \quad \text{and} \quad \alpha_* = \inf_{z \in E} \frac{\psi^\top \kappa (\theta - z)}{\phi + \psi^\top z}, \]
and we set $\alpha = \alpha^*$, provided this is finite.\(^\text{10}\) The short rate then satisfies
\[ r_t \in [0, \alpha^* - \alpha_*]. \] (8)

Notice that $\alpha^*$ and $\alpha_*$ depend on the drift parameters of $Z_t$, which are estimated from data. Therefore, a crucial step in the model validation process is to verify that the range of possible short rates is sufficiently wide. Finally, whenever the eigenvalues of $\kappa$ have positive real part, one can verify that $(-1/\tau) \log F(\tau, z)$ converges to $\alpha$ when $\tau$ goes to infinity. That is, $\alpha$ can be interpreted as the infinite-maturity zero-coupon bond yield.

\(^{10}\) More generally, via an appropriate choice of $\alpha$, one can impose any lower bound on the short rate. One can also treat $\alpha$ as a free parameter and estimate the lower bound from term structure data. In general, the short rate satisfies $r_t \in [\alpha - \alpha^*, \alpha - \alpha_*]$. 

B. Term Structure Factors

The functional form of the term structure (5) depends only on the drift of $Z_t$. This leaves freedom to specify exogenous factors feeding into the martingale part of $Z_t$. Such factors would be unspanned by the term structure and give rise to USV. For a clear distinction between spanned and unspanned factors, we now provide conditions such that $Z_t$ does not itself exhibit unspanned components. Specifically, we first describe any direction $\xi \in \mathbb{R}^m$ such that the term structure remains unchanged when $Z_t$ moves along $\xi$.

**Definition 1:** The term structure kernel, denoted by $\mathcal{U}$, is given by

$$
\mathcal{U} = \bigcap_{\tau \geq 0, z \in E} \ker \nabla_z F(\tau, z),
$$

where $\nabla_z F(\tau, z)$ denotes the gradient with respect to the $z$ variables.

That is, $\mathcal{U}$ consists of all $\xi \in \mathbb{R}^m$ that are orthogonal to the factor loadings of the term structure in the sense that $\nabla_z F(\tau, z)^\top \xi = 0$ for all $\tau \geq 0$ and $z \in E$. Therefore, the location of $Z_t$ along the direction $\xi$ cannot be recovered solely from knowledge of the time-$t$ bond prices $P(t, T)$, $T \geq t$. The following result characterizes $\mathcal{U}$ in terms of the parameters $\kappa$ and $\psi$.

**Theorem 1:** Assume that the short rate $r_t$ is not constant. Then $\mathcal{U}$ is the largest subspace of $\ker \psi^\top$ that is invariant under $\kappa$. Formally, this is equivalent to

$$
\mathcal{U} = \text{span}\{\psi, \kappa^\top \psi, \ldots, \kappa^{(m-1)}\psi\}^\perp.
$$

If the term structure kernel is zero, $Z_t$ exhibits no unspanned directions and can be reconstructed from a snapshot of the term structure at time $t$, under mild technical conditions. In this case we refer to the components of $Z_t$ as term structure factors. The following theorem formalizes this fact.

**Theorem 2:** The term structure $F(\tau, z)$ is injective if and only if the term structure kernel is zero, $\mathcal{U} = \{0\}$, $\kappa$ is invertible, and $\phi + \psi^\top \theta \neq 0$.\(^{13}\)

Finally, note that even if the term structure kernel is zero, the short end of the term structure may nonetheless be insensitive to movements of $Z_t$ along

\(^{11}\) In view of (6), the short rate $r_t$ is constant if and only if $\psi$ is an eigenvector of $\kappa^\top$ with eigenvalue $\lambda$ satisfying $\lambda(\phi + \psi^\top \theta) = 0$. In this case, we have $r_t \equiv \alpha + \lambda$, and $\mathcal{U} = \mathbb{R}^m$, while the right-hand side of (9) equals $\ker \psi^\top$. The assumption that the short rate is not constant will be in force throughout the paper.

\(^{12}\) If the term structure kernel is nonzero with dimension $k$, we can linearly transform $Z_t$ such that the unspanned directions correspond to the last $k$ components of the transform of $Z_t$. The state price density and zero-coupon bond prices are then functions of the $m' = m - k$ first components, say $Z_t$, of the transform of $Z_t$. The process $Z_t$ has a linear drift and thus gives rise to a linear-rational model with a zero term structure kernel, which is observationally equivalent to the original model (2) and (3). Details are provided in the Internet Appendix.

\(^{13}\) Injectivity here means that if $F(\tau, z) = F(\tau, z')$ for all $\tau \geq 0$, then $z = z'$. In other words, if $F(\tau, Z_t)$ is known for all $\tau \geq 0$, we can back out $Z_t$.\(^{13}\)
certain directions. In view of Theorem 1, for \( m \geq 3 \) we can have \( \mathcal{U} = \{0\} \) while there still exists a nonzero vector \( \xi \) such that \( \psi^\top \xi = \psi^\top \kappa \xi = 0 \). This implies that the short rate is constant along \( \xi \); see (6).

C. Unspanned Stochastic Volatility Factors

We now specialize the linear-rational framework (2) and (3) to the case in which \( Z_t \) has diffusive dynamics of the form

\[
\text{d}Z_t = \kappa(\theta - Z_t)\text{d}t + \sigma(Z_t, U_t)\text{d}B_t
\]

for some \( n \)-dimensional USV factor process \( U_t \), some \( d \)-dimensional Brownian motion \( B_t \), and some \( \mathbb{R}^{m \times d} \)-valued dispersion function \( \sigma(z, u) \). An extension to more general martingales \( M_t \) including jumps is straightforward. The state space of the joint factor process \((Z_t, U_t)\) is a subset \( \mathcal{E} \subset \mathbb{R}^{m+n} \). We denote the diffusion matrix of \( Z_t \) by \( a(Z_t, U_t) = \sigma(Z_t, U_t)\sigma(Z_t, U_t)^\top \), and we assume that it is differentiable on \( \mathcal{E} \).

A short calculation using Itô’s formula shows that the dynamics of the state price density can be written as

\[
\frac{\text{d}\zeta_t}{\zeta_t} = -r_t \text{d}t - \lambda_t^\top \text{d}B_t,
\]

where the short rate \( r_t \) is given by (6), and

\[
\lambda_t = -\frac{\sigma(Z_t, U_t)^\top \psi}{\phi + \psi^\top Z_t}
\]

is the market price of risk. It then follows that the dynamics of \( P(t, T) \) are given by

\[
\frac{\text{d}P(t, T)}{P(t, T)} = (r_t + \nu(t, T)^\top \lambda_t) \text{d}t + \nu(t, T)^\top \text{d}B_t,
\]

where

\[
\nu(t, T) = \frac{\sigma(Z_t, U_t)^\top \nabla_z F(T - t, Z_t)}{F(T - t, Z_t)}.
\]

To see why the term structure is unaffected by the USV factors, consider the following bond price decomposition. The pricing formula (1) implies that the time-

\( t \)

price of a zero-coupon bond with maturity \( T \) equals its discounted expected price at a nearby future date \( t + dt \) plus the risk premium,

\[
P(t, T) = P(t, t + dt)\mathbb{E}_t[P(t + dt, T)] + \text{Cov}_t \left[ \frac{\zeta_{t + dt}}{\zeta_t}, P(t + dt, T) \right].
\]

While \( P(t, t + dt) = 1 - r_t dt \) does not depend on \( U_t \), the expected time-

\( t \)

price does depend on \( U_t \) due to the nonlinear dependence on \( Z_{t + dt} \),

\[
\mathbb{E}_t[P(t + dt, T)] = P(t, T) + \left( -\frac{\partial}{\partial t} F(T - t, Z_t) + \nabla_z F(T - t, Z_t)^\top \kappa(\theta - Z_t) + \frac{1}{2} \text{tr} \left( \nabla_z^2 F(T - t, Z_t) a(U_t, Z_t) \right) \right) dt.
\]
But this dependence is offset by the risk premium on the right-hand side of (13),

$$\text{Cov}_t \left[ \frac{\zeta_t + dt}{\zeta_t}, P(t + dt, T) \right] = -P(t, T)\nu(t, T)^\top \lambda_t dt.$$ (15)

Indeed, straightforward verification shows that, with market price of risk given by (11), we have

$$\frac{1}{2} \text{tr} \left( \nabla_z^2 F(T - t, Z_t) a(U_t, Z_t) \right) - P(t, T)\nu(t, T)^\top \lambda_t = 0.$$ (16)

We now refine the discussion of USV by identifying those USV factors that directly affect the instantaneous bond return covariances,

$$\text{Cov}_t \left[ \frac{dP(t, T_1)}{P(t, T_1)}, \frac{dP(t, T_2)}{P(t, T_2)} \right] = G(T_1 - t, T_2 - t, Z_t, U_t),$$ (16)

where

$$G(\tau_1, \tau_2, z, u) = \frac{\nabla_z F(\tau_1, z)^\top a(z, u) \nabla_z F(\tau_2, z)}{F(\tau_1, z) F(\tau_2, z)}.$$ (17)

To this end we describe any direction $\xi \in \mathbb{R}^n$ such that the instantaneous bond return covariance matrix remains unchanged when $U_t$ moves along $\xi$.

**Definition 2:** *The covariance kernel, denoted by $\mathcal{W}$, is given by*

$$\mathcal{W} = \bigcap_{\tau_1, \tau_2 \geq 0, (z, u) \in \mathcal{E}} \text{ker} \nabla_u G(\tau_1, \tau_2, z, u).$$

That is, $\mathcal{W}$ consists of all $\xi \in \mathbb{R}^n$ such that $\nabla_u G(\tau_1, \tau_2, z, u)^\top \xi = 0$ for all $\tau_1, \tau_2 \geq 0$ and $(z, u) \in \mathcal{E}$. Therefore, the location of $U_t$ along the direction $\xi$ cannot be recovered solely from knowledge of the time-$t$ instantaneous bond return covariances (16), $T_1, T_2 \geq t$. However, movements of $U_t$ along this direction could affect expected future bond return covariances in which case the location of $U_t$ can be recovered from time-$t$ bond derivatives prices.

The extent to which USV factors directly affect the instantaneous bond return covariances depends on how the $u$-gradient of the diffusion matrix $a(z, u)$ transmits to the $u$-gradient of $G(\tau_1, \tau_2, z, u)$ through the defining relation (17). This is formalized in the following theorem.

**Theorem 3:** *The number of USV factors that directly affect the instantaneous bond return covariances is less than or equal to the dimension $p$ of*

$$\text{span}\{\nabla_u a_{ij}(z, u) : 1 \leq i, j \leq m, (z, u) \in \mathcal{E}\}.$$ 

*Equality holds if the term structure kernel is zero, $\mathcal{U} = \{0\}$, $\kappa$ is invertible, and $\phi + \psi^\top \theta \neq 0$.*

We emphasize that the concepts of term structure and covariance kernels are generic. Definitions 1 and 2 carry over and can be applied to any factor model.
D. Swaps and Swaptions

Linear-rational term structure models have the important advantage of allowing for tractable swaption pricing. A fixed versus floating interest rate swap (IRS) on \([T_0, T_n]\) is specified by a tenor structure of reset and payment dates \(T_0 = t_0 < t_1 < \cdots < t_N = T_n\) for the floating leg and a tenor structure \(T_0 < T_1 < \cdots < T_n\) for the fixed leg. We let \(\delta = t_i - t_{i-1}\) and \(\Delta = T_i - T_{i-1}\) denote the lengths between tenor dates.\(^{14}\) Throughout most of the paper, we make the simplifying assumption that the discount factors implied by the state price density reflect the same credit and liquidity characteristics as LIBOR. In this case, we obtain the textbook valuation formula for an IRS. Specifically, the value of a payer swap (paying fixed and receiving floating) at time \(t \leq T_0\) is given by

\[
\Pi_t^{\text{swap}} = P(t, T_0) - P(t, T_n) - \Delta K \sum_{i=1}^n P(t, T_i),
\]

where \(K\) is the fixed annualized rate. This valuation formula was market standard until the financial crisis. In Section IV.I, we give a more general valuation formula that is consistent with current market practice.

The time-\(t\) forward swap rate, \(S_t^{T_0, T_n}\), is the rate \(K\) that makes the value of the swap equal to zero. It is given by

\[
S_t^{T_0, T_n} = \frac{P(t, T_0) - P(t, T_n)}{\Delta \sum_{i=1}^n P(t, T_i)},
\]

which is linear-rational in \(Z_t\). The forward swap rate becomes the spot swap rate at time \(T_0\).

A payer swaption is an option to enter into an IRS, paying the fixed leg at a predetermined rate and receiving the floating leg.\(^{15}\) A European payer swaption expiring at \(T_0\) on a swap with the characteristics described above has a value at expiration of

\[
C_{T_0} = \left(\Pi_{T_0}^{\text{swap}}\right)^+ = \left(\sum_{i=0}^n c_i P(T_0, T_i)\right)^+ = \frac{1}{\xi_{T_0}} \left(\sum_{i=0}^n c_i E_{T_0}[\xi_{T_i}]\right)^+.
\]

for coefficients \(c_i\) that can be easily read off the expression (18).

\(^{14}\) In the USD market fixed-leg payments occur at a semi-annual frequency, while floating-leg payments occur at a quarterly frequency. The valuation formula in this section depends only on the frequency of the fixed-leg payments, while the more general valuation formula in Section IV.I depends on the payment frequencies of both legs.

\(^{15}\) Conversely, a receiver swaption gives the right to enter into an IRS, receiving the fixed leg at a predetermined rate and paying the floating leg.
In a linear-rational term structure model, the conditional expectations $E_{T_0}[\zeta_{T_i}]$ are explicit linear functions of $Z_{T_0}$; see (4). Specifically, we have $C_{T_0} = p_{\text{swap}}(Z_{T_0})^+/\zeta_{T_0}$, where $p_{\text{swap}}(z)$ is the explicit linear function

$$p_{\text{swap}}(z) = \sum_{i=0}^{n} c_i e^{-\alpha_{T_i}} \left( \phi + \psi^\top \theta + \psi^\top e^{-\kappa (T_i - T_0)} (z - \theta) \right).$$

The swaption price at time $t \leq T_0$ is then obtained by an application of the fundamental pricing formula (1), which yields

$$\Pi^\text{swaption}_t = \frac{1}{\zeta_t} \mathbb{E}_t [\zeta_{T_0} C_{T_0}] = \frac{1}{\zeta_t} \mathbb{E}_t \left[ p_{\text{swap}}(Z_{T_0})^+ \right]. \quad (20)$$

To compute the price, one has to evaluate the conditional expectation on the right-hand side of (20). If the conditional distribution of $Z_{T_0}$ is known, this can be done via direct numerical integration over $\mathbb{R}^m$. This is a challenging problem in general; fortunately there is an efficient alternative approach based on Fourier transform methods.

**Theorem 4:** Define $\hat{q}(x) = \mathbb{E}_t [\exp(x p_{\text{swap}}(Z_{T_0}))]$ for $x \in \mathbb{C}$ and let $\mu > 0$ be such that $\hat{q}(\mu) < \infty$. Then the swaption price is given by

$$\Pi^\text{swaption}_t = \frac{1}{\zeta_t} \pi \int_0^\infty \Re \left[ \frac{\hat{q}(\mu + i\lambda)}{(\mu + i\lambda)^2} \right] d\lambda.$$

**Theorem 4** reduces the problem of computing an integral over $\mathbb{R}^m$ to that of computing a simple line integral. Of course, there is a price to pay: we now have to evaluate $\hat{q}(\mu + i\lambda)$ efficiently as $\lambda$ varies over $\mathbb{R}_+$. This problem can be approached in different ways depending on the specific class of factor processes under consideration. In our empirical analysis we focus on square-root factor processes, for which computing $\hat{q}(\mu + i\lambda)$ amounts to solving a system of ordinary differential equations. We provide further details in Section II.

Finally, we note that the swaption pricing formula does not presume a perfect fit to the term structure at time $t$. In the Internet Appendix, we show how to extend the pricing formula to achieve a perfect fit to the term structure. However, in our empirical analysis we find that the effect on the overall fit to swaptions is negligible.

**E. Normalized Term Structure Factors**

To interpret the linear-rational model it may be helpful to consider the normalized factors $\bar{Z}_t = Z_t / (\phi + \psi^\top Z_t)$. A simple algebraic transformation shows that zero-coupon bond prices (5) become linear in $\bar{Z}_t$,

$$P(t, T) = e^{-\alpha(T-t)} (A(T-t) + B(T-t) \bar{Z}_t) \quad (21)$$
with $A(\tau) = 1 + \psi^\top (\text{Id} - e^{-\kappa \tau}) \frac{\theta}{\phi}$ and $B(\tau) = -\psi^\top (\text{Id} - e^{-\kappa \tau}) (\text{Id} + \frac{\theta}{\phi} \psi^\top)$, where we define $\frac{\theta}{\phi} = 0$ if $\phi = 0$.\footnote{Henceforth we assume that $\theta = 0$ whenever $\phi = 0$. Note that swap rates (19) remain linear-rational in $\tilde{Z}_t$.} Similarly, the short rate is linear in $\tilde{Z}_t$,

$$r_t = \alpha - \psi^\top \kappa \frac{\theta}{\phi} + \psi^\top \kappa \left( \text{Id} + \frac{\theta}{\phi} \psi^\top \right) \tilde{Z}_t. \tag{22}$$

Zero-coupon bond yields $y(t, T) = (-1/(T - t)) \log P(t, T)$ are not linear in $\tilde{Z}_t$. However, linearizing around $\theta/(\phi + \psi^\top \theta)$ gives the linear expression\footnote{Linearizing around an arbitrary state vector $\bar{z}$ gives \[
y(t, T) \approx \alpha - \frac{A(T - t) - 1}{T - t} - \frac{B(T - t)}{T - t} \bar{Z}_t. \tag{23}\]
Here we use the fact that $A(\tau) + B(\tau) \bar{z} = 1$ for $\bar{z} = \theta/(\phi + \psi^\top \theta)$.}

$$y(t, T) \approx \alpha - \frac{A(T - t) - 1}{T - t} - \frac{B(T - t)}{T - t} \bar{Z}_t. \tag{24}$$

In the case in which $Z_t$ has diffusive dynamics (10), an application of Itô's formula to the map $\mathcal{H}(z) = z/(\phi + \psi^\top z)$ shows that $\tilde{Z}_t = \mathcal{H}(Z_t)$ satisfies

$$d\tilde{Z}_t = D\mathcal{H}(Z_t) \kappa (\theta - Z_t) dt + D\mathcal{H}(Z_t) \sigma(Z_t, U_t) (dB_t + \lambda_t dt),$$

where $\lambda_t$ is the market price of risk given in (11) and $D\mathcal{H}(z)$ denotes the derivative of $\mathcal{H}(z)$. Hence, $\bar{\mu}_t = D\mathcal{H}(Z_t) \kappa (\theta - Z_t) \bar{Z}_t$ is the risk-neutral drift of $\tilde{Z}_t$. Some algebraic transformations show that it is quadratic in $\tilde{Z}_t$ of the form

$$\bar{\mu}_t = \kappa \frac{\theta}{\phi} + \left( r_t - \alpha - \kappa - \kappa \frac{\theta}{\phi} \psi^\top \right) \tilde{Z}_t. \tag{25}$$

There are several reasons for not specifying the process $\tilde{Z}_t$ directly. First, the drift and martingale characteristics of $\tilde{Z}_t$ are nonlinear. Second, the range of $\tilde{Z}_t$ is restricted by the requirement that the implied bond prices (21) are positive and the short rate (22) is nonnegative. Taken together this makes it difficult to find a priori conditions on the $\tilde{Z}_t$ parameters such that the resulting term structure model is well defined. Third, deflated bond prices, $\zeta_t P(t, T)$, are nonlinear in $\tilde{Z}_t$ but linear in the original factors $Z_t$, which in view of the discussion in Section 1.D is a precondition for tractable swaption pricing.

\section*{F. Comparison with Other Models}

Linear-rational term structure models are related to the LG processes studied in Gabaix (2009). Indeed, the process $\zeta_t(1, \tilde{Z}_t)$ is an LG process; see Definition 4 in Gabaix (2009). Conversely, every LG process $\zeta_t(1, \tilde{Z}_t)$ can be represented as a linear-rational model with factor process $Z_t = (\zeta_t, \zeta_t \tilde{Z}_t)$ and parameters $\phi = 0, \psi = (1, 0, \ldots, 0)^\top$, and $\alpha = 0$. The drift of $Z_t$ becomes strictly
linear, $dZ_t = -\kappa Z_t dt + dM_t$. In the specific term structure model studied in Carr, Gabaix, and Wu (2009), the martingale part of $Z_t$ is given by $dM_t = e^{-\kappa t} \beta dN_t$, where $\beta$ is a vector in $\mathbb{R}^m$ and $N_t$ is a scalar exponential martingale of the form $dN_t/N_t = \sum_{i=1}^d \sqrt{\nu_{ii}} dB_i$ for independent Brownian motions $B_i$, and processes $\nu_{ii}$ following square-root dynamics. The process $Z_t$ is nonstationary due to the time-inhomogeneous volatility specification. The eigenvalues of $\kappa$ have positive real part, so that the volatility of $Z_t$ tends to zero as time goes to infinity, and $Z_t$ itself converges to zero almost surely. Further, as $N_t$ is scalar, bond prices are perfectly correlated in their model. The linear-rational models we consider in our empirical analysis are time-homogeneous and stationary, and have a volatility structure that is different from the specification in Carr, Gabaix, and Wu (2009), generating a truly multifactor structure for bond prices.

When the factor process $Z_t$ is Markovian, the linear-rational models fall in the broad class of models contained under the potential approach laid out in Rogers (1997). There the state price density is modeled by the expression $\zeta_t = e^{-\alpha t} R_t g(Z_t)$, where $R_t$ is the resolvent operator corresponding to the Markov process $Z_t$, and $g$ is a suitable function. In our setting we would have $R_t g(z) = \phi + \psi^\top z$, and thus $g(z) = (\alpha - \mathcal{G}) R_t g(z) = \alpha \phi - \psi^\top \kappa \theta + \psi^\top (\alpha + \kappa) z$, where $\mathcal{G}$ is the generator of $Z_t$.

Another related setup that slightly predates the potential approach is the framework of Flesaker and Hughston (1996). The state price density now takes the form $\zeta_t = \int_0^\infty M_t \mu(u) du$, where for each $u$, $(M_t)_{0 \leq t \leq u}$ is a martingale. The Flesaker-Hughston framework is related to the potential approach (and thus to the linear-rational framework) via the representation $e^{-\alpha t} R_t g(Z_t) = \int_0^\infty \mathbb{E}_t [e^{-\alpha u} g(Z_u)] du$, which implies $M_t \mu(u) = \mathbb{E}_t [e^{-\alpha u} g(Z_u)]$. The linear-rational framework fits into this template by taking $\mu(u) = e^{-\alpha u}$ and $M_t = \mathbb{E}_t [g(Z_u)] = \alpha \phi + \alpha \psi^\top \theta + \psi^\top (\alpha + \kappa) e^{-\kappa (u-t)} (Z_t - \theta)$, where $g(z) = \alpha \phi - \psi^\top \kappa \theta + \psi^\top (\alpha + \kappa) z$ is chosen as above. One member of this class, introduced in Flesaker and Hughston (1996), is the one-factor rational log-normal model. The simplest time-homogeneous version of this model, in the notation of (2) and (3), is obtained by taking $\phi$ and $\psi$ to be positive, setting $\kappa = \theta = 0$, and letting the martingale part $M_t$ of the factor process $Z_t$ be a geometric Brownian motion.

II. The Linear-Rational Square-Root Model

The primary example of a linear-rational diffusion model (10) with term structure state space $E = \mathbb{R}^m_+$ is the linear-rational square-root (LRSQ) model. In this section, we show that USV can easily be incorporated, and swaptions can be priced efficiently, in this model. This lays the groundwork for our empirical analysis.

The LRSQ model is based on a $(m+n)$-dimensional square-root diffusion process $X_t$ taking values in $\mathbb{R}^{m+n}_+$ of the form

$$dX_t = (b - \beta X_t) dt + \text{Diag} \left( \sigma_1 \sqrt{X_{1t}}, \ldots, \sigma_{m+n} \sqrt{X_{m+n,t}} \right) dB_t \tag{26}$$
Linear-Rational Term Structure Models

with volatility parameters $\sigma_i$, where $0 \leq n \leq m$ represents the desired number of USV factors. Define $(Z_t, U_t) = SX_t$ as a linear transform of $X_t$ with state space $\mathcal{E} = S(\mathbb{R}^{m+n})$. We thus need to specify a $(m + n) \times (m + n)$-matrix $S$ such that the implied term structure state space is $E = \mathbb{R}^m$ and the drift of $Z_t$ does not depend on $U_t$, while $U_t$ feeds into the martingale part of $Z_t$. Many such constructions are possible, but the one given here is more than sufficient for the applications we are interested in.

Specifically, we let the matrix $S$ be given by

$$S = \begin{pmatrix} \text{Id}_m & A \\ 0 & \text{Id}_n \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} \text{Id}_n \\ 0 \end{pmatrix}.$$

In coordinates this reads $Z_{it} = X_{it} + X_{m+i,t}$ and $U_{it} = X_{m+i,t}$ for $1 \leq i \leq n$, and $Z_{it} = X_{it}$ for $n + 1 \leq i \leq m$. For $Z_t$ to have an autonomous linear drift, the $(m + n) \times (m + n)$-matrix $\beta$ in (26) is chosen to be upper block-triangular of the form

$$\beta = S^{-1} \begin{pmatrix} \kappa & 0 \\ 0 & A^\top \kappa \end{pmatrix} S = \begin{pmatrix} \kappa A - AA^\top \kappa A \\ 0 \\ A^\top \kappa A \end{pmatrix}$$

for some $\kappa \in \mathbb{R}^{m \times m}$. Note that $A^\top \kappa A$ is the upper left $n \times n$ block of $\kappa$. The constant drift term in (26) is specified as

$$b = \beta S^{-1} \begin{pmatrix} \theta \\ \theta_U \end{pmatrix} = \begin{pmatrix} \kappa \theta - AA^\top \kappa A \theta_U \\ A^\top \kappa A \theta_U \end{pmatrix}$$

for some $\theta \in \mathbb{R}^m$ and $\theta_U \in \mathbb{R}^n$. The interpretation of the parameters $\kappa, \theta$, and $\theta_U$ follows readily from the implied dynamics of the joint factor process $(Z_t, U_t)$,

$$dZ_t = \kappa(\theta - Z_t)dt + \sigma(Z_t, U_t)dB_t$$

$$dU_t = A^\top \kappa A(\theta_U - U_t)dt + \text{Diag}(\sigma_{m+1} \sqrt{U_{1,t}} dB_{m+1,t}, \ldots, \sigma_{m+n} \sqrt{U_{n,t}} dB_{m+n,t}),$$

(27)

with the dispersion function of $Z_t$ given by $\sigma(z, u) = (\text{Id}_m, A) \text{Diag}(\sigma_1 \sqrt{z_1 - u_1}, \ldots, \sigma_{m+n} \sqrt{U_{n,t}})$.

As a next step we show that the LRSQ model admits a canonical representation.

**Theorem 5:** The short rate (6) is bounded from below if and only if, after a coordinatewise scaling of $Z_t$, we have $\phi = 1$ and $\psi = 1$, where we write $1 = (1, \ldots, 1)^\top$. In this case, the extremal values in (7) are given by $\alpha^* = \max S$ and $\alpha_* = \min S$, where $S = \{1^\top \kappa \theta, -1^\top \kappa 1, \ldots, -1^\top \kappa m\}$, and $\kappa_i$ denotes the $i^{th}$ column vector of $\kappa$.

In accordance with this result, we always let the state price density be given by $\zeta_t = e^{-\alpha t}(1 + 1^\top Z_t)$ when considering the LRSQ model. By Theorem 1, the term structure kernel is zero if and only if

$$\text{span} \left\{1, \kappa^\top 1, \ldots, \kappa^{(m-1)^\top} 1\right\} = \mathbb{R}^m.$$

(28)

This construction motivates the following definition.
Definition 3: The LRSQ(m,n) specification is obtained by choosing \( \kappa \in \mathbb{R}^{m \times n} \) with nonpositive off-diagonal elements and such that (28) holds. The mean-reversion levels \( \theta \) and \( \theta_U \) are chosen such that \( b \in \mathbb{R}_+^{m+n} \) and the volatility parameters are \( \sigma_1, \ldots, \sigma_{m+n} \geq 0 \).

This definition guarantees that a unique solution to (26), and thus (27), exists; see, for example, Filipović (2009), Theorem 10.2. Indeed, note that \( \beta \) has nonpositive off-diagonal elements by construction.

As for the number of USV factors directly affecting the instantaneous bond return covariances, we have the following corollary of Theorem 3.

Corollary 1: Assume that \( \kappa \) is invertible and \( 1 + 1^\top \theta \neq 0 \). Then the number of USV factors directly affecting the instantaneous bond return covariances equals the number of indices \( 1 \leq i \leq n \) such that \( \sigma_i \neq \sigma_{m+i} \).

To illustrate, in the example below we consider the LRSQ(1,1) specification, where we have one term structure factor and one USV factor. This example shows in particular that a linear-rational term structure model may exhibit USV even in the two-factor case.\(^{18}\)

Example 1: In the LRSQ(1,1) specification, the mean-reversion matrix of \( X_t \) is given by

\[
\beta = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix}.
\]

The term structure factor and USV factor become \( Z_t = X_{1t} + X_{2t} \) and \( U_t = X_{2t} \), respectively. The corresponding dispersion function is

\[
\sigma(z, u) = \left( \sigma_1 \sqrt{z - u} \sigma_2 \sqrt{u} \right).
\]

Corollary 1 implies that \( U_t \) directly affects the instantaneous bond return covariances if \( \kappa \neq 0, 1 + \theta \neq 0 \), and \( \sigma_1 \neq \sigma_2 \).

Swaption pricing becomes particularly tractable in the LRSQ model. Since \( (Z_t, U_t) = SX_t \) is the linear transform of a square-root diffusion process, the function \( \hat{q}(\mu + i \lambda) \) in Theorem 4 can be expressed using the exponential-affine transform formula that is available for such processes. Computing \( \hat{q}(\mu + i \lambda) \) then amounts to solving a well-known system of ordinary differential equations; see, for example, Duffie, Pan, and Singleton (2000) and Filipović (2009), Theorem 10.3. For convenience the relevant expressions are reproduced in the Internet Appendix. For Theorem 4 to be applicable, it is necessary that some exponential moments of \( p_{\text{swap}}(Z_{T_0}) \) be finite. We therefore note that, for any \( v \in \mathbb{R}^m \), there is always some \( \mu > 0 \) (depending on \( v, Z_0, U_0, T_0 \)) such that \( \mathbb{E}[\exp(\mu v^\top Z_{T_0})] < \infty \). While it may be difficult a priori to decide how small \( \mu \) should be, the choice is easy in practice since numerical methods diverge if \( \mu \) is too large, resulting in easily detectable outliers.

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\(^{18}\)This contradicts Proposition 3 in Collin-Dufresne and Goldstein (2002), which states that a two-factor Markov model of the term structure cannot exhibit USV.
III. Extended State Price Density Specification

Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) show that a state price density can be factorized into a transitory and a permanent component. Our specification of $\zeta_t$ so far captures only the transitory component.\(^{19}\) As shown in the empirical section below, this is too restrictive to match observed dynamics of bond risk premiums. In this section, we extend the state price density specification (3) to incorporate the permanent component, which allows for much richer risk premium dynamics.

The starting point is the observation that the linear-rational framework can be equally well developed under some auxiliary probability measure $\mathbb{A}$ that is equivalent to the historical probability measure $\mathbb{P}$ with Radon–Nikodym density process $E^\mathbb{P}_t[d\mathbb{A}/d\mathbb{P}]$. The state price density $\zeta_t \equiv \zeta_t^\mathbb{A} = e^{-\alpha t}(\phi + \psi^\top Z_t)$ and the martingale $M_t \equiv M_t^\mathbb{A}$ are then understood with respect to $\mathbb{A}$. The dynamics of $Z_t$ read $dZ_t = \kappa(\theta - Z_t)dt + dM_t^\mathbb{A}$, and the basic pricing formula (1) becomes $\Pi(t, T) = E^\mathbb{P}_t[\zeta_T^\mathbb{A} C_T]/\zeta_t^\mathbb{P}$. From this, using Bayes’s rule, we obtain the state price density with respect to $\mathbb{P}$,

$$\zeta_t^\mathbb{P} = \zeta_t^\mathbb{A} E^\mathbb{P}_t \left[ \frac{d\mathbb{A}}{d\mathbb{P}} \right] , \quad (29)$$

so that $\Pi(t, T) = E^\mathbb{P}_t[\zeta_T^\mathbb{P} C_T]/\zeta_t^\mathbb{P}$.\(^{20}\) Bond prices $P(t, T) = F(T - t, Z_t)$ are still given as functions of $Z_t$, with the same $F(t, z)$ as in (5).

In the diffusion setup of Section I.C, the martingale $M_t^\mathbb{A}$ is given by $dM_t^\mathbb{A} = \sigma(Z_t, U_t)dB_t^\mathbb{A}$ for some $\mathbb{A}$-Brownian motion $B_t^\mathbb{A}$, and the market price of risk $\lambda_t \equiv \lambda_t^\mathbb{A} = -\sigma(Z_t, U_t)^\top \psi/(\phi + \psi^\top Z_t)$ is understood with respect to $\mathbb{A}$. Having specified the model under the auxiliary measure $\mathbb{A}$, we now have full freedom in choosing an equivalent change of measure from $\mathbb{A}$ to the historical measure $\mathbb{P}$. Specifically, $\mathbb{P}$ can be defined using a Radon–Nikodym density process of the form

$$E^\mathbb{A}_t \left[ \frac{d\mathbb{P}}{d\mathbb{A}} \right] = \exp \left( \int_0^t \delta_s^\top dB_s^\mathbb{A} - \frac{1}{2} \int_0^t \|\delta_s\|^2 ds \right)$$

for some appropriate integrand $\delta_t$. The $\mathbb{P}$-dynamics of $Z_t$ become

$$dZ_t = \left( \kappa(\theta - Z_t) + \sigma(Z_t, U_t)\delta_t \right) dt + \sigma(Z_t, U_t)dB_t^\mathbb{P}$$

\(^{19}\) Equation (21) in Hansen and Scheinkman (2009) for the multiplicative decomposition of the stochastic discount factor reads $S_t = \exp(\rho(t)\hat{M}_t^\mathbb{A} X_t)$ for some positive martingale $\hat{M}_t$ representing the permanent component and $X_t$ being a Markov state process. The function $\phi(x)$ is a positive eigenfunction of the extended generator of the pricing semigroup with eigenvalue $\rho$, capturing the transitory component. In view of our state price density specification (3), this corresponds to $\rho = -\alpha$, $\hat{M}_t = 1$, and $\phi(x) = 1/(\phi + \psi^\top z)$.

\(^{20}\) Equation (29) corresponds to the state price density factorization into a transitory component, $\zeta_t^\mathbb{A}$, and a permanent component, $E^\mathbb{P}_t[d\mathbb{A}/d\mathbb{P}]$, as given in Alvarez and Jermann (2005) and Hansen and Scheinkman (2009).
for the $\mathbb{P}$-Brownian motion $dB_t^\mathbb{P} = dB_t^\mathbb{A} - \delta_t dt$. The state price density with respect to $\mathbb{P}$ follows the dynamics $d\xi_t^\mathbb{P}/\xi_t^\mathbb{P} = -r_t dt - (\lambda_t^\mathbb{P})^T dB_t^\mathbb{P}$, where the market price of risk $\lambda_t^\mathbb{P}$ is now given by

$$
\lambda_t^\mathbb{P} = \lambda_t^\mathbb{A} + \delta_t = -\frac{\sigma(Z_t, U_t)^T \psi}{\phi + \psi^T Z_t} + \delta_t.
$$

The exogenous choice of $\delta_t$ gives us the freedom to introduce additional unspanned factors.21 Such unspanned factors would affect risk premiums but would not constitute USV factors. Examples of such unspanned risk premium factors in Gaussian exponential-affine models include the “hidden factors $h_t$” in Duffee (2011) and the “unspanned components of the macro variables $M_t$” in Joslin, Priebsch, and Singleton (2014). They enter through the equivalent change of measure from the risk-neutral measure $Q$ to $\mathbb{P}$ and thus affect the distribution of future bond prices under $\mathbb{P}$ but not under $Q$.

The intuitive argument for why the term structure is unaffected by USV factors also illustrates why it is unaffected by unspanned risk premium factors. When the expectation and covariance in (13) are taken with respect to $\mathbb{P}$, it can be readily seen that the terms containing $\delta_t$ cancel out. Indeed, the change of measure from $\mathbb{A}$ to $\mathbb{P}$ adds the term $P(t, T)\nu(t, T)^T \delta_t dt$ to the expected future bond price (14) and $-P(t, T)\nu(t, T)^T \delta_t dt$ to the risk premium (15).

In our empirical analysis the focus is on USV, and our specification of $\delta_t$ in Section IV.B below does not introduce unspanned risk premium factors to the model.

IV. Empirical Analysis

In this section, we perform an extensive empirical analysis of the LRSQ model. We focus in particular on how the model captures key features of term structure, volatility, and risk premium dynamics.

A. Data

The empirical analysis is based on a panel data set consisting of swaps and swaptions. At each observation date, we observe rates on spot-starting swap contracts with maturities of 1, 2, 3, 5, 7, and 10 years, respectively. We also observe prices of swaptions with the same six swap maturities, option expiries of 3 months and 1, 2, and 5 years, and strikes equal to the forward swap rates. Such at-the-money-forward swaptions are the most liquid. The data come from Bloomberg and consist of composite quotes based on quotes from major banks and interdealer brokers. Each time series consists of 866 weekly observations from January 29, 1997 to August 28, 2013.

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21 It allows us to generate essentially any drift $\mu_t^\mathbb{P}$ of $Z_t$ under the historical measure $\mathbb{P}$ as long as $\mu_t^\mathbb{P} - \kappa(\theta - Z_t)$ is in the range of $\sigma(Z_t, U_t)$. This situation is similar to the discrete-time framework studied in Le, Singleton, and Dai (2010), where they use the conditional version of the Esscher transform for the conditional risk-neutral distribution of the state vector.
It is more convenient to represent swaption prices in terms of implied volatilities. In the USD market, the market standard is the “normal” implied volatility (NIV), which is the volatility parameter that matches a given price when plugged into the pricing formula that assumes a normal distribution for the underlying forward swap rate. For an at-the-money-forward swaption there is a particularly simple relation between the swaption price and the NIV, \( \sigma_{N,t} \), which is given by

\[
\Pi_{\text{swaption}}^t = \sqrt{T_0 - t} \cdot \frac{1}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} \Delta P(t, T_i) \right) \sigma_{N,t};
\]

see, for example, Corb (2012).

Time series of the 1-year, 5-year, and 10-year swap rates are displayed in Panel A1 of Figure 2. The 1-year swap rate fluctuates between a minimum of 0.30% (on May 1, 2013) and a maximum of 7.51% (on May 17, 2000), while the longer-term swap rates exhibit less variation. A principal component analysis (PCA) of weekly changes in swap rates shows that the first three factors explain 90%, 7%, and 2%, respectively, of the variation. Panel B1 of Figure 2 displays time series of NIVs for three “benchmark” swaptions: the 3-month option on the 2-year swap, the 2-year option on the 2-year swap, and the 5-year option on the 5-year swap. Of these, the 3-month NIV of the 2-year swap rate is the most volatile, fluctuating between a minimum of 18 bps (on December 12, 2012) and a maximum of 213 bps (on October 8, 2008). Swaptions also display a high degree of commonality, with the first three factors from a PCA of weekly changes in NIVs explaining 77%, 8%, and 5%, respectively, of the variation. Summary statistics of the data are given in the Internet Appendix.

**B. Model Specifications**

We restrict attention to the \( \text{LRSQ}(m,n) \) specification; see Definition 3. We always set \( m = 3 \) (three term structure factors) and consider specifications with \( n = 1 \) (volatility of \( Z_{1t} \) containing a USV component), \( n = 2 \) (volatilities of \( Z_{1t} \) and \( Z_{2t} \) containing USV components), and \( n = 3 \) (volatilities of all term structure factors containing USV components). The Internet Appendix provides the explicit factor dynamics in these specifications.

We develop the model under the \( \mathbb{A} \)-measure and obtain the \( \mathbb{P} \)-dynamics of the factor process \( X_t \) in (26) by specifying \( \delta_t \) parsimoniously as

\[
\delta_t = \left( \delta_{1t} \sqrt{X_{1t}}, \ldots, \delta_{m+n} \sqrt{X_{m+n,t}} \right)^\top.
\]

This choice is convenient as \( X_t \) remains a square-root process under \( \mathbb{P} \), facilitating the use of standard estimation techniques from the vast body of literature

---

22 This is sometimes also referred to as the “absolute” or “basis point” implied volatility. Alternatively, a price may be represented in terms of “log-normal” (or “percentage”) implied volatility, which assumes a log-normal distribution for the underlying forward swap rate.
Figure 2. Data and fit. Panel A1 shows time series of the 1-year, 5-year, and 10-year swap rates (displayed as thick light-grey, thick dark-grey, and thin black lines, respectively). Panel B1 shows time series of the normal implied volatilities on three “benchmark” swaptions: the 3-month option on the 2-year swap, the 2-year option on the 2-year swap, and the 5-year option on the 5-year swap (displayed as thick light-grey, thick dark-grey, and thin black lines, respectively). Panels A2 and B2 show the fit to swap rates and implied volatilities, respectively, in the case of the $LRSQ(3,3)$ specification. Panels A3 and B3 show time series of the root-mean-squared pricing errors (RMSEs) of swap rates and implied volatilities, respectively, in the case of the $LRSQ(3,3)$ specification. The units in Panels B1, B2, A3, and B3 are basis points. The grey areas mark the two NBER-designated recessions from March 2001 to November 2001 and from December 2007 to June 2009. Each time series consists of 866 weekly observations from January 29, 1997 to August 28, 2013.
on affine models (note, however, that $X_t$ is not a square-root process under $\mathbb{Q}$).\footnote{Alternatively, one could use a measure change from $\mathbb{Q}$ to $\mathbb{P}$, similar to the one suggested by Cheridito, Filipovic, and Kimmel (2007). In this case $X_t$ would also be a square-root process under $\mathbb{P}$, but the model would be less parsimonious.}

Specifically, we estimate the model by quasi-maximum likelihood in conjunction with Kalman filtering; details are provided in the Internet Appendix.

We also evaluate two variants of the $LRSQ(3,3)$ specification. First, we impose $\delta_t = 0$, that is, we omit the permanent component of the state price density. Second, we treat $\alpha$ as a free parameter, that is, we do not impose the ZLB on interest rates. These two variants are denoted $LRSQ(3,3)^{\parallel}$ and $LRSQ(3,3)^{\perp}$, respectively, and are discussed in Sections IV.D and IV.H below.

We consider two approaches to fitting the model to observed NIVs. The first approach makes the assumption that the NIVs apply to swaptions that are at-the-money-forward from the perspective of the model, that is, we price swaptions using the pricing formula in Section I.D with strikes set equal to the model-implied forward swap rates. The resulting swaption prices are then converted to NIVs using (30) with model-implied discount factors. In the second approach we price swaptions using the extended pricing formula in the Internet Appendix (which fits the observed term structure perfectly) with strikes set equal to the observed forward swap rates. The resulting swaption prices are then converted to NIVs using (30) with observed discount factors. The Internet Appendix shows that the two approaches give very similar results in terms of estimated factors and overall pricing errors. However, the first approach has the advantage that it is computationally faster and preserves the time-homogeneity of the model, so we use this approach for estimation.

In preliminary analyses, we find that the upper-triangular elements of $\kappa$ are always very close to zero. The same is true of the lower left element of $\kappa$. As a first step toward obtaining more parsimonious specifications, we reestimate the models after setting these elements to zero, that is, imposing that $\kappa$ is lower bi-diagonal. Furthermore, several of the parameters in (31) are very imprecisely estimated. As a second step toward obtaining more parsimonious specifications, we follow Duffee (2002) and Dai and Singleton (2002) in reestimating the models after setting to zero those market price of risk parameters for which the absolute $t$-statistics do not exceed one. In all cases, the likelihood functions are virtually unaffected by these constraints.

C. Maximum Likelihood Estimates

Table II displays parameter estimates and their asymptotic standard errors. It is straightforward to verify that (28) holds true in all model specifications, which implies that the number of term structure factors cannot be reduced to less than three. Also, a robust feature across all specifications is that the drift parameters align such that to a close approximation

$$\kappa \theta = (\alpha, 0, 0)^{\top} \quad \text{and} \quad \alpha = 1^{\top} \kappa \theta = -1^{\top} \kappa_1 = -1^{\top} \kappa_2 > -1^{\top} \kappa_3 = \alpha_s,$$

(32)
The table reports parameter estimates with asymptotic standard errors in parentheses. $\sigma_{\text{rates}}$ denotes the standard deviation of swap rate pricing errors and $\sigma_{\text{swaptions}}$ denotes the standard deviation of swaption pricing errors in terms of normal implied volatilities. Both $\sigma_{\text{rates}}$ and $\sigma_{\text{swaptions}}$ are measured in basis points. $\alpha$ is chosen as the smallest value that guarantees a nonnegative short rate. $\sup r_t$ is the upper bound on possible short rates. $L$ denotes the log-likelihood value. The sample period consists of 866 weekly observations from January 29, 1997 to August 28, 2013.

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<tr>
<th></th>
<th>$\kappa_{11}$</th>
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<th>$\kappa_{21}$</th>
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<td>$LRSQ(3,1)$</td>
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<tr>
<td></td>
<td>(0.0002)</td>
<td>(0.0008)</td>
<td>(0.0004)</td>
<td>(0.0003)</td>
<td>(0.0009)</td>
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<tr>
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<td></td>
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<td>(0.0002)</td>
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<tr>
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<td>(0.1009)</td>
<td>(1.0823)</td>
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<td>$LRSQ(3,2)$</td>
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<td>(2.2028)</td>
<td>(0.4018)</td>
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<td>$LRSQ(3,3)$</td>
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<td>(0.6048)</td>
<td>(0.0976)</td>
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<th>$\sigma_{\text{rates}}$</th>
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<th>$\sup r_t$</th>
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<td>$LRSQ(3,1)$</td>
<td>8.3978</td>
<td>7.1879</td>
<td>0.0746</td>
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<td>14.6913</td>
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<td>(0.0558)</td>
<td>(0.0196)</td>
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<tr>
<td>$LRSQ(3,2)$</td>
<td>5.0389</td>
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<td>0.0688</td>
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<td>15.0869</td>
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<td>(0.0393)</td>
<td>(0.0131)</td>
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<tr>
<td>$LRSQ(3,3)$</td>
<td>4.8086</td>
<td>5.6982</td>
<td>0.0566</td>
<td>0.7181</td>
<td>15.3100</td>
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<tr>
<td></td>
<td>(0.0487)</td>
<td>(0.0137)</td>
<td></td>
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where $\kappa_i$ denotes the $i^{th}$ column vector of $\kappa$. Because $\mathbf{1}^\top \kappa_3 = \kappa_{33}$, the range of the short rate (8) is given by $r_t \in [0, \alpha + \kappa_{33}]$, and the expression (6) for $r_t$ effectively reduces to

$$r_t = (\alpha + \kappa_{33}) \frac{Z_{3t}}{1 + \mathbf{1}^\top Z_t}.$$  \hspace{1cm} (33)

It is immediately clear from this expression that the ZLB on the short rate is attained when $Z_{3t} = 0$. The table reports the upper bound on the short rate, which is 20.04%, 146.13%, and 71.81% for the LRSQ(3,1), LRSQ(3,2), and LRSQ(3,3) specification, respectively. As such, the upper bound is not a restrictive feature of the model.\footnote{The historical maximum for the effective federal funds rate is 22.36%. However, such levels of short rates were only reached during the monetary policy experiment (MPE) in the early 1980s when the Federal Reserve was conducting monetary policy in a fundamentally different way by targeting monetary aggregates instead of interest rates. Regime-switching models such as Gray (1996) and Dai, Singleton, and Yang (2007) also identify the period around the MPE as a very different regime from the post-MPE period.}

Simulations show that the likelihood of observing very high short rates is negligible in all specifications; in contrast, as shown below, there is a significant likelihood of observing very low short rates. The parameter $\alpha$ ranges between 5.66% and 7.46% across model specifications, which appears reasonable given that $\alpha$ has the economic interpretation as the model-implied infinite-maturity zero-coupon bond yield (all specifications are stationary since all eigenvalues of $\kappa$ are positive).\footnote{As an out-of-sample check on the value of $\alpha$, we bootstrapped the swap curve out to 30 years on each observation date (recall that only swap maturities up to 10 years are used in the estimation). The sample mean of the 30-year zero-coupon bond yield is 5.26%.

From a practical perspective, the simple structure for the short rate gives the model flexibility to capture significant variation in longer-term interest rates during the latter part of the sample period, when policy rates are effectively zero. The reason is that a low value of $Z_{3t}$ constrains the short rate to be close to zero, which allows $Z_{1t}$ and $Z_{2t}$ freedom to affect longer-term interest rates without having much impact on the short rate.

Finally, note that every USV factor directly affects the instantaneous bond return covariances; see Corollary 1. Specifically, for all $1 \leq i \leq n$, we have $\sigma_{i+3} > \sigma_i$. Since the instantaneous volatility of the $i^{th}$ term structure factor is given by $\sqrt{\sigma_i^2 Z_{it} + (\sigma_{i+3}^2 - \sigma_i^2) U_{it}}$, term structure factor volatilities are increasing in the USV factors.

### D. Specification Analysis

For each of the model specifications, we compute the fitted swap rates and NIVs based on the estimated factors. We then compute weekly root-mean-squared pricing errors (RMSEs) for swap rates and RMSEs for NIVs—both across all swaptions and for each option expiry separately. Panel A of Table III compares the specifications as we increase the number of USV factors. The first three rows report the sample means of the RMSE time series, while the next
Table III
Comparison of Model Specifications

The table reports means of time series of the root-mean-squared pricing errors (RMSEs) of swap rates and normal implied swaption volatilities. For swaptions, results are reported for the entire volatility surface as well as for the volatility term structures at the four option maturities in the sample (3 months, 1 year, 2 years, and 5 years). Units are basis points. Panel B compares the baseline LRSQ(3,3) specification with the variant that imposes $\delta_t = 0$ (denoted LRSQ(3,3)$^\dagger$). Panel C compares the baseline LRSQ(3,3) specification with the variant that treats $\alpha$ as a free parameter (denoted LRSQ(3,3)$^\ddagger$). t-statistics, corrected for heteroscedasticity and serial correlation up to 26 lags (i.e., 6 months) using the method of Newey and West (1987), are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% level, respectively. The sample period consists of 866 weekly observations from January 29, 1997 to August 28, 2013.

<table>
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<th>Specification</th>
<th>Swaps</th>
<th>Swaps 3 mths</th>
<th>Swaps 1 yr</th>
<th>Swaps 2 yrs</th>
<th>Swaps 5 yrs</th>
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<tr>
<td><strong>Panel A: Different number of USV factors</strong></td>
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<td></td>
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<tr>
<td>LRSQ(3,1)</td>
<td>7.11</td>
<td>6.63</td>
<td>8.27</td>
<td>5.54</td>
<td>5.25</td>
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<tr>
<td>LRSQ(3,2)</td>
<td>3.83</td>
<td>5.77</td>
<td>7.87</td>
<td>5.12</td>
<td>3.98</td>
</tr>
<tr>
<td>LRSQ(3,3)</td>
<td>3.72</td>
<td>5.19</td>
<td>7.20</td>
<td>4.40</td>
<td>3.88</td>
</tr>
<tr>
<td>LRSQ(3,2) − LRSQ(3,1)</td>
<td>−3.28***</td>
<td>−0.86**</td>
<td>−0.40</td>
<td>−0.42</td>
<td>−1.27***</td>
</tr>
<tr>
<td></td>
<td>(−8.95)</td>
<td>(−2.18)</td>
<td>(−0.74)</td>
<td>(−1.04)</td>
<td>(−3.66)</td>
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<tr>
<td>LRSQ(3,3) − LRSQ(3,2)</td>
<td>−0.12</td>
<td>−0.58**</td>
<td>−0.67*</td>
<td>−0.72***</td>
<td>−0.11</td>
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<tr>
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<td>(−0.78)</td>
<td>(−2.52)</td>
<td>(−1.82)</td>
<td>(−2.97)</td>
<td>(−0.46)</td>
</tr>
<tr>
<td><strong>Panel B: Imposing $\delta_t = 0$</strong></td>
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<tr>
<td>LRSQ(3,3)$^\dagger$</td>
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<td>7.24</td>
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<td>(0.56)</td>
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<td><strong>Panel C: Estimating $\alpha$</strong></td>
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<td>LRSQ(3,3)$^\ddagger$</td>
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<tr>
<td>LRSQ(3,3)$^\ddagger$ − LRSQ(3,3)</td>
<td>−0.64***</td>
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<td>(−1.93)</td>
<td>(0.48)</td>
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two rows report the mean differences in RMSEs between model specifications along with the associated Newey and West (1987) t-statistics in parentheses. The most parsimonious LRSQ(3,1) specification has a reasonable fit to the data, with mean RMSEs for swap rates and NIVs equal to 7.11 bps and 6.63 bps, respectively. Adding one more USV factor decreases the mean RMSEs by 3.28 bps and 0.86 bps, respectively, which is both economically important and statistically significant (t-statistics of −8.95 and −2.18, respectively). The improvement in the fit to swaptions mainly occurs at the 2-year and 5-year option expiries. Adding an additional USV factor has a negligible impact on the mean RMSE for swap rates, but decreases the mean RMSE for NIVs by a further 0.58 bps, which again is both economically important and statistically significant (t-statistic of −2.52). In this case, the improvement in the fit to swaptions mainly occurs at the 3-month and 1-year option expiries.
The impact of restricting the state price density specification can be seen in Panel B of Table III, which compares the baseline $LRSQ(3,3)$ specification with the variant that imposes $\delta_t = 0$. This restriction has a negligible adverse impact on the fit to swaps and swaptions. On the other hand, in Section IV.H we document a significant impact on model-implied risk premiums. This reflects the fact that the likelihood function puts more weight on fitting the cross-sectional properties of the model than its time-series dynamics.\footnote{This is a common finding in empirical studies of dynamic term structure models; see, for example, Joslin, Priebsch, and Singleton (2014), who show that restrictions on market prices of risk significantly impact the model’s $\mathbb{F}$-dynamics but leave the $\mathbb{Q}$-dynamics virtually unaffected.}

The impact of not imposing the ZLB can be seen in Panel C of Table III, which compares the baseline $LRSQ(3,3)$ specification with the variant that treats $\alpha$ as a free parameter. The mean RMSE for swap rates decreases by 0.64 bps, which is both economically important and statistically significant ($t$-statistic of $-3.90$), while the improvement in the fit to swaptions is relatively minor. The parameter $\alpha$ is estimated at 5.01% and the lower and upper bounds on the short rate become $-0.49\%$ and $74.68\%$, respectively, so that this model variant allows for a slightly negative short rate. Given that money market rates have generally remained nonnegative during the sample period and beyond, we prefer the more parsimonious baseline specification that imposes the ZLB.

The performance of the $LRSQ(3,3)$ specification over time is illustrated in Figure 2, which shows time series of fitted swap rates (Panel A2) and NIVs (Panel B2) as well as time series of RMSEs for swap rates (Panel A3) and NIVs (Panel B3). The RMSEs are relatively stable over time, except during the financial crisis, when the RMSEs are generally larger. The transitory spikes in RMSEs are mostly associated with familiar crisis events, such as the collapse of Long-Term Capital Management in September 1998, the bond market sell-off in June 2003 driven by MBS convexity hedging, and the aftermath of the Lehman Brothers default in September 2008.

To investigate the performance of the model when policy rates are close to the ZLB, we also split the sample period into a ZLB period and a pre-ZLB period. The beginning of the ZLB period is taken to be December 16, 2008, when the Federal Reserve reduced the federal funds rate from 1% to a target range of 0 to 25 bps. Taking the $LRSQ(3,3)$ specification as an example, the mean RMSEs for swap rates and NIVs are 2.97 bps and 4.78 bps, respectively, during the pre-ZLB period compared with 5.59 bps and 6.21 bps, respectively, during the ZLB period. Given the challenging market conditions during the ZLB period, including potentially distortive effects from the Federal Reserve’s quantitative easing programs, this performance seems respectable. The Internet Appendix contains additional details on model performance.

E. Factors

Figure 3 displays the estimated factors. The first, second, and third column shows the factors of the $LRSQ(3,1)$, $LRSQ(3,2)$, and $LRSQ(3,3)$ model specification, respectively. The first, second, and third row corresponds to $(Z_{1t}, U_{1t})$, \ldots
Figure 3. Estimated factors. The figure displays time series of the estimated factors. The first, second, and third column shows the factors of the $LRSQ(3,1)$, $LRSQ(3,2)$, and $LRSQ(3,3)$ specification, respectively. The first row displays $Z_{1,t}$ and $U_{1,t}$. The second row displays $Z_{2,t}$ and possibly $U_{2,t}$. The third row displays $Z_{3,t}$ and possibly $U_{3,t}$. The thin black lines show the term structure factors, $Z_{1,t}$, $Z_{2,t}$, and $Z_{3,t}$. The thick grey lines show the unspanned stochastic volatility factors, $U_{1,t}$, $U_{2,t}$, and $U_{3,t}$. The grey areas mark the two NBER-designated recessions from March 2001 to November 2001 and from December 2007 to June 2009. Each time series consists of 866 weekly observations from January 29, 1997 to August 28, 2013.

The factors are highly correlated across specifications, which is indicative of a stable factor structure. The USV factors are occasionally large relative to the term structure factors, which highlights the importance of allowing for USV.

As an additional test of the model structure, we consider the correlation between the term structure factors. By construction, the term structure factors are instantaneously uncorrelated, while the unconditional correlations may be nonnegative due to the feedback via the drift. The same is true within the branch of exponential-affine term structure models for which the factor process
is of the square-root type (the $A_m(m)$ model in the notation of Dai and Singleton (2000)). However, there the unconditional correlations among the estimated factors are often reported to be strongly negative in contradiction to the model (see, for example, the discussion in Dai and Singleton (2000) of the Duffie and Singleton (1997) model). In our setting, this appears not to be the case. Taking the $LRSQ(3,3)$ specification as an example, the unconditional correlation is strongly positive between $Z_{2t}$ and $Z_{3t}$ (0.82), somewhat positive between $Z_{1t}$ and $Z_{2t}$ (0.19), and virtually zero between $Z_{1t}$ and $Z_{3t}$ (0.01). This is consistent with the lower bi-diagonal structure of $\kappa$.

In terms of the economic interpretation of the term structure factors, we consider the normalized factors $\bar{Z}_t$ defined in Section I.E. First, in light of (33) it is natural to scale $\bar{Z}_t$ by $(\alpha + \kappa_{33})$ such that the third factor becomes the short rate, $\bar{Z}_{3t} = r_t$. Second, from (25) together with (32) it follows that the risk-neutral drift $\bar{\mu}_t$ of $(\bar{Z}_{1t}, \bar{Z}_{2t}, r_t)$ is

$$\bar{\mu}_{1t} = \alpha(\alpha + \kappa_{33} - \bar{Z}_{1t} - \bar{Z}_{2t} - r_t) - (\alpha + \kappa_{11} - r_t)\bar{Z}_{1t}$$
$$\bar{\mu}_{2t} = (\alpha + \kappa_{11})\bar{Z}_{1t} - (\alpha + \kappa_{22} - r_t)\bar{Z}_{2t}$$
$$\bar{\mu}_{3t} = (\alpha + \kappa_{22})\bar{Z}_{2t} - (\alpha + \kappa_{33} - r_t)r_t.$$

(34)

The risk-neutral drift has an intuitive structure with the short rate mean-reverting toward a level determined by $\bar{Z}_{2t}$, which in turn mean-reverts toward a level determined by $\bar{Z}_{1t}$. Note that the speeds of mean-reversion are decreasing in the short rate.\(^{27}\)

Panels A, B, and C in Figure 4 show the loadings of zero-coupon bond yields on $\bar{Z}_{1t}$, $\bar{Z}_{2t}$, and $r_t$, respectively, in the case of the $LRSQ(3,3)$ specification. In each panel, the grey lines show the factor loadings when the yield expression is linearized around each state vector during the sample period (i.e., the loadings from (23) normalized by $(\alpha + \kappa_{33})$), and the black line shows the factor loadings when the yield expression is linearized around $(\alpha + \kappa_{33})\theta/(\phi + \psi^\top\theta)$ (i.e., the loadings from (24) normalized by $(\alpha + \kappa_{33})$). In general, the factor loadings obtained from (24) constitute a reasonable approximation to the state-dependent factor loadings. A shock to $r_t$ lifts all yields, with the effect being strongest for short maturities. The effect of a shock to $\bar{Z}_{1t}$ is zero at the short end and increases with maturity, and hence it constitutes a slope factor. The effect of a shock to $\bar{Z}_{2t}$ is zero at the short end and is most pronounced for intermediate maturities, and hence it constitutes a curvature factor.

In the time series, the correlation between $r_t$ and the 1-year swap rate is 0.85 in weekly changes (0.99 in levels). For the other two term structure factors, $\bar{Z}_{1t}$ is most highly correlated with the slope of the term structure (in weekly changes, the correlation with the spread between the 10-year and 1-year swap rates is 0.62), while $\bar{Z}_{2t}$ is most highly correlated with the curvature of the term structure (in weekly changes, the correlation with the 1-5-10 “butterfly” spread—twice the 5-year rate swap minus the sum of the 1-year and 10-year swap rates—is 0.73).

\(^{27}\) The Internet Appendix shows that the system has a unique interior stationary point for which $\bar{\mu}_t = 0$. 
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Panel A, $\bar{Z}_{1t}$

Panel B, $\bar{Z}_{2t}$

Panel C, $r_t$

Panel D, $\bar{Z}_{1t}$ (lhs) and $\tilde{Z}_{1t}$ (rhs)

Panel E, $\bar{Z}_{2t}$ (lhs) and $\tilde{Z}_{2t}$ (rhs)

Panel F, $r_I$

**Figure 4. Factor loadings.** Panels A, B, and C show the loadings of zero-coupon bond yields on the normalized factors $\bar{Z}_t$ in the case of the LRSQ(3,3) specification. $\bar{Z}_t$ is scaled by $(\alpha + \kappa_{33})$ such that the third factor becomes the short rate, $\bar{Z}_3t = r_t$. In each panel, the grey lines show the factor loadings when the yield expression is linearized around each state vector during the sample period (i.e., the loadings from (23) normalized by $(\alpha + \kappa_{33})$), and the black line shows the factor loadings when the yield expression is linearized around $(\alpha + \kappa_{33})\theta / (\phi + \psi^T \theta)$ (i.e., the loadings from (24) normalized by $(\alpha + \kappa_{33})$). Panels D, E, and F show the loadings on $\bar{Z}_t$ for the LRSQ(3,0) model (solid lines, left-hand scale) and on $\tilde{Z}_t$ for the $A_3(3)$ model (dashed-dotted lines, right-hand scale) when both models are estimated on swap rates only. $\bar{Z}_t$ is scaled by $(\alpha + \kappa_{33})$ and $\tilde{Z}_t$ is scaled by $\gamma_3$ such that in both cases the third factor becomes the short rate, $\bar{Z}_3t = r_t$ and $\tilde{Z}_3t = r_t$.

**F. Term Structure Dynamics Near the Zero Lower Bound**

A key characteristic of the recent history of U.S. interest rates is the extended period of near-zero policy rates. Even more striking, Japan has experienced near-zero policy rates since the early 2000s. A challenge for term structure models is to generate such extended periods of low short rates. A related challenge, as emphasized by Kim and Singleton (2012), is that near the ZLB the distribution of future short rates becomes highly asymmetric with the most likely (modal) values being significantly lower than the mean values. Kim and Singleton (2012) show that models that belong to the shadow-rate and exponential-quadratic frameworks are able to generate these patterns. Here we investigate if this is also the case for the LRSQ model.\(^{28}\) For this purpose, we simulate 50,000 years of weekly data (2,600,000 observations) from

\(^{28}\)Note that Kim and Singleton (2012) focus on the conditional $Q$-distribution, while the focus here is on the conditional $P$-distribution.
Panel A: Conditional dist., 1 yr

Panel B: Conditional dist., 2 yrs

Panel C: Conditional dist., 5 yrs

Panel D: Mean and median

Figure 5. Conditional distribution of the short rate. Conditional on the short rate being between 0 and 25 basis points, Panels A, B, and C display histograms showing the frequency distribution of the future short rate at the 1-year, 2-year, and 5-year horizon, respectively. Panel D displays the mean and median paths of the short rate. The frequency distributions are obtained from 2,600,000 weekly observations (50,000 years) of the short rate simulated from the $LRSQ(3,3)$ specification.

We use simulated data instead of fitted data because the simulated data are much more revealing about the true properties of the model (this also applies to the analyses in the subsequent sections).

Taking the $LRSQ(3,3)$ specification as an example and conditioning on the short rate being between 0 and 25 bps, Panels A, B, and C in Figure 5 display histograms showing the frequency distributions of the future short rate at the horizon.
1-year, 2-year, and 5-year horizon, respectively. Clearly, the model generates persistently low short rates; conditional on the short rate being between 0 and 25 bps at a given point in time, the likelihood of the short rate being in the same interval after 1, 2, and 5 years is 64%, 47%, and 24%, respectively. Also, the conditional distributions of the future short rates are highly asymmetric—the most likely interval for the short rate remains 0 to 25 bps at all displayed horizons, while Panel D shows the mean (median) value of the short rate rising to 1.12% (0.80%) at a 5-year horizon.

The model’s ability to generate persistently low short rates can be understood from the short-rate dynamics discussed in Section IV.E. When all factors are close to zero, market prices of risk are close to zero, and the $\mathbb{P}$-drift of $(\tilde{Z}_{1t}, \tilde{Z}_{2t}, r_t)$ is closely approximated by (34). Therefore, $r_t$ does not immediately drift away from the ZLB; rather, the drift of $r_t$ is constrained by a low value of $\tilde{Z}_{2t}$, the drift of which is constrained by a low value of $\tilde{Z}_{1t}$. It appears that this “cascading” structure causes $r_t$ to very slowly move away from the ZLB.

Extended periods of low short rates also have an impact on the loadings on the first principal component (PC) of the term structure. We construct PC loadings from the covariance matrix of weekly changes in swap rates. Figure 6 displays PC loadings when the short rate is away from the ZLB (black lines) and close to the ZLB (grey lines). Solid lines correspond to the data. During the pre-ZLB period, the PC loadings are relatively flat (in fact somewhat hump-shaped). For this reason, the first PC is traditionally referred to as a “level” factor. However, during the ZLB period, the PC loadings increase strongly with maturity, rising from 0.13 for the 1-year maturity to 0.52 for the 10-year maturity. As such, the first PC effectively becomes a “slope” factor.

The figure also shows the PC loadings in the simulated weekly data, in which the ZLB sample consists of those observations for which the short rate is less than 25 bps. The dashed, dashed-dotted, and dotted line corresponds to the $LRSQ(3,1)$, $LRSQ(3,2)$, and $LRSQ(3,3)$ specification, respectively. In general, the model-implied PC loadings are close to those observed in the data, particularly for the ZLB sample. As such, the model captures how the first PC transitions into a slope factor when the short rate is near the ZLB.

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30 Note that the short rate does not literally stay at zero; it merely has a high probability of remaining very close to the ZLB. Shadow rate models have the advantage that the short rate can stay at zero for extended periods.

31 The impact on the loadings on the second and third PCs appear less pronounced, and thus we focus on the loadings on the first PC.

32 In contemporaneous work, Kim and Priebsch (2013) also note a change in the loadings on the first PC during the ZLB period.

33 The PC loadings are not to be confounded with the loadings on the normalized term structure factors that are shown in Figure 4. It is the combination of state-dependent factor loadings and a state-dependent instantaneous covariance matrix of the factors that generate the change in PC loadings across ZLB and non-ZLB periods.
**Figure 6. Loadings on the first principal component of the term structure.** The figure shows the loadings on the first principal component (PC) of the term structure when the short rate is away from the zero lower bound (ZLB, thin black lines) and close to the ZLB (thick grey lines). The PC loadings are constructed from the eigenvector corresponding to the largest eigenvalue of the covariance matrix of weekly changes in swap rates. The solid lines show the PC loadings in the data, where the non-ZLB sample period consists of 620 weekly observations from January 29, 1997 to December 16, 2008, and the ZLB sample period consists of 246 weekly observations from December 16, 2008 to August 28, 2013. The dashed, dashed-dotted, and dotted line shows the PC loadings implied by the LRSQ(3,1), LRSQ(3,2), and LRSQ(3,3) specification, respectively. The model-implied PC loadings are obtained from simulated data, where each time series consists of 2,600,000 weekly observations (50,000 years). In the simulated data, the non-ZLB (ZLB) sample consists of those observations where the short rate is larger (less) than 25 basis points.

**G. Volatility Dynamics Near the Zero Lower Bound**

A large literature investigates the dynamics of interest rate volatility. A particular focus has been on the extent to which variation in volatility is related to variation in the term structure. Using data that predate the current
environment of very low interest rates, several papers find that a large component of volatility is effectively unrelated to the term structure. Here, we provide an important qualification to this result: volatility becomes compressed and gradually more level-dependent as interest rates approach the ZLB. This is illustrated by Figure 1, which, as we mention in the introduction, plots the 3-month NIV of the 1-year swap rate (in bps) against the level of the 1-year swap rate. The grey area in Figure 1 marks the possible range of implied volatilities in the case of the LRSQ(3,3) specification and shows that the model qualitatively captures the observed pattern.

More formally, for each swap maturity, we regress weekly changes in the 3-month NIV of the swap rate on weekly changes in the level of the swap rate (including a constant),

\[ \Delta \sigma_{N,t} = \beta_0 + \beta_1 \Delta S_t + \epsilon_t. \]

We run these regressions unconditionally as well as conditional on swap rates being in the intervals 0% to 1%, 1% to 2%, 2% to 3%, 3% to 4%, and 4% to 5%, respectively. Results are displayed in Table IV, where Panel A shows \( \hat{\beta}_1 \)s with Newey and West (1987) \( t \)-statistics in parentheses and Panel B shows \( R^2 \)s. Within each panel, the first row displays results of the unconditional regressions, while the second to sixth rows display results of the conditional regressions. The right-most column reports average \( \hat{\beta}_1 \)s and \( R^2 \)s across swap maturities. In the unconditional regressions, the \( \hat{\beta}_1 \)s are positive but relatively small in magnitudes (between 0.16 and 0.18), and while the coefficients are statistically significant (\( t \)-statistics between 2.80 and 5.49), the \( R^2 \)s are small (between 0.05 and 0.11). That is, unconditionally, there is a relatively small degree of positive level-dependence in volatility.

A more nuanced picture emerges from the conditional regressions. Conditional on swap rates being in the interval 0% to 1%, the \( \hat{\beta}_1 \)s are positive, large in magnitude (between 0.48 and 1.20), and very highly statistically significant (\( t \)-statistics between 7.83 and 8.79 despite fewer observations than in the unconditional regressions), and the \( R^2 \)s are large (between 0.44 and 0.54). In other words, there is a strong and positive relation between volatility and swap rate

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34 See Collin-Dufresne and Goldstein (2002) and subsequent papers by Heidari and Wu (2003), Andersen and Benzoni (2010), Li and Zhao (2006, 2009), Trolle and Schwartz (2009), and Collin-Dufresne, Goldstein, and Jones (2009), among others.

35 In the Internet Appendix, we overlay data from the four largest swap markets—the United States, the Eurozone, the United Kingdom, and Japan—and show that a similar pattern is observed for all the swap maturities considered in the paper (we include international data to increase the number of data points with very low interest rates).

36 We run the regressions in first-differences to avoid spurious results due to the persistence of implied volatilities and interest rates.

37 Trolle and Schwartz (2014) also document that, unconditionally, there is a positive level-dependence in NIVs. An earlier literature estimates generalized diffusion models for the short-term interest rate; see, for example, Chan et al. (1992), Ait-Sahalia (1996), Conley et al. (1997), and Stanton (1997). These papers generally find a relatively strong level-dependence in interest rate volatility. However, much of this level-dependence can be attributed to the monetary policy experiment in the early 1980s, which is not representative of the current monetary policy regime.
Table IV
Level-Dependence in Volatility

For each available swap maturity, the table reports results from regressing weekly changes in the 3-month normal implied volatility of the swap rate on weekly changes in the level of the swap rate (including a constant). Panel A shows the slope coefficients with t-statistics in parentheses, and Panel B shows the $R^2$s. Within each panel, the first row displays unconditional results, while the second to sixth rows display results conditional on swap rates being in the intervals 0% to 1%, 1% to 2%, 2% to 3%, 3% to 4%, and 4% to 5%, respectively. Each underlying time series consists of 866 weekly observations from January 29, 1997 to August 28, 2013. t-statistics are corrected for heteroscedasticity and serial correlation up to 12 lags using the method of Newey and West (1987). *, **, and *** denote significance at the 10%, 5%, and 1% level, respectively.

<table>
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<th>2 yrs</th>
<th>3 yrs</th>
<th>5 yrs</th>
<th>7 yrs</th>
<th>10 yrs</th>
<th>Mean</th>
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<td>0.16***</td>
<td>0.16***</td>
<td>0.16***</td>
<td>0.16***</td>
<td>0.16***</td>
<td>0.16</td>
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<tr>
<td>(2.38)</td>
<td>(2.88)</td>
<td>(3.31)</td>
<td>(4.12)</td>
<td>(4.59)</td>
<td>(4.97)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0%-1%</td>
<td>1.20***</td>
<td>0.74***</td>
<td>0.62***</td>
<td>0.48***</td>
<td></td>
<td></td>
<td>0.76</td>
</tr>
<tr>
<td>(8.03)</td>
<td>(8.79)</td>
<td>(8.19)</td>
<td>(7.83)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1%-2%</td>
<td>0.54***</td>
<td>0.64***</td>
<td>0.46***</td>
<td>0.52***</td>
<td>0.45***</td>
<td>0.26***</td>
<td>0.48</td>
</tr>
<tr>
<td>(2.70)</td>
<td>(6.21)</td>
<td>(6.77)</td>
<td>(5.02)</td>
<td>(5.23)</td>
<td>(8.24)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2%-3%</td>
<td>0.28***</td>
<td>0.11**</td>
<td>0.30***</td>
<td>0.36***</td>
<td>0.40***</td>
<td>0.40***</td>
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<tr>
<td>(3.10)</td>
<td>(1.97)</td>
<td>(3.77)</td>
<td>(5.08)</td>
<td>(5.62)</td>
<td>(4.93)</td>
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<tr>
<td>3%-4%</td>
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<td>0.11</td>
<td>0.06</td>
<td>0.05</td>
<td>0.11*</td>
<td>0.17*</td>
<td>0.08</td>
</tr>
<tr>
<td>(−0.22)</td>
<td>(1.21)</td>
<td>(0.92)</td>
<td>(0.80)</td>
<td>(1.82)</td>
<td>(1.96)</td>
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<td>4%-5%</td>
<td>0.04</td>
<td>−0.07</td>
<td>0.01</td>
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<tr>
<td>(0.31)</td>
<td>(−0.82)</td>
<td>(0.08)</td>
<td>(1.59)</td>
<td>(1.76)</td>
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<table>
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<tr>
<th>Panel B: $R^2$</th>
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<th>7 yrs</th>
<th>10 yrs</th>
<th>Mean</th>
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<td>(0.52)</td>
<td>(0.54)</td>
<td>(0.54)</td>
<td>(0.44)</td>
<td>(0.55)</td>
<td>(0.55)</td>
<td>(0.27)</td>
<td>0.51</td>
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<td>0%-1%</td>
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<td>0.45</td>
<td>0.55</td>
<td>0.55</td>
<td>0.27</td>
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<td>(0.16)</td>
<td>(0.06)</td>
<td>(0.28)</td>
<td>(0.37)</td>
<td>(0.44)</td>
<td>(0.45)</td>
<td>(0.29)</td>
<td></td>
</tr>
<tr>
<td>1%-2%</td>
<td>0.00</td>
<td>0.03</td>
<td>0.01</td>
<td>0.01</td>
<td>0.07</td>
<td>0.12</td>
<td>0.04</td>
</tr>
<tr>
<td>(0.00)</td>
<td>(0.01)</td>
<td>(0.00)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.02)</td>
<td></td>
</tr>
<tr>
<td>2%-3%</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.02</td>
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</table>

changes when swap rates are close to the ZLB. However, as the conditioning interval increases in level, the relation between volatility and swap rate changes becomes progressively weaker, and volatility exhibits very little level-dependence at moderate levels of swap rates. For instance, conditional on swap rates being in the interval 4% to 5%, the $\hat{\beta}_1$s are close to zero (between −0.07 and 0.08) and mostly statistically insignificant (t-statistics between −0.82 and 1.76), and the $R^2$s are very small (between 0.00 and 0.03).

We next perform the same analysis on the simulated weekly data. To succinctly summarize the results, we focus on the $LRSQ(3,3)$ specification and its ability to capture the average $\hat{\beta}_1$s and $R^2$s across swap maturities. Figure 7 shows the average $\hat{\beta}_1$s and $R^2$s in the data (Panels A and C) and those implied

---

38 In Japanese data, Kim and Singleton (2012) also note the high degree of level-dependence in volatility when interest rates are close to the ZLB.
Figure 7. Level-dependence in volatility. For each swap maturity, weekly changes in the 3-month normal implied volatility of the swap rate are regressed on weekly changes in the level of the swap rate (including a constant). Regressions are run unconditionally as well as conditional on swap rates being in the intervals 0% to 1%, 1% to 2%, 2% to 3%, 3% to 4%, and 4% to 5%, respectively. Panels A and C show the average (across swap maturities) slope coefficients and $R^2$s, respectively. Panels B and D show the average (across swap maturities) model-implied slope coefficients and $R^2$s, respectively. In each panel, the first bar corresponds to the unconditional regressions, while the second to sixth bars correspond to the conditional regressions. Model-implied values are obtained by running the regressions on data simulated from the LRSQ(3,3) specification, where each time series consists of 2,600,000 weekly observations (50,000 years).

by the model (Panels B and D). In the unconditional regressions, the average $\hat{\beta}_1$ and $R^2$ in the data are 0.16 and 0.08, respectively, while the corresponding model-implied values are 0.15 and 0.15, respectively. In the conditional regressions, the average $\hat{\beta}_1$ and $R^2$ in the data are 0.76 and 0.51 when swap rates are between 0% and 1%, decreasing to 0.03 and 0.02 when swap rates
are between 4% and 5%. This is closely matched by the model, which generates values of 0.72 and 0.52 when swap rates are between 0% and 1%, decreasing to 0.10 and 0.12 when swap rates are between 4% and 5%. As such, the model largely matches the changing volatility dynamics as interest rates approach the ZLB.39

**H. Risk Premium Dynamics**

Next, we study risk premiums in swap contracts. We first establish key properties of swap risk premiums. We then investigate whether the model is able to capture those properties. Returns on swap contracts can be computed in several ways. To facilitate comparison with existing literature on Treasury risk premiums, we consider excess returns on zero-coupon bonds bootstrapped from the swap term structure. In the Internet Appendix, we report results for excess returns on synthetic coupon bonds constructed from swap contracts. We always consider nonoverlapping monthly excess returns (relative to 1-month LIBOR) computed using closing prices on the last business day of each month.

The top panel in Table V shows the unconditional means and volatilities of excess returns in addition to the unconditional Sharpe ratios. All statistics are annualized. Both the mean and volatility increase with bond maturity, while the Sharpe ratio peaks at 0.91 for a 2-year bond and then decreases with bond maturity. In the case of Treasury bonds, Duffee (2010) and Frazzini and Pedersen (2014) also note that the Sharpe ratio decreases with bond maturity. The mean excess returns and Sharpe ratios are high in our sample, but are inflated by the downward trend in swap rates over the sample period.

We also consider conditional expected excess returns. For a $\tau$-year bond, we regress its excess return on the previous month’s term structure slope and implied volatility (including a constant),

$$R_{t+1}^e = \beta_0 + \beta_{Slp} Slp_t + \beta_{Vol} Vol_t + \epsilon_{t+1},$$

where $Slp_t$ is the difference between the $\tau$-year swap rate and 1-month LIBOR, and $Vol_t$ is the NIV of a swaption with a 1-month option expiry and a $\tau$-year swap maturity.40 We use a 1-month option expiry to match the horizon in the predictive regression. Excess returns are in percent, and both $Slp_t$ and $Vol_t$ are standardized to facilitate comparison of regression coefficients. Results are reported in the top panel in Table VI with Newey and West (1987) $t$-statistics given in parentheses. Many papers, typically using long samples of Treasury

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39 The Internet Appendix shows that the $LRSQ(3,2)$ specification generates similar volatility dynamics, while the $LRSQ(3,1)$ specification generates too high a degree of level-dependence in volatility, both unconditionally and conditionally, as a consequence of having only one USV factor.

40 Cochrane and Piazzesi (2005) propose an alternative predictor variable given as a linear combination of forward rates. In the Internet Appendix, we show that the predictive power of their factor is not fully captured by the first three PCs of swap rates and implied volatilities. Hence, there is a dimension of excess return predictability that is unlikely to be generated by the LRSQ specifications studied in this paper given that they have three term structure factors and at most three USV factors.
Table V

Unconditional Excess Returns on Zero-Coupon Bonds

The table reports the annualized means and volatilities of nonoverlapping monthly excess returns on zero-coupon bonds bootstrapped from swap rates. Also reported are the annualized Sharpe ratios (SR). Excess returns are in percent. The top panel shows results in the data, where each time series consists of 200 monthly observations from February 1997 to August 2013. The second, third, and fourth panel shows results in simulated data from the $LRSQ(3,1)$, $LRSQ(3,2)$, and $LRSQ(3,3)$ specification, respectively. The bottom panel shows results in simulated data from the $LRSQ(3,3)$ specification that imposes $\delta_t = 0$ (denoted $LRSQ(3,3)^\dagger$). Each simulated time series consists of 600,000 monthly observations (50,000 years).

<table>
<thead>
<tr>
<th></th>
<th>1 yr</th>
<th>2 yrs</th>
<th>3 yrs</th>
<th>5 yrs</th>
<th>7 yrs</th>
<th>10 yrs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>Mean</td>
<td>0.58</td>
<td>1.56</td>
<td>2.39</td>
<td>3.61</td>
<td>4.46</td>
</tr>
<tr>
<td></td>
<td>Vol</td>
<td>0.71</td>
<td>1.72</td>
<td>2.82</td>
<td>4.96</td>
<td>6.96</td>
</tr>
<tr>
<td></td>
<td>SR</td>
<td>0.82</td>
<td>0.91</td>
<td>0.85</td>
<td>0.73</td>
<td>0.64</td>
</tr>
<tr>
<td>$LRSQ(3,1)$</td>
<td>Mean</td>
<td>0.37</td>
<td>0.74</td>
<td>1.10</td>
<td>1.77</td>
<td>2.39</td>
</tr>
<tr>
<td></td>
<td>Vol</td>
<td>0.57</td>
<td>1.28</td>
<td>2.14</td>
<td>4.02</td>
<td>5.83</td>
</tr>
<tr>
<td></td>
<td>SR</td>
<td>0.64</td>
<td>0.58</td>
<td>0.51</td>
<td>0.44</td>
<td>0.41</td>
</tr>
<tr>
<td>$LRSQ(3,2)$</td>
<td>Mean</td>
<td>0.37</td>
<td>0.70</td>
<td>1.01</td>
<td>1.60</td>
<td>2.14</td>
</tr>
<tr>
<td></td>
<td>Vol</td>
<td>0.53</td>
<td>1.21</td>
<td>1.97</td>
<td>3.54</td>
<td>5.04</td>
</tr>
<tr>
<td></td>
<td>SR</td>
<td>0.69</td>
<td>0.58</td>
<td>0.51</td>
<td>0.45</td>
<td>0.42</td>
</tr>
<tr>
<td>$LRSQ(3,3)$</td>
<td>Mean</td>
<td>0.25</td>
<td>0.58</td>
<td>0.91</td>
<td>1.53</td>
<td>2.04</td>
</tr>
<tr>
<td></td>
<td>Vol</td>
<td>0.57</td>
<td>1.19</td>
<td>1.92</td>
<td>3.51</td>
<td>5.06</td>
</tr>
<tr>
<td></td>
<td>SR</td>
<td>0.43</td>
<td>0.48</td>
<td>0.47</td>
<td>0.44</td>
<td>0.40</td>
</tr>
<tr>
<td>$LRSQ(3,3)^\dagger$</td>
<td>Mean</td>
<td>−0.03</td>
<td>0.01</td>
<td>0.10</td>
<td>0.34</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>Vol</td>
<td>1.01</td>
<td>1.71</td>
<td>2.35</td>
<td>3.75</td>
<td>5.23</td>
</tr>
<tr>
<td></td>
<td>SR</td>
<td>−0.03</td>
<td>0.01</td>
<td>0.04</td>
<td>0.09</td>
<td>0.11</td>
</tr>
</tbody>
</table>

data, find that the slope of the term structure predicts excess bond returns with a positive sign; see, for example, Campbell and Shiller (1991) and Dai and Singleton (2002). In our more recent (and shorter) sample of swap data, the predictive power of the slope of the term structure is relatively weak. Indeed, the regression coefficient is never statistically significant and is even negative for 1-year and 2-year bond maturities. In contrast, the predictive power of implied volatility is much stronger. The regression coefficient is statistically significant for bond maturities up to 3 years and is positive for all bond maturities indicating a positive risk-return tradeoff in the swap market.\footnote{For the Treasury market, there is mixed evidence for a risk-return tradeoff. Engle, Lilien, and Robins (1987) and, more recently, Ghysels et al. (2014) find evidence of a positive risk-return tradeoff in an ARCH-in-mean framework, while Duffee (2002) runs regressions similar to those in Table VI and finds that volatility only weakly predicts returns. The differences between the results of Duffee (2002) and our results are likely due to some combination of differences in sample period (his sample ends where our sample begins), his use of historical volatilities versus our use of forward-looking implied volatilities, and perhaps structural differences between the Treasury and swap markets. Note also that many equilibrium term structure models including those within the long-run-risk framework (such as Bansal and Shaliastovich (2012)) and the habit-based framework (such as Wachter (2006)) generally imply a positive risk-return tradeoff; see also the discussion in Le and Singleton (2013).} Economically, volatility is also a more important predictor of excess returns. Taking the
Table VI

Conditional Excess Returns on Zero-Coupon Bonds

The table reports results from regressing nonoverlapping monthly excess zero-coupon bond returns on the previous month's term structure slope and implied volatility (including a constant). Consider the results for the 5-year maturity. The excess return is on a 5-year zero-coupon bond bootstrapped from swap rates. The term structure slope is the difference between the 5-year swap rate and 1-month LIBOR. The implied volatility is the normal implied volatility of a swaption with a 1-month option expiry and a 5-year swap maturity. Excess returns are in percent, and the term structure slopes and implied volatilities are standardized. The top panel shows results in the data, where each time series consists of 200 monthly observations from February 1997 to August 2013. t-statistics, corrected for heteroscedasticity and serial correlation up to 12 lags using the method of Newey and West (1987), are in parentheses. *, **, and *** denote significance at the 10%, 5%, and 1% level, respectively. The second, third, and fourth panel shows results in simulated data from the \( LRSQ(3,1) \), \( LRSQ(3,2) \), and \( LRSQ(3,3) \) specification, respectively. The bottom panel shows results in simulated data from the \( LRSQ(3,3) \) specification that imposes \( \delta_t = 0 \) (denoted \( LRSQ(3,3)^\dagger \)). Each simulated time series consists of 600,000 monthly observations (50,000 years).

<table>
<thead>
<tr>
<th></th>
<th>1 yr</th>
<th>2 yrs</th>
<th>3 yrs</th>
<th>5 yrs</th>
<th>7 yrs</th>
<th>10 yrs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>( \hat{\beta}_{\text{Slp}} )</td>
<td>-0.025</td>
<td>-0.009</td>
<td>0.027</td>
<td>0.092</td>
<td>0.121</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\beta}_{\text{Vol}} )</td>
<td>0.058***</td>
<td>0.114***</td>
<td>0.144**</td>
<td>0.169</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R^2 )</td>
<td>(4.549)</td>
<td>(3.409)</td>
<td>(2.506)</td>
<td>(1.546)</td>
</tr>
<tr>
<td>LRSQ(3,1)</td>
<td>( \hat{\beta}_{\text{Slp}} )</td>
<td>0.004</td>
<td>0.003</td>
<td>-0.004</td>
<td>-0.032</td>
<td>-0.065</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\beta}_{\text{Vol}} )</td>
<td>0.012</td>
<td>0.017</td>
<td>0.026</td>
<td>0.058</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R^2 )</td>
<td>0.007</td>
<td>0.003</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>LRSQ(3,2)</td>
<td>( \hat{\beta}_{\text{Slp}} )</td>
<td>0.000</td>
<td>0.002</td>
<td>0.008</td>
<td>0.018</td>
<td>0.021</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\beta}_{\text{Vol}} )</td>
<td>0.016</td>
<td>0.033</td>
<td>0.049</td>
<td>0.072</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R^2 )</td>
<td>0.011</td>
<td>0.009</td>
<td>0.008</td>
<td>0.005</td>
</tr>
<tr>
<td>LRSQ(3,3)</td>
<td>( \hat{\beta}_{\text{Slp}} )</td>
<td>0.025</td>
<td>0.038</td>
<td>0.046</td>
<td>0.055</td>
<td>0.059</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\beta}_{\text{Vol}} )</td>
<td>0.031</td>
<td>0.054</td>
<td>0.074</td>
<td>0.112</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R^2 )</td>
<td>0.082</td>
<td>0.054</td>
<td>0.035</td>
<td>0.020</td>
</tr>
<tr>
<td>LRSQ(3,3)(^\dagger)</td>
<td>( \hat{\beta}_{\text{Slp}} )</td>
<td>-0.002</td>
<td>-0.001</td>
<td>0.001</td>
<td>0.006</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\beta}_{\text{Vol}} )</td>
<td>-0.004</td>
<td>-0.002</td>
<td>0.005</td>
<td>0.026</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R^2 )</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
</tr>
</tbody>
</table>

5-year maturity as an example, a one-standard-deviation increase in volatility (the term structure slope) increases 1-month expected excess returns by 17 bps (9 bps), which should be put in relation to the unconditional mean of the 1-month excess return of 30 bps.

To investigate whether our model is able to capture these characteristics of risk premiums, we simulate 50,000 years of monthly data (600,000 observations) from each of the three model specifications. Note that swaptions with 1-month option expiries are not included in the estimation, giving an out-of-sample flavor to the exercise. The results are reported in the second, third, and fourth panels in Tables V and VI. Table V shows that all model specifications capture the pattern that the mean and volatility of excess returns increase with bond maturity, while the Sharpe ratio decreases with bond maturity (the \( LRSQ(3,3) \) specification even replicates the small hump in the
Sharpe ratio term structure). The mean excess returns and Sharpe ratios are lower than those observed in the data, which is to be expected given that we are simulating stationary samples of interest rates, while actual interest rates exhibit a downward trend over the sample period.

Table VI shows that the model qualitatively captures the predictive power of implied volatility for excess returns. The size of the regression coefficients as well as the $R^2$’s increase with the number of USV factors. For the $LRSQ(3,3)$ specification, the ratio between model-implied and actual $\hat{\beta}_{\text{Vol}}$ ($R^2$) lies between 0.47 and 0.87 (0.70 and 1.06) across bond maturities. This performance appears reasonable considering the parsimonious market price of risk specification that we employ.\footnote{As a further “sanity check” of the estimates of the market prices of risk, we follow Duffee (2010) in considering the model-implied maximum conditional Sharpe ratio, which is given by $\sqrt{\lambda_{\text{Vol}}^\top \lambda_{\text{Vol}}^t}$ in our setting. Based on the simulated data, we infer that the population means of this quantity are 1.40, 0.91, and 0.64 for the $LRSQ(3,1)$, $LRSQ(3,2)$, and $LRSQ(3,3)$ specifications, respectively. For comparison, Duffee (2010) reports that, for a standard three-factor Gaussian term structure model, and considering log-returns, the population mean is 1.07 (when annualized). As such, our market price of risk estimates appears reasonable.}

A different perspective on the pricing of risk comes from decomposing the yield curve into expectations about future short-term interest rates and term premiums. As an example, the black line in Figure 8 shows the time series of the 5-year term premium implied by the $LRSQ(3,3)$ specification. The term premium is defined as the difference between the fitted 5-year zero-coupon bond yield, $-\frac{1}{5} \log P(t, t + 5)$, and the conditional $\mathbb{P}$-expectation of the average future short rate over the next five years, $\mathbb{E}_t \left[ \int_t^{t+5} r_s ds \right]$ (obtained via simulation). For comparison, the light-grey line shows the equivalent 5-year term premium in the Treasury market as estimated in a recent paper by Adrian, Crump, and Moench (2013).\footnote{Many papers estimate term premiums in the Treasury market. We benchmark against the Adrian, Crump, and Moench (2013) model because estimated term premiums are publicly available (http://www.newyorkfed.org/research/data_indicators/term_premia.html) and often referred to by commentators; see, for example, Bernanke (2015).} They use a high-dimensional Gaussian model in which risk premiums depend largely on higher-order principal components that have negligible impact on the term structure; in contrast, we use a model in which risk premiums depend explicitly on volatility factors. The average estimated term premiums in the swap and Treasury markets are 94 bps and 70 bps, respectively. Relative to the average zero-coupon bond yields of 4.11% and 3.65%, respectively, the average term premiums are roughly similar in the two markets. Although the Treasury term premium is more volatile than the swap term premium, the two are highly correlated with a correlation coefficient of 0.78.\footnote{The significant decrease in the term premium toward the end of the sample period coincides with the Federal Reserve engaging in multiple rounds of quantitative easing (QE). From the announcement of the first round of QE (on November 25, 2008) to the first indication by Ben Bernanke that the third round of QE would be tapered (on May 22, 2013), the term premium decreased by 125 bps (versus 130 bps in the Treasury market). While this is suggestive of QE...} Indeed, Adrian, Crump, and Moench (2013) note that their term premium estimates correlate strongly with volatility during a sample period...
Figure 8. Five-year term premium. The 5-year term premium is defined as the difference between the fitted 5-year zero-coupon bond yield, $-\frac{1}{5}\log P(t, t + 5)$, and the conditional $\mathbb{P}$-expectation of the average future short rate over the next five years, $\mathbb{E}_t^\mathbb{P}[\frac{1}{5}\int_{t}^{t+5} r_s ds]$. At each date, the latter is obtained by simulation using 2,000 paths (1,000 plus 1,000 antithetic). The black line shows the term premium for the $LRSQ(3,3)$ specification. The dark-grey line shows the term premium for the $LRSQ(3,3)$ specification with $\delta_t = 0$. The light-grey line shows the 5-year term premium in the Treasury market as estimated by Adrian, Crump, and Moench (2013). The grey areas mark the two NBER-designated recessions from March 2001 to November 2001 and from December 2007 to June 2009. The first vertical dotted line marks the announcement of the first round of quantitative easing on November 25, 2008. The second vertical dotted line marks Ben Bernanke’s congressional testimony on May 22, 2013, which indicated that asset purchases would be tapered later that year. Each time series consists of 866 weekly observations from January 29, 1997 to August 28, 2013.

that largely overlaps with ours. This supports the use of a stochastic volatility model for studying risk premiums.

To illustrate the importance of the extended state price density specification for achieving our results for risk premium dynamics, we also report results for the $LRSQ(3,3)$ specification that imposes $\delta_t = 0$. The bottom panel in Table V shows that $\delta_t = 0$ significantly reduces the mean excess returns and Sharpe ratios. For instance, at the 5-year maturity the Sharpe ratio decreases from 0.44 to 0.09. The bottom panel in Table VI shows that $\delta_t = 0$ results in significantly less predictability in excess returns. For instance, at the 5-year maturity $\hat{\beta}_{\text{vol}} (\hat{\beta}_{\text{slp})} [R^2]$ drops from 0.112 (0.055) [0.020] to 0.06 (0.026) [0.001]. Finally, the dark-grey line in Figure 8 shows that $\delta_t = 0$ significantly reduces the level and variability of the 5-year term premium, with the mean decreasing from 94 bps having an effect on the swap market, a full analysis of this issue is beyond the scope of the present paper.
to 16 bps and the standard deviation decreasing from 37 bps to 9 bps. Therefore, incorporating the permanent component of the state price density is critical for generating realistic risk premium dynamics.

I. Pricing Collateralized Interest Rate Derivatives

Swap contracts are virtually always collateralized, and collateralized cash flows should be discounted using the rate that is paid on collateral, which in the case of USD-denominated swaps is the effective federal funds (FF) rate; see, for example, Filipović and Trolle (2013). Because FF-based discount factors reflect less credit and liquidity risk than LIBOR, the valuation of LIBOR-based derivatives becomes more involved. Indeed, in recent years the market has adopted a “multi-curve” approach, where discount factors used for valuing LIBOR-based derivatives are inferred from overnight indexed swaps (OISs)—swap contracts with floating-rate payments indexed to the compounded FF rate. Our framework has the advantage of being able to capture this new market reality in a tractable and parsimonious way. Details are given in the Internet Appendix; here we give the main results. We let $\zeta_t$ denote the state price density used for valuing collateralized cash flows so that $r_t$ is the (instantaneous) FF rate and $P(t, T)$ refers to the price of a collateralized zero coupon bond. The value of a fixed-rate payer OIS at time $t \leq T_0$, $\Pi_t^{\text{OIS}}$, is given by

$$\Pi_t^{\text{OIS}} = P(t, T_0) - \Delta K \sum_{i=1}^{n} P(t, T_i) + \sum_{j=1}^{N} A(t, t_{j-1}, t_j),$$

where $A(t, t_{j-1}, t_j)$ is the time-$t$ value of the $\delta$-maturity LIBOR-OIS spread that fixes at $t_{j-1}$ and is paid at $t_j$. Empirically, LIBOR-OIS spreads exhibit little dependence on the OIS term structure; therefore, we extend the $LRSQ(m,n)$ specification with an $s$-dimensional square-root diffusion, $Y_t$, independent of $X_t$ and let LIBOR-OIS spreads (and, therefore $A(t, t_{j-1}, t_j)$) depend linearly on $Y_t$. The extended model specification is denoted $LRSQ(m,n,s)$. By construction, $U_t$ is unspanned by both the OIS and IRS term structures. The swaption price at time $t \leq T_0$ is $\Pi_t^{\text{swaption}} = \mathbb{E}_t[p_{\text{swaption}}(Z_{T_0}, Y_{T_0})]/\zeta_t$, where $p_{\text{swaption}}(Z_{T_0}, Y_{T_0}) = \zeta_{T_0} \Pi_{T_0}^{\text{IRS}}$ is linear in the factors. As in the original model, $\Pi_t^{\text{swaption}}$ can be computed using Fourier transform.

As an illustration of model performance we compare the $LRSQ(3,3)$ specification estimated on the original data set with the $LRSQ(3,3,1)$ specification estimated on an extended data set that includes rates on spot-starting OISs.

---

45 Filipović and Trolle (2013) develop a model in which the LIBOR-OIS spread arises because of the risk of credit quality deterioration of current LIBOR panel banks and/or interbank market illiquidity. Here we are agnostic about the sources of the LIBOR-OIS spread.
The sample period is August 15, 2007 to August 28, 2013 (316 weekly observations). The \textit{LRSQ}(3,3,1) specification has a slightly better fit to the IRS term structure and swaptions (the mean RMSEs for IRS rates and NIVs are lower by 0.09 bps and 0.05 bps, respectively) and has a very good fit to the IRS-OIS spread term structure with a mean RMSE of 3.62 bps. Simulations show that the two model specifications exhibit very similar dynamics for IRS rates and NIVs, implying that the results in the previous sections also hold true for the extended specification.

\textit{J. Comparison with Exponential-Affine Model}

It is instructive to compare the LRSQ model with the exponential-affine model for which the factor process is also of the square-root type (the $A_m(m)$ model). The two models share the property that interest rates are nonnegative, but the latter model is severely restricted by not being able to accommodate USV and not admitting semi-analytical solutions to swaptions. For a clean comparison, we contrast the \textit{LRSQ}($m,0$) model with the $A_m(m)$ model and focus solely on their abilities to price swaps.

In the $A_m(m)$ model, the short rate is a linear function of the factors, $r_t = \gamma^\top \tilde{Z}_t$ for some $\gamma \in \mathbb{R}_+^m$, and the $\mathbb{Q}$-dynamics of the factor process are given by

$$d\tilde{Z}_t = \tilde{\kappa}(\tilde{\theta} - \tilde{Z}_t)dt + \text{Diag}(\sqrt{\tilde{Z}_{1t}}, \ldots, \sqrt{\tilde{Z}_{mt}})d\mathbb{B}_Q^2,$$

where $\tilde{\kappa}$ and $\tilde{\theta}$ satisfy standard admissibility conditions and $\mathbb{B}_Q^2$ denotes a $\mathbb{Q}$-Brownian motion. Zero-coupon bond prices are exponential-affine—instead of linear-rational—functions of the factors with $P(t, T) = e^{A(T-t) + B(T-t)^\top \tilde{Z}_t}$, where $A(\tau)$ and $B(\tau)$ solve a well-known system of ordinary differential equations. The $\mathbb{P}$-dynamics of the factor process are obtained by setting the market price of risk equal to $\lambda_t = (\lambda_1\sqrt{\tilde{Z}_{1t}}, \ldots, \lambda_m\sqrt{\tilde{Z}_{mt}})^\top$. This aligns the two models as closely as possible—they have the same $\mathbb{P}$-dynamics of the factor processes and the same number of model parameters—making it easier to isolate their structural differences.

We estimate both models with $m = 3$ using the swap data described in Section IV.A and again applying quasi-maximum likelihood together with Kalman filtering. Parameter estimates and other details are given in the Internet Appendix. The \textit{LRSQ}(3,0) model is similar to its USV extensions in that $\kappa$ has a lower bi-diagonal structure, and the drift parameters align such that (32) holds true. Therefore, after scaling $\tilde{Z}_t$ by $(\alpha + \kappa_{33})$, the risk-neutral drift of $(\tilde{Z}_{1t}, \tilde{Z}_{2t}, r_t)$ is of the form given in (34).

\textit{46} We start on August 15, 2007 because the LIBOR-OIS spread exhibits a regime switch around this date from being tight and essentially constant to being much wider and very volatile; see, for example, Filipović and Trolle (2013). The OIS data come from Bloomberg.

\textit{47} We follow Kim and Singleton (2012) in normalizing the diffusion parameters in (35) to one (to achieve identification) and setting the constant term in the short-rate expression to zero (to impose a lower bound of zero on the short rate as we do for the LRSQ model).
For the $A_3(3)$ model, $\tilde{k}$ has a lower bi-diagonal structure and, to a close approximation, $\tilde{k} \tilde{\theta} = (\varphi, 0, 0)$ and $\gamma = (0, 0, \gamma_3)$. Therefore, after scaling $\tilde{Z}_t$ by $\gamma_3$, the third factor becomes the short rate, $\tilde{Z}_{3t} = r_t$, and the risk-neutral drift $\tilde{\mu}_t$ of $(\tilde{Z}_{1t}, \tilde{Z}_{2t}, r_t)$ is

$$
\tilde{\mu}_{1t} = \varphi \gamma_3 - \tilde{k}_{11} \tilde{Z}_{1t}, \quad \tilde{\mu}_{2t} = -\tilde{k}_{21} \tilde{Z}_{1t} - \tilde{k}_{22} \tilde{Z}_{2t}, \quad \text{and} \quad \tilde{\mu}_{3t} = -\tilde{k}_{32} \tilde{Z}_{2t} - \tilde{k}_{33} r_t.
$$

The structure of the risk-neutral drift is similar to that of the $LRSQ(3,0)$ model in that the short rate mean-reverts toward a level determined by $\tilde{Z}_{2t}$, which in turn mean-reverts toward a level determined by $\tilde{Z}_{1t}$—only now the speeds of mean-reversion (and the long-run mean of the first factor) are constant.

Panels D, E, and F in Figure 4 compare the loadings of zero-coupon bond yields on $(\bar{Z}_{1t}, \bar{Z}_{2t}, r_t)$ (solid lines, left-hand scale) with those on $(\tilde{Z}_{1t}, \tilde{Z}_{2t}, r_t)$ (dashed-dotted lines, right-hand scale). The factor loadings are similar across the two models with $\bar{Z}_{1t}$ and $\tilde{Z}_{1t}$ constituting slope factors and $\bar{Z}_{2t}$ and $\tilde{Z}_{2t}$ constituting curvature factors.

The two models differ in the set of parameters that can be identified from the term structure, which is $(\tilde{k}, \tilde{\theta}, \gamma)$ in the exponential-affine model compared with $(k, \theta)$ in the linear-rational model. However, despite the linear-rational model having a more parsimonious description of the term structure, the pricing performance of the two models is virtually identical both for the entire sample period (mean RMSE of 2.73 bps for $LRSQ(3,0)$ versus 2.72 bps for $A_3(3)$) and for the ZLB period (mean RMSE of 4.48 bps for $LRSQ(3,0)$ versus 4.53 bps for $A_3(3)$).

V. Conclusion

We introduce the class of linear-rational term structure models, where the state price density is modeled such that bond prices become linear-rational functions of the factors. This class is highly tractable with several distinct advantages: (i) it ensures nonnegative interest rates, (ii) it easily accommodates unspanned factors affecting volatility and risk premiums, and (iii) it admits semi-analytical solutions to swaptions. A parsimonious model specification within the linear-rational class has a very good fit to both interest rate swaps and swaptions over the period from 1997 to 2013 and captures many features of term structure, volatility, and risk premium dynamics—including when interest rates are close to the ZLB.

Many extensions are possible. The $LRSQ(m,n)$ specification is constructed to be as parsimonious as possible. Following the recipe in Section II, more complex specifications can be constructed. In addition, we focus on unspanned factors that affect volatility but also show up as factors affecting risk premiums. As discussed in Section III, it is possible to introduce pure unspanned risk premium factors in the sense of Duffee (2011) and Joslin, Priebsch, and Singleton (2014).

48 For $(\tilde{Z}_{1t}, \tilde{Z}_{2t}, r_t)$, we display the factor loadings when the yield expression is linearized around $(\alpha + k_{33}) \tilde{\theta}/(\varphi + \psi^\top \tilde{\theta})$ (i.e., the loadings from (24) normalized by $(\alpha + k_{33})$).
More generally, the linear-rational framework can be extended or modified to be applicable to equity, foreign-exchange, credit, and commodity markets.

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Appendix A: Proofs

This Appendix provides the proofs of all theorems and a corollary of Theorem 1. Proofs of the lemmas are deferred to the Internet Appendix.

A. Proof and Corollary of Theorem 1

By the Cayley-Hamilton theorem (see Horn and Johnson (1990), Theorem 2.4.2) we know that any power $\kappa^p$ can be written as a linear combination of $\text{Id}, \kappa^\top, \ldots, \kappa^{(m-1)\top}$. Taking orthogonal complements in (9), we thus need to prove

$$\text{span}\left\{\nabla z F(\tau, z) : \tau \geq 0, z \in E\right\} = \text{span}\left\{\kappa^p \psi : p \geq 0\right\}.$$  \hspace{1cm} (A1)

Denote the left-hand side by $S$. Some calculation shows that the $z$-gradient of $F(\tau, z)$ is given by

$$\nabla z F(\tau, z) = \frac{e^{-\alpha \tau}}{\psi^\top z} \left[ e^{-\kappa^\top \tau} \psi - e^{\alpha \tau} F(\tau, z) \psi \right],$$  \hspace{1cm} (A2)

and hence $S = \text{span}\{e^{-\kappa^\top \tau} \psi - e^{\alpha \tau} F(\tau, z) \psi : \tau \geq 0, z \in E\}$. By the assumption that the short rate is not constant, there exist $z, z' \in E$ and $\tau \geq 0$ such that $F(\tau, z) \neq F(\tau, z')$. It follows that $e^{\alpha \tau}(F(\tau, z) - F(\tau, z'))\psi$, and hence $\psi$ itself, lies in $S$. We deduce that $S = \text{span}\{e^{-\kappa^\top \tau} \psi : \tau \geq 0\}$, which coincides with the right-hand side of (A1). This proves the formal expression (9).

It remains to show that $U$ given by (9) equals $U'$, defined as the largest subspace of $\ker \psi^\top$ that is invariant under $\kappa$. It follows from (9), for $p = 0$, that $U$ is a subspace of $\ker \psi^\top$. Moreover, $U$ is invariant under $\kappa$. Let $\xi \in U$. Then $\kappa^p \xi \in \ker \psi^\top \kappa^{p-1}$ for all $p \geq 1$ and hence $\kappa^p \xi \in U$. Since $U'$ is the largest subspace of $\psi^\top$ with this property, we conclude that $U \subseteq U'$. Conversely, let $\xi \in U'$. By invariance we have $\kappa^p \xi \in U' \subseteq \ker \psi^\top$ and hence $\xi \in \ker \psi^\top \kappa^p$ for any $p \geq 0$. Thus, $\xi \in U$ and we conclude that $U' \subseteq U$. This completes the proof of Theorem 1.

As a corollary of Theorem 1, in the case in which $\kappa$ is diagonalizable, we have the following equivalent characterization of a zero term structure kernel.

**Corollary A1:** Assume that $\kappa$ is diagonalizable with real eigenvalues, that is, $\kappa = S^{-1} \Lambda S$ with $S$ invertible and $\Lambda$ diagonal and real. Then the term structure kernel is zero, $U = \{0\}$, if and only if all eigenvalues of $\kappa$ are distinct and all components of $S^{-\top} \psi$ are nonzero.
**Proof:** Write $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_m)$ and consider the matrix $A = [\psi \kappa^\top \psi \ldots \kappa^{(m-1)\top} \psi]$. Writing $\hat{\psi} = S^{-\top} \psi$, the determinant of $A$ is given by

$$
\det A = \det(S^\top) \det \left( \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \cdots & \lambda_{m-1} \end{pmatrix} \right)
$$

$$
= \det(S^\top) \hat{\psi}_1 \cdots \hat{\psi}_m \prod_{1 \leq i < j \leq m} (\lambda_j - \lambda_i),
$$

where the last equality uses the formula for the determinant of the Vandermonde matrix. Theorem 1 now shows that the term structure kernel is zero, $U = \{0\}$, precisely when all eigenvalues of $\kappa$ are distinct and all components of $\hat{\psi}$ are nonzero, as was to be shown.

**B. Proof of Theorem 2**

The subsequent proof will build on the following lemma.

**Lemma A1:** The following conditions are equivalent:

(i) $\text{span}\{\kappa^\top \psi, \ldots, \kappa^{m\top} \psi\} = \mathbb{R}^m$;

(ii) $\kappa$ is invertible and $U = \{0\}$;

(iii) $\psi \in \text{span}\{\kappa^\top \psi, \ldots, \kappa^{m\top} \psi\}$ and $U = \{0\}$.

We can now proceed to the proof of Theorem 2. Let $z, y \in E$. The power series expansion of $F(\tau, z)$ in $\tau$,

$$
F(\tau, z) = e^{-\alpha \tau} \left( 1 + \sum_{p \geq 1} \frac{\psi^\top \kappa^p (z - \theta) (-\tau)^p}{\phi + \psi^\top z} \right), \quad (A3)
$$

shows that $F(\tau, z) = F(\tau, y)$ for all $\tau \geq 0$ if and only if

$$
\frac{\psi^\top \kappa^p (z - \theta)}{\phi + \psi^\top z} = \frac{\psi^\top \kappa^p (y - \theta)}{\phi + \psi^\top y}, \quad p \geq 1. \quad (A4)
$$

To prove sufficiency, assume the term structure kernel is zero, $U = \{0\}$, $\kappa$ is invertible, and $\phi + \psi^\top \theta \neq 0$. Then pick $z, y$ such that (A4) is satisfied. We must prove that $z = y$. Lemma A1(ii)–(i) implies that we may find coefficients $a_1, \ldots, a_m$ so that $\psi = \sum_{p=1}^{m} a_p \kappa^p \psi$. Multiplying both sides of (A4) by $a_p$ and summing over $p = 1, \ldots, m$ yields

$$
\frac{\psi^\top (z - \theta)}{\phi + \psi^\top z} = \frac{\psi^\top (y - \theta)}{\phi + \psi^\top y},
$$
or, equivalently, \( \psi^\top (z - y)(\phi + \psi^\top \theta) = 0 \). Since \( \phi + \psi^\top \theta \neq 0 \), we deduce from (A4) that \( \psi^\top \kappa^p(z - y) = 0 \) for all \( p \geq 0 \), which by the aforementioned spanning property of \( \psi^\top \kappa^p, p \geq 0 \), implies \( z = y \) as required. This finishes the proof of the sufficiency assertion.

That the term structure kernel is zero, \( \mathcal{U} = \{0\} \), is obviously a necessary condition for injectivity of the term structure. We now argue by contradiction and suppose that \( \mathcal{U} = \{0\} \) while it is not true that \( \kappa \) is invertible and \( \phi + \psi^\top \theta \neq 0 \). There are two cases. First, assume \( \kappa \) is not invertible. We claim that there is an element \( \eta \in \ker \kappa \) such that \( \theta + s\eta \) lies in the set \( \{z \in \mathbb{R}^m : \phi + \psi^\top z \neq 0\} \) for all large \( s \). Indeed, if this were not the case we would have \( \ker \kappa \subseteq \ker \psi^\top \), which would contradict \( \mathcal{U} = \{0\} \). So such an \( \eta \) exists. Now simply take \( z = \theta + s_1\eta, y = \theta + s_2\eta \) for large enough \( s_1 \neq s_2 \)—clearly (A4) holds for this choice, proving that injectivity fails.

In the second case \( \kappa \) is invertible but \( \phi + \psi^\top \theta = 0 \). In particular \( \phi + \psi^\top z = \psi^\top(z - \theta) \). Together with the fact that \( \kappa^\top \psi, \ldots, \kappa^m\top \psi \) span \( \mathbb{R}^m \) (Lemma A1(ii)–(i)), this shows that (A4) is equivalent to

\[
\frac{z - \theta}{\psi^\top(z - \theta)} = \frac{y - \theta}{\psi^\top(y - \theta)}.
\]

We deduce that \( F(\tau, z) \) is constant along rays of the form \( \theta + s(z - \theta) \), where \( z \) is any point in the state space, and thus that injectivity fails.

C. Proof of Theorem 3

The subsequent proof will build on the following lemma.

**Lemma A2**: Assume \( \phi + \psi^\top \theta \neq 0 \), and consider any \( z \in E \). The following conditions are equivalent.

(i) \( \psi \in \text{span}\{\kappa^\top \psi, \ldots, \kappa^m\top \psi\} \),

(ii) \( \text{span}\{\nabla_z F(\tau, z) : \tau \geq 0\} = \mathcal{U}^\perp \).

We can now proceed to the proof of Theorem 2. Elementary calculus shows that the covariance kernel satisfies

\[
\mathcal{W} = \bigcap_{\tau_1, \tau_2 \geq 0, (z, u) \in E} \ker \nabla_u \left( \nabla_z F(\tau_1, z)^\top a(z, u) \nabla_z F(\tau_2, z) \right)
\supseteq \bigcap_{1 \leq i, j \leq m, (z, u) \in E} \ker \nabla_u a_{ij}(z, u). \tag{A5}
\]

This proves that the number of USV factors directly affecting the instantaneous bond return covariances is less than or equal to \( p \).

Assume now that the term structure kernel is zero, \( \mathcal{U} = \{0\} \), \( \kappa \) is invertible, and \( \phi + \psi^\top \theta \neq 0 \). Combining Lemma A1(ii)–(iii) and Lemma A2, we infer that \( \text{span}\{\nabla_z F(\tau, z) : \tau \geq 0\} = \mathbb{R}^m \) for any \( z \in E \). This shows that equality holds in (A5) and proves the theorem.
D. Proof of Theorem 4

The proof uses the following identity from Fourier analysis, valid for any \( \mu > 0 \) and \( s \in \mathbb{R} \) (see, for instance, Bateman and Erdélyi (1954), Formula 3.2(3)):

\[
s^+ = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(\mu + i\lambda)s} \frac{1}{(\mu + i\lambda)^2} d\lambda.
\]

(A6)

Let \( q(ds) \) denote the conditional distribution of the random variable \( p_{\text{swap}}(Z_{T_0}) \) so that \( \widehat{q}(x) = \int_{\mathbb{R}} e^{xs} q(ds) \) for \( x \in \mathbb{C} \). Let \( \mu > 0 \) be such that \( \widehat{q}(\mu) < \infty \). Then

\[
\int_{\mathbb{R}^2} |e^{(\mu + i\lambda)s}| \frac{1}{(\mu + i\lambda)^2} d\lambda \, q(ds) = \int_{\mathbb{R}} \frac{e^{i\mu s}}{\mu^2 + \lambda^2} d\lambda \, q(ds) = \int_{\mathbb{R}} e^{i\mu s} \, q(ds) \int_{\mathbb{R}} \frac{1}{\mu^2 + \lambda^2} d\lambda < \infty,
\]

where the second equality follows from Tonelli’s theorem. This justifies applying Fubini’s theorem in the following calculation, which uses identity (A6) on the second line:

\[
\mathbb{E}_t [p_{\text{swap}}(Z_{T_0})^+] = \int_{\mathbb{R}} s^+ q(ds) = \int_{\mathbb{R}} \left( \frac{1}{2\pi} \int_{\mathbb{R}} e^{(\mu + i\lambda)s} \frac{1}{(\mu + i\lambda)^2} d\lambda \right) q(ds) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{q}(\mu + i\lambda) (\mu + i\lambda)^2 d\lambda = \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \widehat{q}(\mu + i\lambda) \right] d\lambda.
\]

Here the last equality uses the fact that the left, and hence right, side is real, together with the observation that the real part of \((\mu + i\lambda)^{-2} \widehat{q}(\mu + i\lambda)\) is an even function of \( \lambda \) (this follows from a brief calculation). The resulting expression for the conditional expectation, together with (20), gives the result.

E. Proof of Theorem 5

The following lemma is used in the proof of Lemma A4 below.

**Lemma A3:** Assume that \( Z_t \) is of the form (2) with integrable starting point \( Z_0 \). Then, for any bounded stopping time \( \rho \) and any deterministic \( \tau \geq 0 \), the random variable \( Z_{\rho+\tau} \) is integrable, and we have

\[
\mathbb{E}_\rho [Z_{\rho+\tau}] = \theta + e^{-\kappa \tau} (Z_\rho - \theta).
\]

The following result, which is valid for a linear-rational term structure model (2) and (3) with a factor process \( Z_t \) whose minimal state space is the nonnegative orthant \( \mathbb{R}_m^+ \). Hereafter, we say that the state space \( E \) is minimal if \( \mathbb{P}(Z_t \in U \text{ for some } t \geq 0) > 0 \) holds for any relatively open subset \( U \subset E \).

**Lemma A4:** Assume \( Z_t \) is of the form (2) and with minimal state space \( \mathbb{R}_m^+ \). Then \( \kappa_{ij} \leq 0 \) for all \( i \neq j \).
Now consider a linear-rational term structure model (2) and (3) with a factor process $Z_t$ whose minimal state space is $\mathbb{R}^m_+$. Since $\phi + \psi^T z$ is assumed positive on $\mathbb{R}^m_+$, we must have $\psi \in \mathbb{R}^m_+$ and $\phi > 0$. Dividing $\zeta_t$ by $\phi$ does not affect any model prices, so we may take $\phi = 1$. Moreover, after permuting and scaling the components, $Z_t$ is still of the form (2) with minimal state space $\mathbb{R}^m_+$, so we can assume $\psi = 1_p$. Here, we let $1_p$ denote the vector in $\mathbb{R}^m$ whose first $p \leq m$ components are ones, and the remaining components are zeros. As before, we write $1 = 1_m$. The short rate is then given by $r_t = \alpha - \rho(Z_t)$, where

$$\rho(z) = \frac{1_p^T \kappa \theta - 1_p^T \kappa z}{1 + 1_p^T z} = \frac{1_p^T \kappa \theta + \sum_{i=1}^m (-1_p^T \kappa_i) z_i}{1 + \sum_{i=1}^p z_i} \quad (A7)$$

and where $\kappa_i$ denotes the $i$th column vector of $\kappa$.

**Lemma A5:** The short rates are bounded from below, $\alpha^* = \sup_{z \in \mathbb{R}^m_+} \rho(z) < \infty$, if and only if $1_p^T \kappa_i = 0$ for $i > p$. In this case, $\alpha^* = \max S_p$ and $\alpha_* = \min S_p$, where $S_p = \{1_p^T \kappa \theta, -1_p^T \kappa_1, \ldots, -1_p^T \kappa_p\}$, and the submatrix $\kappa_{1:p, p+1:m}$ is zero.

We can now prove Theorem 5. To this end, we first observe that $(Z_t, U_t)$ remains a transformed square-root diffusion process after coordinatewise scaling and permutation of $Z_t$. Hence, as above, we can assume that $\phi = 1$ and $\psi = 1_p$ for some $p \leq m$. Lemma A5 then shows that short rates are bounded from below if and only if submatrix $\kappa_{1:p, p+1:m}$ vanishes. If $p = m$ there is nothing left to prove. So assume now that $p < m$. This implies that $(Z_{1t}, \ldots, Z_{pt}, U_t)$ is an autonomous transformed square-root process with the smaller term structure state space $\mathbb{R}^p_+$. Since the state price density $\zeta_t$ depends only on the first $p$ components of $Z_t$, the pricing model is unaffected if we exclude the last $m - p$ components of $Z_t$, and this proves that we may always take $p = m$, as desired.

Finally, the expressions for $\alpha^*$ and $\alpha_*$ follow directly from Lemma A5. This completes the proof of Theorem 5.

**F. Proof of Corollary 1**

In view of the spanning condition (28), the $LRSQ(m,n)$ specification exhibits $m$ term structure factors and $n$ USV factors. A calculation shows that

$$a_{ij}(z, u) = \begin{cases} \sigma_i^2 z_i + (\sigma_{m+i}^2 - \sigma_i^2) u_i, & \text{if } 1 \leq i = j \leq n, \\ \sigma_i^2 z_i, & \text{if } n + 1 \leq i = j \leq m, \\ 0, & \text{otherwise}. \end{cases}$$

This implies

$$\nabla_u a_{ij}(z, u) = \begin{cases} (\sigma_{m+i}^2 - \sigma_i^2) e_i, & \text{if } 1 \leq i = j \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

where $e_i$ denotes the $i$th standard basis vector in $\mathbb{R}^n$. The assertion about the number of USV factors directly affecting the instantaneous bond return covariances now follows from Theorem 3.
REFERENCES


**Supporting Information**

Additional Supporting Information may be found in the online version of this article at the publisher’s website:

**Appendix S1:** Internet Appendix