Arbitrage

- The absence of arbitrage, defined as the possibility of simultaneously buying and selling the same security at different prices, is the most fundamental concept of finance.

- To make a parrot into a trained financial economist it suffices to teach him a single word: arbitrage.

  S. Ross (1987)
Anomalies

- **But significant violations** of this basic paradigm are often observed in real world markets.
- A famous example is the simultaneous trading of **Royal Dutch and Shell** in Amsterdam and London:
  - The two companies merged in 1907 on a 60/40 basis
  - Cash flows are attributed to the stocks in these proportions
  - Despite this RD traded at a significant premium relative to Shell throughout most of the 1990’s.
- Other examples: Molex, Unilever NV/PLC, 3Com/Palm...
Bubbles and portfolio constraints
Theory

- Neo-classical theory has little to say:
  - The workhorse model of modern asset pricing is the representative agent model of Lucas (1974).
  - In this model mispricing on positive net supply assets is incompatible with the existence of an equilibrium.
- Most of the work on the origin of bubbles is behavioral
  - Common feature: **partial equilibrium setting**.
  - Different definition of the fundamental value which implies that bubbles are not connected to arbitrage activity.
Portfolio constraints

- There are some models where arbitrages arise endogenously due to portfolio constraints.
  - Common feature: all agents are constrained, *riskless arbitrage*
  - If the constraints are lifted for some agents then mispricing becomes inconsistent with equilibrium.
- This need not be the case with *risky arbitrage*: portfolio constraints can generate bubbles in equilibrium even if there are unconstrained arbitrageurs in the economy.

This paper

- Continuous-time model with two groups of agents:
  - Unconstrained agents,
  - Constrained agents with logarithmic utility.
- Necessary and sufficient conditions under which portfolio constraints generate bubbles in equilibrium.
- When there are multiple stocks, the presence of bubbles may give rise to *multiplicity* and *real indeterminacy*.
- Examples of innocuous portfolio constraints, including limited market participation, that generate bubbles in equilibrium.
Related literature

- Behavioral models:
- Equilibrium under constraints:
- Equilibrium mispricing:
- Partial equilibrium:
  - Cox and Hobson (2005), Jarrow et al. (2008,2010),...

Outline

1. The model
2. Equilibrium bubbles
3. Limited participation
4. Multiplicity
The model

- Continuous–time economy on $[0, T]$.
- One perishable consumption good and $n + 1$ traded securities:
  - A locally riskless asset in zero net supply,
  - $n$ risky assets in positive net supply of one unit each.
- The price of the riskless asset evolves according to
  \[
  dS_{0t} = r_t S_{0t} dt
  \]
  where the instantaneously risk free rate process $r_t$ is to be determined endogenously in equilibrium.

Risky assets

- Dividends evolve according to
  \[
  d\delta_t = \text{diag}(\delta_t) (\mu_{\delta t} dt + \sigma_{\delta t} dB_t)
  \]
  for some exogenous $(\mu_{\delta}, \sigma_{\delta})$ where $B$ is a BM in $\mathbb{R}^n$.
- The stock prices evolve according to
  \[
  dS_t + \delta_t dt = \text{diag}(S_t) (\mu_t dt + \sigma_t dB_t).
  \]
  where the initial price $S_0$, the drift $\mu_t$ and the volatility $\sigma_t$ are to be determined endogenously in equilibrium.
Agents

- Two agents indexed by $a = 1, 2$.
- The preferences of agent $a$ are represented by
  \[ U_a(c) = E_0 \left[ \int_0^T e^{-\rho \tau} u_a(c_\tau) d\tau \right] \]
  where $\rho$ is a nonnegative discount rate, $u_2 \equiv \log$ and $u_1$ is a utility function satisfying textbook regularity conditions.
- Agent 2 is initially endowed with $\beta$ units of the riskless asset and a positive fraction $\alpha_i$ of the supply of stock $i$.

Trading strategies

- A **trading strategy** is a process $(\phi, \pi) \in \mathbb{R} \times \mathbb{R}^n$.
- The strategy $(\phi, \pi)$ is self financing for agent $a$ given a consumption plan $c$ if the corresponding **wealth process**
  \[ W_t = W_t(\phi, \pi) \equiv \phi_t + 1^* \pi_t \]
  satisfies the dynamic budget constraint
  \[ W_t = w_a + \int_0^t (\phi_\tau r_\tau + \pi_\tau^* \mu_\tau - c_\tau) d\tau + \int_0^t \pi_\tau^* \sigma_\tau dB_\tau \]
  where the constant $w_a$ denotes the agent’s initial wealth computed at equilibrium prices.
Portfolio constraints

• Agent 1 is unconstrained (except for $W_t \geq 0$)
• Agent 2 is constrained: I assume that the trading strategy that he chooses must satisfy

$$\text{Amount in stocks} = \pi_t \in W_t C_t$$

as well as $W_t \geq 0$ where $C_t \subseteq \mathbb{R}^n$ is a closed convex set.

• A wide variety of constraints, including constraints on short selling, collateral constraints, borrowing and participation constraints can be modeled in this way.

Equilibrium

• An equilibrium is a collection of prices, consumption plans and trading strategies such that:
  (a) $c_a$ maximizes $U_a$ and is financed by $(\phi_a, \pi_a)$,
  (b) The securities and goods markets clear

$$\phi_1 + \phi_2 = 0,$$
$$\pi_1 + \pi_2 = S,$$
$$c_1 + c_2 = 1^* \delta \equiv e.$$

• I will restrict the analysis to the class of non redundant equilibria in which the stock volatility is invertible.
Rational stock bubbles

• A traded security is said to have a bubble if its market price differs from its fundamental value: \( B_{it} \equiv S_{it} - F_{it} \).

• Since markets are complete for Agent 1, the fundamental value of a stock is unambiguously defined as

\[
F_{it} = \frac{1}{\xi_t} E_t \left[ \int_t^T \xi_\tau \delta_{i\tau} d\tau \right]
\]

where the process

\[
\xi_t = \frac{1}{S_{0t}} \exp \left( -\int_0^t \theta^*_\tau dB_\tau - \frac{1}{2} \int_0^t \|\theta_\tau\|^2 d\tau \right)
\]

is the SPD and \( \theta \) is the market price of risk.

Basic properties

• A bubble is nonnegative and satisfies \( B_{iT} = 0 \).

• A bubble cannot be born: if \( B_{it} = 0 \) then \( B_{i\tau} = 0 \) for all \( \tau \geq t \).

• A bubble is not an arbitrage: The strategy which

  • Sells the stock short,
  • Buys the replicating portfolio,
  • Invests the remainder in the riskless asset,

has wealth process

\[
W_t = B_{i0} S_{0t} - B_{it}
\]

and thus is not admissible on its own (even if the positive wealth constraint is relaxed to allow for bounded credit).
Riskless asset bubble

- Over $[0, T]$ the riskless asset can be seen as a European derivative security with pay–off $S_{0T}$ at the terminal time.
- The fundamental value of such a security is
  \[ F_{0t} = E_t \left[ \frac{\xi_T}{\xi_t} S_{0T} \right] = S_{0t} E_t \left[ \frac{M_T}{M_t} \right] \]
  where $M_t \equiv \xi_t S_{0t}$.
- The existence of a bubble on the riskless asset is equivalent to the non existence of the EMM.

The equilibrium SPD

- **Proposition.** In equilibrium
  \[ \xi_t = e^{-\rho t} \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} \]
  where $e_t$ is the aggregate dividend process, $\lambda_t$ is the ratio of the agents’ marginal utilities and
  \[ u(e, \lambda_t) = \max_{c_1 + c_2 = e} \{ u_1(c_1) + \lambda_t u_2(c_2) \}. \]
- Since the allocation is inefficient, $\lambda$ is not a constant but a stochastic process that acts as an endogenous state variable.
Bubble on the market portfolio

\[
\sum_{i=1}^{n} B_{it} = \sum_{i=1}^{n} S_{it} - E_t \left[ \int_t^T e^{-\rho(\tau-t)} \frac{u_c(e_\tau, \lambda_\tau)}{u_c(e_t, \lambda_t)} e_\tau d\tau \right]
\]

\[
= W_{1t} + W_{2t} - E_t \left[ \int_t^T e^{-\rho(\tau-t)} \frac{u_c(e_\tau, \lambda_\tau)}{u_c(e_t, \lambda_t)} e_\tau d\tau \right]
\]

\[
= W_{2t} - E_t \left[ \int_t^T e^{-\rho(\tau-t)} \frac{u_c(e_\tau, \lambda_\tau)}{u_c(e_t, \lambda_t)} c_{2\tau} d\tau \right]
\]

\[
= E_t \left[ \int_t^T e^{-\rho(\tau-t)} \frac{u_c(e_\tau, \lambda_\tau)}{u_c(e_t, \lambda_t)} \left( \frac{\lambda_t}{\lambda_\tau} - 1 \right) c_{2\tau} d\tau \right]
\]

\[
= \frac{1}{u_c(e_t, \lambda_t)} E_t \left[ \int_t^T e^{-\rho(\tau-t)} (\lambda_t - \lambda_\tau) d\tau \right] \quad (u_2 = \log)
\]

Equilibrium bubbles

• **Proposition.** In equilibrium,

\[
\lambda_t = \lambda_0 - \int_0^t \lambda_\tau \left( \theta_\tau - \Pi(\theta_\tau | \sigma_\tau^* \Psi_\tau) \right)^* dB_\tau
\]

where \( \Pi \) is the projection operator and \( \theta \) solves

\[
\theta_t = \sigma et R_t + s_t R_t \left( \theta_t - \Pi(\theta_t | \sigma_t^* \Psi_t) \right)
\]

with

\[
R_t = -\frac{u_{cc}(e_t, \lambda_t)}{u_c(e_t, \lambda_t)} e_t, \quad s_t = \frac{c_{2t}}{e_t} = \frac{\lambda_t}{u_c(e_t, \lambda_t)}.
\]

The weighting process is a local martingale and it is a martingale if and only if the stock prices do not include bubbles.
Limited participation

- Consider the following specification:
  - There is a single stock,
  - Both agents have logarithmic utility,
  - The dividend is a GBM with drift $\mu_\delta$ and volatility $\sigma_\delta$,
  - $\mathcal{X}_t = [0, 1 - \varepsilon]$ for some $0 \leq \varepsilon \leq 1$.

- Assume $\beta < (1 - \alpha)\delta_0 T$ to guarantee that the unconstrained agent is not so deeply in debt that he can never repay.

- **Special cases** include
  - Unconstrained economy ($\varepsilon = 0$).
  - Restricted participation model of Basak and Cuoco ($\varepsilon = 1$).

Equilibrium

- **Proposition.** Let $\lambda$ denote the unique solution to

$$\lambda_t = \frac{w_2}{w_1} - \int_0^t \lambda_\tau (1 + \lambda_\tau) \sigma_\lambda dB_\tau$$

with $\sigma_\lambda = \varepsilon \sigma_\delta$. In the **unique** equilibrium, the consumption plans and trading strategies are given by

$$\phi_{1t} = -\varepsilon \lambda_t W_{1t}, \quad \pi_{1t} = (1 + \varepsilon \lambda_t) W_{1t}, \quad c_{1t} = \frac{e_t}{1 + \lambda_t},$$

$$\phi_{2t} = \varepsilon W_{2t}, \quad \pi_{2t} = (1 - \varepsilon) W_{2t}, \quad c_{2t} = \frac{e_t \lambda_t}{1 + \lambda_t},$$

and the stock price is

$$S_t / e_t = \int_t^T e^{-\rho(\tau-t)} d\tau \equiv \eta(t).$$
Equilibrium bubbles

- The weighting process is a **strict local martingale**!
- **Proposition.** The riskless asset and the stock both include bubble components that are given by

\[ \frac{B_t}{S_t} = b(t, s_t) \leq b_0(t, s_t) = \frac{B_{0t}}{S_{0t}} \]

where the bounded process

\[ s_t = \frac{c_2 t}{\epsilon t} = \frac{\lambda_t}{1 + \lambda_t} \]

represents the constrained agent’s share of aggregate consumption and \( b, b_0 \) are known functions.

Bubbles

- The bubbles are explicitly given by

\[ b_0(t, T, s) \equiv s^{-1/\epsilon} H(T - t, s; a_0), \]

\[ b(t, s) = \frac{1}{\rho \eta(t)} H(T - t, s; a_1) + \frac{\eta'(t)}{\rho \eta(t)} H(T - t, s; 1), \]

where \( a_0, a_1 \) are constants

\[ H(\tau, s; a) \equiv s^{1+2 \epsilon} \Phi(d_+(\tau, s; a)) + s^{1+2 \epsilon} \Phi(d_-(\tau, s; a)), \]

\[ d_\pm(\tau, s; a) \equiv \frac{1}{\|v_\lambda\|\sqrt{\tau}} \log s \pm \frac{a}{2} \|v_\lambda\|\sqrt{\tau}, \]

and \( \Phi \) denotes the normal cdf.
Strict local martingale

Expected value $E[\lambda_T | \lambda_0]$

Initial value $\lambda_0$

Horizon $T$

Bubbles and portfolio constraints
Strict local martingale

Bubbles and portfolio constraints

Strict local martingale
Strict local martingale

Equilibrium bubbles
Mechanism

- Agent 2 must keep some wealth in the bank.
- Agent 1 must find it optimal to hold a leveraged position.
- This implies that the **short rate must decrease** and the **market price of risk must increase**. Indeed:

\[
\begin{align*}
    r_t &= \rho + \mu \delta - (1 + \varepsilon \lambda_t) |\sigma \delta|^2 = r_{tc} - \varepsilon \lambda_t |\sigma \delta|^2, \\
    \theta_t &= (1 + \varepsilon \lambda_t) \sigma \delta = \theta_{tc} + \varepsilon \lambda_t |\sigma \delta|.
\end{align*}
\]

- But this is **not sufficient** to entice Agent 1 to hold the highly leveraged portfolio necessary to clear markets.

Equilibrium portfolio

- The equilibrium portfolio of Agent 1 can be decomposed into: A **short position** of size

\[
m_t \equiv \frac{S_t}{1/(\varepsilon s_t) + \partial_s \log b^0(t, s_t)} > 0
\]

in the riskless asset bubble and a long position in the stock.
- The first part is an **arbitrage strategy** with negative value
  - This strategy is not admissible by itself,
  - The bubble on the stock raises its collateral value and allows the agent to scale his position to the required level.
Consumption share

- The equilibrium consumption share of the constrained agent can be explicitly computed as

\[ s_t = \frac{c_{2t}}{c_{1t} + c_{2t}} = \frac{\lambda_t}{1 + \lambda_t} \equiv s(\lambda_t). \]

- Since the weighting process is a nonnegative local martingale and the function \( s \) is increasing and concave, the consumption share is a supermartingale and is thus expected to decrease.

- This would be the case even if the weighting process \( \lambda_t \) was a true martingale (comp. heterogenous beliefs) but the presence of bubbles increases the speed at which \( s \) decreases.

Expected consumption share

![Diagram showing expected consumption share]
Consumption share

Bubbles and portfolio constraints

Consumption share
Consumption share

Transition density

Martingale  
Weighting process

Consumption share

Bubbles and portfolio constraints

Consumption share

Transition density

Martingale  
Weighting process

Consumption share

Bubbles and portfolio constraints
Consumption share

**20 years**

**40 years**

Bubbles and portfolio constraints
Multiple risky assets

- **If there is no bubble** in the market portfolio, then the stock prices are given by the familiar formula

\[ S_t = F_t = E_t \left[ \int_t^T e^{-\rho(T-t)} \frac{u_c(e_t, \lambda_T)}{u_c(e_t, \lambda_T)} \delta_{\tau} d\tau \right]. \]

- The existence of a bubble-free equilibrium is thus equivalent to the existence of a solution to a FBSDE.
- If such a solution does not exist, then only the value of the market portfolio is uniquely determined.
Multiplicity

- **Proposition.** A process $S \in \mathbb{R}_n^+$ with invertible volatility matrix $\sigma$ is an equilibrium price process if and only if

$$
\sum_{i=1}^{n} S_{it} = E_t \left[ \int_t^T e^{-\rho(\tau-t)} \frac{u_c(e_\tau, \lambda_\tau) e_\tau + \lambda_\tau - \lambda_\tau}{u_c(e_t, \lambda_t)} d\tau \right]
$$

and the discounted process

$$
e^{-\rho t} \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} S_t + \int_0^t e^{-\rho \tau} \frac{u_c(e_\tau, \lambda_\tau)}{u_c(e_0, \lambda_0)} \delta_\tau \, d\tau
$$

is a nonnegative local martingale.

- For risk constraints of the form $\mathcal{C}_t = (\sigma_t^*)^{-1} \mathcal{C}_0^t$ the weighting process can be determined independently of the prices.

Volatility constraints

- Consider the following specification:
  - There are two stocks,
  - Agents have logarithmic utility,
  - The aggregate dividend is a GBM with drift $\mu_e$ and volatility $\sigma_e$,
  - The dividend share $x_{1t} = \delta_{1t}/e_t$ is a martingale that is independent from the aggregate dividend process,
  - The portfolio constraint set is

$$
\mathcal{C}_t = \left\{ \rho \in \mathbb{R}^2 : \|\sigma_t^* \rho\| \leq (1 - \varepsilon)\|\sigma_e\| \right\}.
$$

- This constraint restricts the volatility of the agent’s wealth to be less than a fixed fraction of that of the market.
Equilibrium

- **Proposition.** Define $\lambda$ as the unique solution to
  \[
  \lambda_t = \frac{w_2}{w_1} - \int_0^t \lambda_\tau (1 + \lambda_\tau) \tilde{\sigma}^* dB_\tau.
  \]
  In equilibrium, the short rate, the risk premia, the fundamental value of the stocks and the value of the market are
  \[
  r_t = \rho + \mu e - (1 + \varepsilon \lambda_t) \| \sigma_e \|^2, \quad F_{it} = \delta_{it} \eta(t)(1 - b(t, s_t)),
  \]
  \[
  \theta_t = (1 + \varepsilon \lambda_t) \sigma_e, \quad \tilde{S}_t = e_t \eta(t).
  \]
  Furthermore, **bubbles** account for a fraction $b_0(t, s_t)$ of the riskless asset and $b(t, s_t)$ of the market portfolio.

Equilibrium prices

- **Proposition.** Let $s_0 = s_0(\phi) \in [0, 1]$ solve
  \[
  \beta + e_0 \eta(0) \alpha^* (x_0 + (\phi - x_0) b(0, s_0)) = s_0 e_0 \eta(0).
  \]
  and denote by $s_t(\phi)$ the corresponding path of the consumption share process. Then the nonnegative process
  \[
  S_t(\phi) = e_t \eta(t) (x_t + (\phi - x_t) b(t, s_t))
  \]
  is an equilibrium price process for each $\phi \in \Delta^2$. In particular, the set of non redundant equilibria is non empty.
- Since all equilibria are Markovian this shows that we have not only multiplicity but also **real indeterminacy** if $(\alpha_1 \neq \alpha_2)$. 

Bubbles and portfolio constraints
Parameter values

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_e$</td>
<td>Market return</td>
<td>8.25%</td>
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<tr>
<td>$\sigma_e$</td>
<td>Market volatility</td>
<td>16.64%</td>
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<tr>
<td>$\sigma_x$</td>
<td>Vol. dividend share</td>
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</tr>
<tr>
<td>$x_{10}$</td>
<td>Initial dividend share</td>
<td>50.00%</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Initial position in bank</td>
<td>0.00%</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>Initial position in $S_1$</td>
<td>100.00%</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>Initial position in $S_2$</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Real indeterminacy

![Graph](image)
Nominal indeterminacy

Bubbles and portfolio constraints

Some extensions

- **CRRA utility** for Agent 1: bubbles persist if $\gamma \geq 1$

- **Uncollateralized borrowing** (Hugonnier and Prieto (2010)):
  - Equilibrium fails if bound formulated in terms of $S_{0t}$
  - Equilibrium exists if bound formulated in terms of the market portfolio.

- Other types of constraint: Prieto (2010) shows that certain risk-based constraints also give rise to bubbles.

- Bubbles also arise in general equilibrium models with proportional **transaction costs** (Cujean (2011))
Thank you!