Capital supply uncertainty, cash holdings, and investment

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Abstract

We develop a model of investment, financing, and cash management decisions in which investment is lumpy and firms face uncertainty regarding their ability to raise funds in the capital markets. We characterize optimal policies explicitly and show that the smooth-pasting conditions used in prior contributions are necessary, but may not be sufficient, for an optimum. Instead of the standard Miller and Orr (1966) double-barrier policy for financing and payout, firms may optimally raise outside funds before exhausting internal resources and the optimal payout policy may feature several regions, with both smooth and discrete dividend payments. In the model, firms with high investment costs are qualitatively as well as quantitatively different in their investment, financing, and payout behaviors from firms with low investment costs. Finally, investment and payout do not always increase with slack, challenging the use of investment-cash flow sensitivities or payout ratios as measures of financing constraints.

Keywords: Capital supply uncertainty; cash management; lumpy investment; inventory models.

JEL Classification Numbers: D83; G24; G31; G32; G35.
Introduction

Following Modigliani and Miller (1958), standard models of investment under uncertainty assume that capital markets are frictionless so that firms are always able to secure funding for positive net present value projects and cash reserves are irrelevant.\(^1\) This traditional view has recently been called into question by a large number of empirical studies.\(^2\) These studies show that firms often face uncertainty regarding their future access to capital markets and that this uncertainty has important feedback effects on corporate decisions. They also reveal that the resulting liquidity risk has led firms to accumulate large amounts of cash, with an average cash-to-assets ratio for U.S. industrial firms that has increased from 10.5\% in 1980 to 23.2\% in 2006 (see Bates, Kahle, and Stulz, BKS 2009).

While it may be clear to most economists that capital supply frictions can affect corporate policies, it is much less clear exactly how they do so. In this paper, we develop a dynamic model of cash management, financing, and investment decisions in which the Modigliani and Miller assumption of infinitely elastic supply of capital is relaxed and firms have to search for investors when in need of funds. With this model, we seek to understand when and how capital supply uncertainty affects real investment. We are also interested in determining the effects of capital market frictions on firms’ financing and cash management policies, i.e. on the decision to pay out or retain earnings and the decision to issue securities.

Our paper makes three main contributions. First, we show that when capital supply is uncertain the optimal policy choices of the firm are in stark contrast with the theoretical predictions of canonical inventory models applied to liquid assets. In particular, a striking feature of the model is that it may not be optimal for firms to follow the standard Miller and Orr (1966) double-barrier policy for dividend and financing decisions. Second, our analysis of optimal policies breaks some new grounds on the mathematical analysis of inventory models. Notably, we demonstrate that the smooth-pasting conditions used to characterize optimal polices in prior contributions are necessary, but not sufficient, for an optimum and provide a full characterization of optimal polices. Third, we show that accounting for capital supply uncertainty and lumpy investment changes the predictions of models of financing constraints in ways that are consistent with stylized facts concerning firms’ policy choices.


In order to aid in the intuition of the model, consider the following two settings in which capital supply uncertainty and search frictions are likely to be especially important:

1. **Public equity offerings and capital injections for private firms**: Firms first sell their equity to the public through an initial public offering (IPO). One of the main features of IPOs is the book-building process, whereby the lead underwriter and firm management search for investors until it is unlikely that the issue will fail. Yet, the risk of failure is often not eliminated and a number of IPOs are withdrawn every year. For example, Busaba, Benveniste, and Guo (2001) show that between the mid-1980s and mid-1990s almost one in five IPOs was withdrawn. Evidence from more recent periods suggests that this fraction has increased to over one in two in some years (see Dunbar and Foerster, 2008). Search frictions are also important for firms that remain private but need new capital injections. Indeed, when a private company decides to raise equity capital, it must search for investors such as angel investors, venture capital firms, or institutional investors. Even when initial investors are found, the firm will need to search for new investors in every subsequent financing round.

2. **Financial crises and economic downturns**: Search frictions are also important for large, publicly traded firms when capital becomes scarce, e.g. during a financial crisis or an economic downturn. The recent global financial crisis has provided a crisp illustration of the potential effects of capital supply dry ups on corporate behavior, with a number of studies (e.g. Duchin, Ozbas, and Sensoy (2010)) documenting a significant decline in corporate investment following the onset of the crisis (controlling for firm fixed effects and time variation in investment opportunities). A survey of 1,050 CFOs by Campello, Graham, and Harvey (2010) also indicates that the contraction in capital supply during the recent financial crisis led firms to burn through more cash to fund their operations and to bypass attractive investment opportunities.

To illustrate the effects of capital supply frictions on investment, financing, and cash management policies, we consider a firm with assets in place that generate stochastic cash flows as well as a finite number of opportunities to expand operations (i.e. growth options). In the model, the firm faces two types of frictions: capital supply frictions and lumpiness in investment. Notably, we consider that the firm has to search for investors when in need of capital. Therefore, it faces uncertainty regarding its ability to raise funds. In addition, we assume that the firm has to pay a lump-sum cost when investing. Indeed, as noted

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3 We thank Darrell Duffie for suggesting these applications.
in Caballero and Engel (1999), “minor upgrades and repairs aside, investment projects are intermittent and lumpy rather than smooth” (see also Doms and Dunne (1998) and Cooper, Haltiwanger, and Power (1999)). The firm maximizes its value by making three interrelated decisions: How much cash to retain and pay out, when to invest, and whether to finance investment with internal or external funds. Using this model, we first show that capital supply frictions lead firms to value financial slack. Second, and more importantly, we demonstrate that the interplay between lumpiness in investment and capital supply frictions implies that barrier policies for investment, payout, and financing decisions may not be optimal. We then use these results to shed light on existing empirical facts and to generate a rich set of testable predictions.

To understand the effects of capital market frictions on corporate policies, consider first the case of a firm with assets in place but no growth option. For such a firm, cash holdings serve to cover potential operating losses and, thus, avoid inefficient closure. Holding cash however is costly because of the lower return of liquid assets inside the firm. Based on this tradeoff, the paper provides an explicit characterization of the value-maximizing payout and financing policies and shows that in this case, optimal policies are always of barrier type. Specifically, we show that there exists a target level for cash holdings (the barrier) such that (i) the optimal payout policy is to distribute dividends to maintain cash holdings at or below the target level, and (ii) when cash holdings are below the target, it is optimal to retain earnings and search for investors so as to increase cash holdings to the target.

Consider next a firm with both assets in place and growth options. For such a firm, cash holdings generally serve two purposes: Reducing the risk of closure and financing investment. Our analysis demonstrates that when the cost of investment is low (i.e. for “growth firms”), it is again optimal to follow a barrier strategy whereby the firm retains earnings and invests if cash reserves reach some target level or upon obtaining outside funds. We show however that when the cost of investment is high (i.e. for “value firms”), it is no longer optimal to follow a barrier strategy. Notably, we find that when cash reserves are high, the firm optimally retains earnings and invests if cash reserves reach some target level or upon obtaining outside funds. However, if cash reserves go down to a critical level following a series of losses, the firm abandons the option of financing investment internally as it becomes too costly to accumulate enough cash to invest. At this point, the marginal value of cash drops to one and it is optimal to make a lump-sum payment to shareholders. After making this payment, the firm retains earnings again but finances investment exclusively with outside funds.

Our theory of investment with capital supply frictions differs from prior contributions in
two important respects. First, with very few exceptions, the literature on inventory models applied to liquid assets assumes that the cumulated net cash consumption has continuous path and, therefore, does not allow for lumpy investment. Second, unlike prior contributions in which liquid assets holdings may be subject to jumps (see e.g. Alvarez and Lippi (2009, 2012) or Bar-Ilan, Perry, and Stadje (2004)), our paper provides a complete characterization of optimal decision rules. These unique features allow us to provide several insights on corporate policies that do not arise from previous models.

We highlight the main implications. First, we show that capital supply frictions and lumpiness in investment can give rise to convexity of firm value and lead firms to follow financing and payout policies that differ from the Miller and Orr (1966) double barrier policy. Notably, firms may optimally raise funds before exhausting internal resources and the optimal payout policy may feature several payout regions, with both smooth and discrete dividend payments. We also demonstrate that constrained firms with low cash reserves will not finance investment internally and may decide to pay dividends early. By contrast, constrained firms with high cash reserves may finance investment internally and will retain earnings. Therefore, in our model investment and payout do not always increase with slack, challenging the use of investment-cash flow sensitivities and payout ratios as measures of financing constraints.

Second, we show that the choice between internal and external funds for financing investment does not follow a strict pecking order, in that any firm can use both internal and external funds to finance investment. We find however that firms usually wait until external financing arrives before investing, consistent with the studies of Opler, Pinkowitz, Stulz, and Williamson (OPSW, 1999), BKS (2009), and Lins, Servaes, and Tufano (2010). We also find that (i) the probability of investment with internal funds increases with asset tangibility and agency costs and decreases with cash flow volatility and market depth, and (ii) when financing investment with external funds, firms should increase their cash reserves, consistent with Kim and Weisbach (2008) and McLean (2010). Finally, we show that negative capital supply shocks should hamper investment even if firms have enough slack to finance investment, consistent with Gan (2007), Becker (2007), Lemmon and Roberts (2010).

The present paper relates to several strands of literature. First, it relates to the literature on inventory models applied to liquid assets. Classic contributions in this literature include Baumol (1952), Miller and Orr (1966), and Tobin (1968). Recent contributions include Bar-Ilan (1990), Décamps, Mariotti, Rochet, and Villeneuve (2011), and Bolton, Chen, and Wang (2011). This literature generally assumes that liquid assets holdings have continuous
paths. Notable exceptions are Alvarez and Lippi (2009, 2012) and Bar-Ilan, Perry, and Stadje (2004).\footnote{Another exception is the study of Décamps and Villeneuve (2007), that examines the dividend policy of a firm that owns a single growth option and has no access to outside funds.} One important difference between our paper and these contributions is that jumps in our model correspond to endogenous investment or financing decisions. Another key difference is that these papers restrict their attention to barrier strategies and only derive necessary conditions for optimality within that class. By contrast, our paper provides a complete characterization of optimal decision rules and shows that barrier strategies may not be optimal when investment is lumpy.

Second, our paper relates to the large literature on investment under uncertainty, in which it is generally assumed that firms can instantaneously tap capital markets at no cost to finance investment (see footnote 1 and the references therein). In these models, there is no role for cash holdings, investment is financed exclusively with outside funds, and firms may raise funds infinitely many times to cover temporary losses.

Third, our paper relates to the literature analyzing corporate investment and financing decisions in the presence of financing constraints (see e.g. Décamps, Mariotti, Rochet, and Villeneuve (2011), Eisfeldt and Muir (2012), or Morellec, Nikolov, and Zucchi (2013)). In this literature, firms can always access capital markets. As a result, depending on whether the costs of external finance are high or low, firms either never raise funds or are never liquidated. In addition, when the cost of external finance is low, there is a strict inside/outside funds dominance in that firms only raise funds when their cash buffer is completely depleted. The papers that are most closely related to our analysis in this literature are Bolton, Chen, and Wang (2011, 2013). In these papers the authors assume that investment is infinitely divisible and that the firm follows a double-barrier policy for investment and financing decisions. Our analysis considers instead an environment in which investment is lumpy and demonstrates that in this case the optimal policy may not take the form of a double-barrier policy. Another distinctive feature of our analysis is that in our paper firms raise external funds in discrete amounts on a regular basis and some firms can be liquidated even when issuance costs are low. Also, while firms can use both internal and external funds to finance investment in our model, investment is financed exclusively with internal funds in Bolton, Chen, and Wang.

The remainder of the paper is organized as follows. Section 1 presents the model for a firm holding one growth option. Section 2 derives the firm’s optimal financing, investment, and payout policies. Section 3 extends the model to finitely many growth options. Section 4 discusses the implications of the model. The proofs are gathered in the Appendices.
1 Model and assumptions

Throughout the paper, agents are risk neutral and discount cash flows at rate $\rho > 0$. Time is continuous and uncertainty is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions.

We start our analysis by solving a model in which the firm has assets in place and a single growth option and is not required to pay any search costs when looking for outside investors and issuance costs when raising funds. We show in Section 3 that our results naturally extend to the case where the firm has a finite number of growth options and pays both issuance or search costs. We consider that before investment, the firm’s assets in place generate a continuous stream of cash flows $dX_t$ satisfying

$$dX_t = \mu_0 dt + \sigma dB_t,$$

where the process $B_t$ is a Brownian motion and $(\mu_0, \sigma)$ are constant parameters representing the mean and volatility of cash flows. The growth option allows the firm to increase cash flows to $dX_t + (\mu_1 - \mu_0) dt$, where $\mu_1 > \mu_0$. One essential and realistic feature of our model is that the firm needs to pay a lump sum cost $K > 0$ when investing in the project (see e.g. Doms and Dunne (1998), Cooper, Haltiwanger, and Power (1999), or Caballero and Engel (1999) for evidence on lumpy investment).

We also consider that the firm has full flexibility in the timing of investment. Management acts in the best interest of shareholders and chooses not only the firm’s investment policy but also its financing, payout, and liquidation policies. Notably, we allow management to retain earnings inside the firm and denote by $C_t$ the firm’s cash holdings at any time $t \geq 0$ (we use indifferently the terms cash buffer, cash holdings, and cash reserves). Cash holdings earn a constant rate of interest $r < \rho$ inside the firm and can be used to fund investment or to cover operating losses if other sources of funds are costly and/or unavailable. The wedge $\delta = \rho - r > 0$ represents a carry cost of cash.

The firm can increase its cash buffer either by retaining earnings or by raising funds in capital markets. A key difference between our setup and previous contributions is that we explicitly take into account capital supply frictions by considering that it takes time to

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5Bolton, Chen, and Wang (2011) consider instead an $AK$-model in which capital is infinitely divisible, firms can invest at a rate, and all frictions are proportional to the size of the firm (i.e. to its capital stock). They then solve numerically for the optimal financing and payout barrier strategies. We show below that the combination between financing constraints and lumpiness in investment produces interesting and unique empirical implications on corporate behavior and may render barrier strategies suboptimal.
secure outside funding and that capital supply is uncertain. Specifically, we assume that the firm needs to search for investors in order to raise funds and that, conditional on searching, it meets investors at the jump times of a Poisson process $N_t$ with arrival rate $\lambda \geq 0$. Under these assumptions, the cash reserves of the firm evolve according to

$$dC_t = \left( rC_t + \mu_0 + 1_{\{T \leq t\}}(\mu_1 - \mu_0) \right) dt + \sigma dB_t + f_t dN_t - dD_t - 1_{\{t = T\}} K,$$

where $T$ is a stopping time representing the time of investment, $f_t$ is a nonnegative predictable process representing the funds raised upon finding investors, and $D_t$ is a non-decreasing adapted process with $D_0 = 0$ representing the cumulative dividends paid to shareholders. In our model, $T$, $D$, and $f$ are endogenously determined.

Because capital supply is uncertain, new investors may be able to capture part of the surplus generated at refinancing dates. That is, we consider that once management and investors meet, they bargain over the terms of the new issue to determine the cost of capital or, equivalently, the proceeds from the stock issue. We assume that the allocation of this surplus between incumbent shareholders and new investors results from Nash bargaining. Denoting the bargaining power of new investors by $\eta \in [0, 1]$, we therefore have that the amount $\pi^*$ that new investors can extract when the firm raises $f \geq 0$ satisfies

$$\pi^* = \arg\max_{\pi \geq 0} \pi^\eta [S_f V(c) - \pi]^{1-\eta} = \eta S_f V(c),$$

where $V(c)$ gives the value of the firm as a function its cash holdings and

$$S_f V(c) = V(c + f) - f - V(c)$$

gives the financing surplus. This specification implies that whenever $\eta \neq 0$ issuance costs are stochastic, time varying, and depend on the financial health of the firm. In the model, the bargaining power of new investors can be related to the supply of funds in capital markets by assuming for example that $\eta = a/(a + \lambda)$ for some $a > 0$. In this case, the fraction of the surplus captured by outside investors and the cost of capital decrease with capital supply, consistent with the evidence in Gompers and Lerner (2000).

The firm can be liquidated if its cash buffer reaches zero following a series of negative cash flow shocks. Alternatively, it can choose to abandon its assets at any time by distributing all of its cash reserves. We consider that the liquidation value of the assets of a firm with mean cash flow rate $\mu_i$ is given by $\ell_i = \frac{\varphi \mu_i}{\rho}$ where $1 - \varphi \in [0, 1]$ represents a haircut related to
the partial irreversibility of investment. When this constant is one, investment is completely irreversible and the liquidation value of assets is zero. When this constant is zero, investment is costlessly reversible. In the analysis below, we denote by $\tau_0$ the stochastic liquidation time of the firm, refer to $\varphi$ as the tangibility of assets, and consider that $\varphi < 1$.

Because management can decide to pay a liquidating dividend at any point in time and capital supply is uncertain, liquidation is automatically triggered when the cash buffer reaches zero. The problem of management is therefore to maximize the present value of future dividends by choosing the firm’s payout $(D_t)_{t \geq 0}$, financing $(f_t)_{t \geq 0}$, and investment $(T)$ policies. That is, management solves:

$$V(c) = \sup_{(f,D,T)} E_c \left[ \int_0^{\tau_0} e^{-\rho t} (dD_t - (f_t + \eta S_t) V(C_{t-}) ) dN_t + e^{-\rho \tau_0} (\ell_0 + 1_{\{\tau_0 > T\}} (\ell_1 - \ell_0)) \right].$$

The first term in this expression represents the present value of payments to shareholders until liquidation net of the claim of new investors on future cash flows. The second term represents the present value of the cash flow to shareholders in liquidation.

Since firm value appears in the objective function via the surplus generated at refinancing dates, the above optimization problem is akin to a rational expectations problem: When bargaining over the terms of financing, outside investors have to correctly anticipate the strategy that the firm will use in the future. We show in the Appendix that introducing bargaining in the model and solving the corresponding rational expectations equilibrium is equivalent to reducing the arrival rate of investors from $\lambda$ to $\lambda^* \equiv \lambda (1 - \eta)$ in an otherwise similar model where outside investors have no bargaining power.

2 Model solution

2.1 Value of the firm with no growth option

To facilitate the analysis of shareholders’ optimization problem, we start by deriving the value $V_i(c)$ of a firm with mean cash flow rate $\mu_i$ and no growth option. When $i = 1$, this function also gives the value of the firm after the exercise of the growth option.

When there is no growth option, management only needs to determine the firm’s payout, liquidation, and financing policies. In line with previous models in the literature (see e.g. Décamps et al. (2011) and the references therein), we conjecture and later verify that there exists some level $C_i^*$ for the cash buffer below which the marginal value of cash is strictly
higher than one and above which it is equal to one. Accordingly, the optimal payout policy should take the form of a barrier policy whereby the firm makes dividend payments to maintain its cash holdings at or below $C^*$. In addition, since there are no issuance costs other than those generated by the bargaining friction, we expect that below $C^*$ it is optimal for the firm to search for new investors so as to increase its cash buffer back to the target. Finally, since the marginal value of cash is strictly higher than one below the target, our conjecture implies that the firm only liquidates if its cash holdings reach zero.

Denote by $v_i(c; b)$ the value of a firm that follows a barrier policy as above with target level $b \geq 0$. Standard arguments show that in the region $(0, b)$ where the firm retains earnings and searches for investors, $v_i(c; b)$ is twice continuously differentiable and satisfies

$$
\rho v_i(c; b) = v'_i(c; b)(rc + \mu_i) + \frac{\sigma^2}{2} v''_i(c; b) + \lambda^* [v_i(b; b) - b + c - v_i(c; b)],
$$

(1)

with $\lambda^* = \lambda(1 - \eta)$ and the boundary condition

$$
v_i(0; b) = \ell_i
$$

(2)

at the point where the firm runs out of cash. The left-hand side of the differential equation (1) represents the required rate of return for investing in the firm. The first and second terms on the right hand side capture the effects of cash savings and of cash flow volatility. The third term reflects the effect of capital supply uncertainty. This last term is the product of the instantaneous probability of meeting investors and the fraction of the surplus that accrues to incumbent shareholders when raising the cash buffer to the given target level.

Consider next the payout region $[b, \infty)$. In this region, we have

$$
v_i(c; b) = v_i(b; b) + c - b, \quad c \geq b,
$$

(3)

and the fact that the firm pays dividends in a minimal way to maintain its cash holdings at or below $b$ requires that

$$
\lim_{c \to b} v'_i(c, b) = 1.
$$

(4)

This boundary condition allows to uniquely determine the value of the firm at the payout boundary and completes the characterization of firm value under a given barrier strategy.

Denote by $C_{i,t}$ the uncontrolled cash buffer process associated with $\mu_i$, let $\tau_{i,x}$ be the
stopping time at which this process reaches the level \( x \in \mathbb{R} \) for the first time, and define a pair of bounded functions by setting

\[
L_i(c; b) = E_c \left[ e^{-(\rho+\lambda^*)\tau_{i,0}} 1_{\{\tau_{i,0} \leq \tau_{i,b}\}} \right], \tag{5}
\]

\[
H_i(c; b) = E_c \left[ e^{-(\rho+\lambda^*)\tau_{i,0} \wedge \tau_{i,b}} \right] - L_i(c; b) = E_c \left[ e^{-(\rho+\lambda^*)\tau_{i,b}} 1_{\{\tau_{i,b} \leq \tau_{i,0}\}} \right]. \tag{6}
\]

\( L_i(x; b) \) gives the present value of one dollar to be paid in liquidation, should it occur before the uncontrolled cash buffer reaches \( b > 0 \) or before finding new investors. Similarly, \( H_i(c; b) \) gives the present value of one dollar to be received when the uncontrolled cash buffer reaches \( b > 0 \), should this occur before liquidation or before finding new investors. Closed form expressions for these two functions are provided in the Appendix.

Using the above notation, together with basic properties of diffusion processes as found for example in Stockey (2009, Chapter V), it can be shown that the unique solution to equations (1), (2), and (3) over the region \((0, b)\) satisfies

\[
v_i(c; b) = v_i(b; b) H_i(c; b) + l_i L_i(c; b) + \Pi_i(c; b) - \Pi_i(b; b) H_i(c; b) - \Pi_i(0; b) L_i(c; b) \tag{7}
\]

with the function

\[
\Pi_i(c; b) = \frac{\lambda^*}{\rho + \lambda^*} \left( v_i(b; b) - b + c + \frac{\mu_i + rc}{\rho + \lambda^* - r} \right).
\]

To better understand this solution, recall that over \((0, b)\) the firm retains earnings. This implies that over this region firm value is the sum of the present value of the payments that incumbent shareholders obtain if cash holdings reach either the payout trigger or the liquidation point before external financing can be secured (first line), plus the present value of the claim that they receive if the firm gets an opportunity to raise funds from outside investors before its cash holding reach either endpoints of the region (second line).

Equations (1), (2), (3), and (4), or equivalently (4) and (7), characterize the value of a given barrier strategy. To determine the optimal target level \( C_i^* \), we further require that the value of the firm be twice continuously differentiable over the whole positive real line by imposing the high-contact condition (see e.g. Dumas (1991))

\[
\lim_{c \uparrow C_i^*} v_i''(c; C_i^*) = 0 \tag{8}
\]
at the dividend distribution boundary. Given (1) and (4), the high contact condition implies

$$\lim_{c \uparrow C_i^*} v_i(c; C_i^*) = \frac{\mu_i}{\rho} + C_i^* - \left(1 - \frac{r}{\rho}\right) C_i^*.$$ (9)

Thus, the value of the firm at the optimal target level equals the first best value of the firm minus the present value of the cost of keeping the optimal amount of cash.

In the Appendix, we show that there exists a unique target level that solves (9) and, relying on a verification theorem for the Bellman equation (B.1) associated with our optimization problem, we prove that the corresponding barrier strategy is optimal among all strategies. This leads to the following result.

**Proposition 1 (Firm value without growth option)** The value of a firm with mean cash flow rate $\mu_i$ and no growth option is $V_i(c) = v_i(c; C_i^*)$, where $C_i^*$ is the unique solution to (9). The optimal target $C_i^*$ increases with cash flow volatility $\sigma^2$ and decreases with capital supply $\lambda$ and asset tangibility $\varphi$.

### 2.2 Value of the firm with a growth option

We now turn to the analysis of corporate policies when the firm has the option to increase the mean cash flow rate by paying a lump sum cost $K$. The growth option changes the firm’s policy choices and value only if the project has positive net present value. The following proposition provides a necessary and sufficient condition for this to be the case.

**Proposition 2** The option to invest has positive net present value if and only if the cost of investment is lower than $K^*$ defined by:

$$\frac{\mu_1 - \mu_0}{\rho} = K^* + \left(1 - \frac{r}{\rho}\right)(C_1^* - C_0^*),$$ (10)

where $C_i^*$ is the target cash level for a firm with mean cash flow rate $\mu_i$ and no growth option.

The intuition for this result is clear. The left hand side of equation (10) represents the expected present value of the increase in cash flows following the exercise of the growth option. The right hand side represents the total cost of investment, which incorporates both the direct cost of investment and the change in the carry cost of the optimal cash balance. In the following, we consider that the cost of investment is below $K^*$. If this was not the case, the firm would simply follow the barrier strategy described in Proposition 1.
2.2.1 Optimality of a barrier strategy

Following the logic of the previous section, it is natural to conjecture that for a firm with a growth option there exists a cutoff level $C^*_u \geq K$ for the cash buffer such that it is optimal to retain earnings and search for investors when $c < C^*_u$ and to invest when the cash buffer reaches $C^*_u$ or upon obtaining outside funds. As before, the firm is liquidated if a sequence of losses depletes its cash reserves before financing can be secured. The expected shape and properties of the value function under such a policy are illustrated in Figure 1.

To determine when such a barrier strategy is optimal, we start by calculating the value $u(c; b)$ of a firm that follows a barrier strategy with some investment trigger $b \geq K$. Standard arguments show that in the region $(0, b)$ over which the firm retains earnings and searches for investors, $u(c; b)$ is twice continuously differentiable and satisfies the differential equation

$$\rho u(c; b) = u'(c; b)(rc + \mu_0) + \frac{\sigma^2}{2} u''(c; b) + \lambda^* \left[ V_1(C^*_1) - C^*_1 - K + c - u(c; b) \right] \quad (11)$$

subject to the value matching conditions

$$u(0; b) = \ell_0, \quad (12)$$

$$u(c; b) = V_1(c - K), \quad c \geq b, \quad (13)$$

at the point where the firm runs out of cash and in the region where it uses its cash reserves to invest. This equation is similar to that derived in the previous section for the value of the firm after investment. The only difference is that the financing surplus is now given by $V_1(C^*_1) - C^*_1 - K + c - u(c; b)$ since, upon obtaining outside funds, the firm invests and simultaneously readjusts its cash buffer to the level $C^*_1$ that is optimal after investment.

Basic properties of diffusion processes show that the unique solution to (11), (12), and (13) is given by

$$u(c; b) = V_1(b - K)H_0(c; b) + \ell_0L_0(c; b) + \Phi(c) - \Phi(b)H_0(c; b) - \Phi(0)L_0(c; b) \quad (14)$$

with the function

$$\Phi(c) = \frac{\lambda^*}{\rho + \lambda^*} \left( V_1(C^*_1) - C^*_1 - K + c + \frac{\mu_0 + rc}{\rho + \lambda^* - r} \right). \quad (15)$$

The interpretation of this solution is similar to that of (7). The first two terms give the present value of the payments that incumbent shareholders receive if the firm invests with
internal funds or is liquidated before external funding can be secured. The other terms gives
the present value of the payments that shareholdes receive if the firm invests with external
funds before its cash holdings reach either zero or the given investment trigger.

Equations (13) and (14) provide a complete characterization of the value associated with
a given barrier strategy for investment and financing. Since the decision to invest with
internal funds can be seen as an optimal stopping problem, it is natural to expect that the
optimal trigger is determined by the smooth pasting condition

\[
\lim_{c \uparrow C_U^*} u'(c; C_U^*) = 1.
\]  

(16)

We show in the Appendix that this is indeed the case if the investment cost is not too small in
that \( K > K^* \), for some threshold \( K < K^* \) for which we provide an explicit characterization.
If the investment cost lies below this threshold, then the smooth pasting pasting no longer
characterizes the optimal investment trigger and we have the corner solution \( C_U^* = K \). In this
case, the growth option is so profitable that it becomes optimal to invest as soon as possible
and liquidate immediately thereafter. In either case, the barrier policy associated with the
investment trigger \( C_U^* \) is optimal in the class of barrier policies but it is not necessarily
optimal among all strategies. A detailed analysis of the Bellman equation presented in the
Appendix allows us to determine the conditions under which this is indeed the case and leads
to the following theorem.

**Theorem 3** There exists a constant \( K^{**} < K^* \) such that a barrier strategy is optimal if and
only if \( K \leq K^{**} \). In this case, the value of the firm is given by \( V(c) = u(c; C_U^*) \) where \( C_U^* \)
the unique solution to (16) when \( K > K^* \) and \( C_U^* = K \) otherwise.

### 2.2.2 Optimality of a band strategy

The function defined by \( U(c) = u(c; C_U^*) \) gives the value of the firm under the optimal barrier
policy and can be constructed for any investment cost. However, Theorem 3 shows that this
barrier strategy is suboptimal if the investment cost is high, in which case the smooth pasting
condition is not sufficient to determine the globally optimal strategy. The intuition for this
finding is that with a sufficiently high investment cost, it becomes too expensive for a firm
with low cash holdings to accumulate the amount of cash necessary to invest with internal
funds. Specifically, we show in the Appendix that with a high investment cost the barrier
strategy fails to be optimal because there exists a level below \( C_U^* \) where the marginal value
of cash \( U'(c) \) drops to one. At that point, incumbent shareholders would rather abandon the
option of financing investment internally and receive dividends, than continue hoarding cash inside the firm. Figure 2 provides an illustration of the marginal value of cash associated with the optimal barrier strategy in this case.

Following this line of argument, we conjecture and later verify that, when the investment cost is high, the optimal strategy includes an intermediate payout region and can be described in terms of thresholds $C_W^* \leq C_L^* \leq C_H^*$ as follows: When cash holdings are in $(C_L^*, C_H^*)$, the firm retains earnings and invests either upon obtaining outside funds or when its cash holdings reach the level $C_H^* \geq K$. If cash holdings drop to the level $C_L^*$ following a sequence of operating losses, the firm abandons the option of financing investment internally and makes a lump-sum payment $C_L^* - C_W^*$ to shareholders. Finally, if cash holdings are at or below the level $C_W^*$, the firm retains earnings, pays dividends to keep its cash reserves in $(0, C_W^*)$, and finances investment exclusively with outside funds. As in the case of a barrier strategy, the firm is liquidated only if its cash buffer reaches zero. The shape and properties of the firm value under such a strategy are illustrated in Figure 3.

To verify our conjecture, we start by constructing the value $v(c; b)$ of a firm that follows a strategy as above with thresholds $b = (b_1, b_2, b_3)$ for some arbitrary constants $b_1 \leq b_2 \leq b_3$ with $b_3 \geq K$. Standard arguments show that in the region $(0, b_1) \cup (b_2, b_3)$ over which the firm retains earnings, $v(c; b)$ is twice continuously differentiable and satisfies the differential equation:

$$\rho v(c; b) = v'(c; b)(rc + \mu_0) + \frac{\sigma^2}{2}v''(c; b) + \lambda^*[V_1(C_1^*) - C_1^* - K + c - v(c; b)],$$

subject to the value matching conditions

$$v(0; b) = \ell_0,$$
$$v(c; b) = V_1(c - K), \quad c \geq b_3.$$  

This differential equation is the same as (11), but the solutions we are looking for differ because the strategy now includes an intermediate payout region $[b_1, b_2]$ over which the value does not satisfy (17). Instead, the value of the firm over this intermediate region is given by

$$v(c; b) = v(b_1; b) + c - b_1, \quad b_1 \leq c \leq b_2.$$  

and the fact that once below $b_1$ the firm distributes dividends so as to maintain its cash
holdings at or below this level implies that we have
\[
\lim_{c \uparrow b_1} v'(c; b_1) = 1.
\]  
(21)

From these boundary conditions, it is clear that since \(v(0; b) = U(0) = \ell_0\) we must necessarily have \(v'(0; b) > U'(0)\) for our candidate strategy to dominate the optimal barrier strategy of the previous section. We show in the Appendix that given the optimal thresholds, this condition is equivalent to the restriction \(K > K^{**}\).

Proceeding as in the two previous cases, it is easily shown that the unique solution to (17), (18), (19), (20), and (21) satisfies
\[
v(c; b) = v(b_1; b) H_0(c; b_1) + \ell_0 L_0(c; b_1)
\]
\[+ \phi(c; b) - \Phi(0) L_0(c; b_1) - \Phi(b_1) H_0(c; b_1)
\]
in the retention interval \((0, b_1)\), and
\[
v(c; b) = V_1(b_3 - K) H_0(c; b_2, b_3) + v(b_2; b) L_0(c; b_2, b_3)
\]
\[+ \phi(c; b) - \Phi(b_2) L_0(c; b_2, b_3) - \Phi(b_3) H_0(c; b_2, b_3)
\]
in the retention interval \((b_2, b_3)\). In these equations, the discount factors are defined as in (5) and (6), but with the first hitting time of the level \(b_2\) instead of the liquidation time, and the function \(\Phi(c)\) is defined as in (15). The first line in (22) gives the present value of the payments that incumbent shareholders receive if the cash buffer reaches either the liquidation point or the payout trigger \(b_1\) before external financing can be secured. The second line gives the value of the payment that they receive if the firm finds new investors before its cash buffer reaches either zero or \(b_1\). Similarly, the first line in (23) gives the present value of the payments that incumbent shareholders receive if the cash buffer reaches either the intermediate payout trigger \(b_2\) or the internal investment trigger \(b_3\) before external financing can be secured. The second line gives the present value of the claims that they receive if external funds are raised before the cash buffer reaches either \(b_2\) or \(b_3\).

Equations (20), (22), and (23) provide a complete characterization of the value of the firm for a given set of thresholds \(b = (b_1, b_2, b_3)\). It remains to determine the triple \(C^* = (C^*_W, C^*_L, C^*_H)\) of optimal thresholds. Following the same logic as in the case without growth option, we determine the point \(C^*_W\) below which the firm invests exclusively with outside
funds by imposing the high-contact condition

\[
\lim_{c \uparrow C^*_W} v''(c; C^*) = 0 \tag{24}
\]

at the lower end of the intermediate payout region. In addition, since the decision to invest with internal funds and the decision to make an intermediate dividend payment can be seen as a joint optimal stopping problem, it is natural to expect that the two remaining thresholds are determined by the smooth pasting conditions

\[
\begin{align*}
\lim_{c \downarrow C^*_L} v'(c; C^*) &= 1, \\
\lim_{c \uparrow C^*_H} v'(c; C^*) &= V_1'(C^*_H - K). \tag{25, 26}
\end{align*}
\]

We show in the Appendix that these boundary conditions uniquely determine a triple of thresholds \(C^*\) and we prove that the corresponding strategy is globally optimal when the investment cost is above \(K^{**}\). The following theorem summarizes our findings.

**Theorem 4** If the investment cost is such that \(K \in (K^{**}, K^*)\), then the value of the firm with a growth option is given by \(V(c) = v(c; C^*)\) where the thresholds \((C^*_W, C^*_L, C^*_H)\) are the unique solutions to (24), (25), and (26).

The above results show that when the investment cost is high, the optimal strategy for shareholders depends on the level of cash reserves. When cash reserves are below \(C^*_W\), it is optimal to finance the capital expenditure exclusively with external funds and to use cash reserves only to cover operating losses. When cash holdings are between \(C^*_W\) and \(C^*_L\), the firm pays a specially designated dividend to lower its cash holdings and abandons the option of investing with internal funds. Finally, when cash holdings are above \(C^*_L\), the firm finances the capital expenditure using either internal or external funds and the optimal policy is to retain earnings until the firm invests or its cash buffer drops to \(C^*_L\).

Interestingly, the change in the optimal financing policy that occurs at \(C^*_L\) implies that \(V'(C^*_L) = 1\). Since the marginal value of cash is constantly above or equal to one, it follows that firm value is not globally concave. In fact, we show in the Appendix that the value of the firm is strictly convex over the whole interval \((C^*_L, C^*_H)\) when the investment cost is high. To understand this feature, recall that in our model cash holdings generally serve two purposes: Reducing the risk of inefficient closure and financing investment. However, when the investment cost is high and cash reserves are below \(C^*_L\), it is optimal to invest exclusively
with outside funds and the value of cash holdings only comes from their mitigating effect on
liquidation risk. At the point $C_L^*$ the marginal value of cash is equal to one and it is optimal
to start paying dividends. Above this point, the possibility to finance investment internally
introduces a second motive for holding cash and pushes the marginal value of cash above
one, leading firm value to be convex. Figure 4 provides an illustration of the marginal value
of cash under the globally optimal band strategy when the cost of investment is high.

3 Finitely many growth options

A key and novel feature of the optimal policy for a firm with a growth option is that it may
include an intermediate payout region where shareholders optimally abandon the option of
investing with internal funds. This feature is unexpected, but we contend that it is in fact
universal in models with fixed costs and capital supply frictions.

To make this point, we consider in this section a firm with assets in place and $N \geq 1$
growth options that arrive sequentially over time. The initial mean cash flow rate of the
firm is $\mu_0$ and the exercise of the $i$’th growth option allows to increase the mean cash flow
rate from $\mu_{i-1}$ to $\mu_i > \mu_{i-1}$ by paying a constant cost $K_i$. To prevent the simultaneous
exercise of multiple growth options, we assume that the firm can hold at most one growth
option at a time and that, after exercising each growth option, the firm enters a waiting
phase in which the next growth option arrives at an exponentially distributed random time
with intensity $\lambda_o$. As in the benchmark model, management seeks to maximize shareholders’
wealth and has full flexibility over the investment, payout, and financing policies of the firm.
The sequential arrival and exercise of the growth options is illustrated in Figure 5. Lastly,
we do not consider for simplicity the case when $\lambda_i$ and $\varphi_i$ may depend on $i$. We provide
comparative static results on this more general case in Propositions 6 and 9 below.

To solve this extension of the model, we use the fact that there are finitely many
investment opportunities and proceed backwards in time starting from the last period where
the firm has exhausted its growth potential. Let $V_{o,i}(c)$ denote the value of the firm as a
function of its cash holdings in the period where it holds the $i$’th growth option, and $V_{n,i}(c)$
denote the value of the firm in the waiting period following the exercise of this growth option.
After the exercise of the last growth option, the value of the firm is $V_{n,N}(c) = V_N(c)$, where
the later is the value of a firm with a mean cash flow rate $\mu_N$ and no growth option that was
derived in section 2.1. Similarly, in the period prior to the exercise of the last growth option,
the value of the firm is given by $V_{o,N}(c) = V(c)$ where the later is the value of a firm with
a single growth option that was derived in section 2.2. To proceed further in this backward recursion, we now have to solve the problem of the firm in the waiting period between the exercise of a growth option and the arrival of the next one.

### 3.1 Optimal policy in the waiting period

In the waiting period following the exercise of the \( i \)’th growth option, the firm may retain earnings to avoid inefficient closure and to exercise not only the next growth option but potentially each of the \( N - i \) growth options that it stands to receive. Following the logic of the previous section, we therefore conjecture that the optimal strategy in the waiting period can be characterized in terms of an optimal target level and up to \( N - i \) intermediate payout intervals, whose upper ends correspond to the points where the firm decides to temporarily stop hoarding cash to finance a future investment opportunity.

In order to describe the class of all such strategies, let \( s = (a, b, x) \) where \( x \geq 0 \) is a constant that represents the target level for the cash holdings of the firm when raising outside funds and \( a, b \in \mathbb{R}^n_+ \) for some \( n \in [0, N - i] \) are vectors with

\[
a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n \leq x
\]

that specify the earnings retention intervals and the intermediate dividend distribution intervals associated with the strategy. Specifically, for every given \( s \) as above, the set

\[
\mathcal{R}(s) = (0, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_{n-1}, a_n) \cup (b_n, x) = \bigcup_{k=0}^{n} \mathcal{R}_k(s)
\]

gives the region over which the firm retains earnings and searches for new investors while its complement in \([0, x]\), that is

\[
\mathcal{D}(s) = \bigcup_{k=1}^{n} \mathcal{D}_k(s) = \bigcup_{k=1}^{n} [a_k, b_k],
\]

gives the collection of intermediate dividend distribution intervals (i.e. the “bands”). When its cash holdings are above the target \( x \), the firm makes a lump sum payment \( c - x \). When its cash holdings are in \( \mathcal{R}_k(s) \), the firm retains earnings, distributes dividends to remain in the same interval, and searches for new investors in order to adjust its cash holdings to the target level \( x \). If its cash holdings fall to the lower endpoint of the interval before outside
funds can be secured, then the firm is liquidated if $k = 0$ and otherwise stops hoarding cash towards the exercise of one of its future growth options. In the latter case, the firm makes a lump sum payment to shareholders given by

$$b_k - a_k = \left| D_k(s) \right| = \inf(\mathcal{R}_k(s)) - \sup(\mathcal{R}_{k-1}(s)) \geq 0,$$

in order to bring its cash buffer down to the next earnings retention interval and then follows an entirely similar payout, financing, and liquidation strategy. Figure 6 provides an illustration of the value of a firm as a function of its cash holdings in the waiting period.

Let $v_{n,i}(c, s)$ denote firm value under such a strategy. Standard arguments show that in the retention region $\mathcal{R}(s)$, this function is twice continuously differentiable and satisfies

$$\rho v_{n,i}(c; s) = (rc + \mu_i)v'_{n,i}(c; s) + \frac{\sigma^2}{2}v''_{n,i}(c; s) + \lambda^* \left[ v_{n,i}(x; s) - x + c - v_{n,i}(c; s) \right] + \lambda_o \left[ V_{o,i+1}(c) - v_{n,i}(c; s) \right],$$

subject to the value matching conditions

$$v_{n,i}(0, s) = \ell_i$$

at the point where the firm is liquidated. This differential equation is similar to (1) with the exception of the last term on the second line which accounts for the change in the value of the firm that occurs upon the arrival of the next investment opportunity.

In each of the dividend distribution intervals, the value of the firm is defined by imposing the value matching condition

$$v_{n,i}(c; s) = c - a_k + v_{n,i}(a_k; s), \quad c \in D_k(s),$$

and the fact that, once in the interval $\mathcal{R}_k(s)$, the firm distributes dividends to maintain its cash holdings at or below the right endpoint of the interval implies that

$$\lim_{c \uparrow a_k} v'_{n,i}(c; s) = \lim_{c \uparrow x} v'_{n,i}(c; s) = 1.$$

The above equations provide a complete characterization of the value associated with a given band strategy and can be solved using techniques similar to those of section 2.2. To
determine the optimal strategy, we further impose the smooth pasting conditions

$$\lim_{c \downarrow b_{i,k}} v_n'(c; s_i^*) = 1,$$  \hspace{1cm} (27)

at each of the points where the firm stops hoarding cash towards the exercise of one of its future growth opportunities, and the high contact conditions

$$\lim_{c \uparrow a_{i,k}} v_n''(c; s_i^*) = \lim_{c \uparrow x_i^*} v_n''(c; s_i^*) = 0.$$  \hspace{1cm} (28)

at the target level $x_i^*$ and each of the intermediate target levels $a_{i,k}^*$. In the Appendix, we show that there always exist a unique $s_i^*$ such that these conditions are satisfied and a detailed analysis of the Bellman equation associated with the problem of the firm in the waiting period allows us to prove that the corresponding strategy is optimal among all the strategies available to the firm. The following proposition summarizes our findings.

**Theorem 5** The value of the firm in the waiting period following the exercise of the $i$’th growth option is given by

$$V_{n,i}(c) = v_{n,i}(c; s_i^*)$$

and satisfies

$$V_{n,i}(x_i^*) = \frac{1}{\rho + \lambda_o} (r x_i^* + \mu_i + \lambda_o V_{o,i+1}(x_i^*))$$

where the triple $s_i^*$ that determines the optimal earnings retention and dividend distribution intervals is the unique solution to (27) and (28).

Theorem 5 shows that in the waiting period that follows the exercise of the $i$’th growth option, the optimal strategy may include up to $N - i$ intermediate dividend distribution intervals (bands). But this upper bound is rough as many of these intervals may actually collapse. While it does not seem possible to determine ex-ante the number of intermediate payout intervals, we provide in the Appendix an explicit algorithm that allows to analytically construct these intervals for each given target level. We also prove the following result.

**Proposition 6** Suppose that the exercise of the $i$’th growth option changes the tangibility of assets from $\varphi_{i-1}$ to $\varphi_i$ and capital supply from $\lambda_{i-1}$ to $\lambda_i$. Then, the dividend distribution region $D$ in the waiting period following the exercise of the $i$’th growth option is increasing.
with respect to \( \varphi_i \) in the inclusion order, and the target cash level \( x_i^* \) is monotone decreasing with respect \( \varphi_i \) and \( \lambda_i \).

### 3.2 Optimal strategy for a firm with a growth option

Having constructed the value of the firm in the waiting period, we now consider the optimal policy of a firm that already holds a growth option. As a first step towards the solution to this problem, the next result provides a sufficient condition for the growth option to have a positive net present value.

**Proposition 7** A sufficient condition for the \( i \)’th growth option to have positive net present value is that

\[
V_{n,i}(x_i^*) - x_i^* - K_i \geq V_{i-1}(C_{i-1}^*) - C_{i-1}^*
\]

where the function \( V_{i-1}(c) \) and the constant \( C_{i-1}^* \) denote the value and optimal target level of a firm with mean cash flow rate \( \mu_{i-1} \) and no growth option.

The intuition for this result is clear. Indeed, the left hand side of the inequality gives the maximal value that the firm can attain by exercising the growth option. The right hand side gives the maximal value that it can achieve by abandoning the growth option. In the later case, the firm not only abandons the next growth option but also all subsequent ones.

To simplify the presentation of our results, we assume below that

\[
K_i \leq K_i^* = \min \left\{ V_{n,i}(x_i^*) - x_i^* + C_{i-1}^* - V_{i-1}(C_{i-1}^*), \frac{\mu_i - \mu_{i-1}}{r} \right\}
\]

for all \( i \). In the single option case, this condition is necessary for a positive net present value but this is not so with multiple options because in that case the net present value of an individual option can no longer be determined on a stand-alone basis. In the present context, this assumption allows us to guarantee that the regions over which the firm invests with internal funds are half-lines instead of unions of disjoint intervals. This assumption can be relaxed at the cost of significantly more involved notation.

When the firm holds a growth option, cash holdings serve two purposes: Reducing the risk of inefficient closure and financing investment. Following the logic of the single option case, we therefore conjecture that the optimal strategy can be described in terms of thresholds \( C_{i,W}^* \leq C_{i,L}^* \leq C_{i,H}^* \) with \( C_{i,H}^* \geq K_i \). When the investment cost is low, it should never be
optimal for the firm to abandon the option of investing with internal funds. We therefore expect the firm to follow a barrier strategy as in section 2.2.1 with \(0 = C^*_{i,W} = C^*_{i,L}\). On the contrary, when the investment cost is high, we expect the firm to follow a strategy similar to that of section 2.2.2, with an intermediate dividend distribution interval at the point where the firm optimally abandons the option to invest with internal funds.

To verify our conjecture, we start by constructing the value \(v_{o,i}(c; b)\) of a firm that follows a strategy as above with thresholds \(b = (b_1, b_2, b_3)\) where \(b_1 \leq b_2 \leq b_3 \geq K\). Standard arguments imply that in the region \((0, b_1) \cup (b_2, b_3)\) over which the firm retains earnings and searches for investors, \(v_{o,i}(c; b)\) is twice continuously differentiable and satisfies the differential equation

\[
\rho v_{o,i}(c; b) = (rc + \mu_i - 1)v'_{o,i}(c; b) + \frac{\sigma^2}{2}v''_{o,i}(c; b) + \lambda^* [V_{n,i}(x^*_i) - x^*_i - K_i + c - v_{o,i}(c; b)]
\]

subject to the value matching conditions

\[
v_{o,i}(0; b) = \ell_{i-1},
\]

\[
v_{o,i}(c; b) = V_{n,i}(c - K_i), \quad c \geq b_3.
\]

This differential equation is similar to that of section 2.2.2 with the exception of the last term which reflect the fact that upon finding new investors the firm raises funds to invest and simultaneously adjust its cash holdings to the target level \(x^*_i\) that is optimal in the waiting period following the exercise of the growth option.

If the thresholds under consideration are such that \([b_1, b_2] = \{0\}\), then the above equations are sufficient to determine the value of the firm and can be solved in closed form using a modification of (14). Otherwise, the value of the firm satisfies

\[
v_{o,i}(c; b) = c - b_1 + v_{o,i}(b_1; b), \quad c \in [b_1, b_2],
\]

and the fact that in the lowest retention region \((0, b_1)\) the firm distributes dividends to maintain its cash holdings at or below \(b_1\) implies that we have

\[
\lim_{c \uparrow b_1} v'_{o,i}(c; b) = 1.
\]

In this case, the value of the firm under the given strategy can be derived in closed-form using a modification of (22) and (23).
To determine the optimal strategy, we distinguish two cases depending on the level of the investment cost. If the investment cost is sufficiently low, then we set $b_i^* = (0, 0, C_{i,H}^*)$ and determine the optimal investment trigger by imposing the smooth pasting condition

$$\lim_{c \uparrow C_{i,H}^*} v_{o,i}(c; b_i^*) = V_{n,i}(C_{i,H}^* - K_i).$$

(30)

If the investment cost is high then, the optimal investment trigger is still determined by the above smooth pasting condition but this equation now needs to be solved in conjunction with the smooth pasting and high contact conditions

$$\lim_{c \uparrow C_{i,L}^*} v''_{o,i}(c; b_i^*) = \lim_{c \downarrow C_{i,L}^*} v'_{o,i}(c; b_i^*) - 1 = 0$$

(31)

that determine the intermediate payout interval.

In the Appendix, we show that in either case the above equations admit a unique solution and a detailed analysis of the Bellman equation associated with the problem of the firm allows us to confirm our conjecture regarding the optimality of the corresponding strategies.

**Theorem 8** Assume that condition (29) holds. Then there exists a constant $K_{i}^{**} \leq K_{i}^{*}$ such that the value of a firm holding a growth option is

$$V_{o,i}(c) = v_{o,i}(c; b_i^*)$$

where the thresholds $b_i^*$ are given by the unique solutions to (30) and (31) when $K_i \geq K_i^{**}$ and by the unique solution to (30) such that $C_{i,L}^* = C_{i,L}^* = 0$ otherwise.

Theorem 8 shows that the results derived in the one growth option case naturally extend to a model in which the firm has multiple growth options. In the Appendix, we show that these results also hold if we incorporate search and issuance costs in the model. In this case however, firms only raise funds when the cash buffer is below some threshold $C_{i,F}^*$, where the financing surplus equals 0. We conclude this section with the following proposition, which provides analytic comparative static results on the optimal investment trigger.

**Proposition 9** Suppose that the exercise of the $i$'th growth option changes the mean cash flow rate from $\mu_{i-1}$ to $\mu_i$, asset tangibility from $\varphi_{i-1}$ to $\varphi_i$, and capital supply from $\lambda_{i-1}$ to $\lambda_i$. Then, the investment trigger $C_{i,H}^*$ is monotone increasing in capital supply $\lambda_{i-1}$, current drift $\mu_{i-1}$ and current asset tangibility $\varphi_{i-1}$, and is decreasing in $\varphi_i$. 

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4 Discussion

4.1 General properties of the model

While our model pertains to the literature analyzing the effects of financing constraints on corporate policies, its predictions regarding payout and financing decisions differ significantly from those of prior contributions (e.g. BCW (2011) or DMRV (2011)). Indeed, in this literature firms always follow a double-barrier policy. In addition, firms either never raise external funds (when the cost of external finance is high) or are never liquidated (when the cost is low). Finally, when the cost of external finance is low, firms only raise funds when their cash buffer is completely depleted. That is, firms never simultaneously hold cash and raise external funds and they only tap capital market following a series of negative shocks.

Our paper considers an environment in which capital supply is uncertain and investment is lumpy and shows that firms only follow a double-barrier policy when the cost of investment is low or, equivalently, when the net present value of the project is high. In our model, firms may raise outside funds before exhausting internal resources, some firms may be liquidated even with low issuance costs, and the optimal payout policy may feature several payout regions, with both smooth and discrete dividend payments. Our analysis also demonstrates that constrained firms with low cash holdings will not finance investment internally and may decide to pay dividends early. By contrast, constrained firms with high cash holdings may finance investment internally and will retain earnings. These results are in sharp contrast with those in standard models of financing constraints, imply that investment and payout do not always increase with slack, and challenge the use of investment-cash flow sensitivities or payout ratios as measures of financing constraints.

Another prediction of our model is that, when financing investment with external funds, the optimal policy is to raise enough funds to finance both the capital expenditure and the potential gap between current cash holdings and the optimal level after investment. That is, firms always increase their cash buffer when raising outside funds. This prediction of the model is consistent with the evidence in Kim and Weisbach (2008) and McLean (2010), who find that firms’ decisions to issue equity are essentially driven by their desire to build up cash reserves. Also, while in prior contributions firms always raise the same amount of cash when accessing financial markets, there exists some time series variation in the amount of funds raised in our model since firms raise $(C_t^* - C_t)^+$ upon meeting investors at time $t$.

Another key implication of the model is to show that, with capital supply uncertainty, the choice between internal and external funds for financing investment does not follow a
strict pecking order, in that any given firm can use both internal and external funds. This is in sharp contrast with the financing policy in prior contributions, in which firms exhaust internal funds before issuing securities and finance investment exclusively with internal funds (see for example Décamps et al. (2011), or Bolton, Chen, and Wang (2011, 2012)).

The next section provides additional implications of our model using numerical examples. Throughout the section we assume for simplicity that the firm has a single growth option.

4.2 Additional model implications

4.2.1 Characterizing the firm’s optimal strategy

A key feature of our model is that firms only follow a double-barrier policy when the cost of investment is low (i.e. $K < K^{**}$). To better understand this feature, Figure 7, Panel A, plots the zero-$NPV$ threshold $K^*$ and the threshold $K^{**}$ that triggers a change in corporate policies, as functions of capital supply $\lambda$, asset tangibility $\varphi$, cash flow volatility $\sigma$, and the carry cost of cash $\delta = \rho - r$. Figure 7, Panel B, plots the internal rate of return of the project when $K = K^{**}$ (solid line) and $K = K^*$ (dashed line) defined as the solution to

$$
\frac{\mu_1 - \mu_0}{R} = K + \left(1 - \frac{r}{R}\right) (C^*_1(R) - C^*_0(R))
$$

where $C^*_i(R)$ is the optimal cash buffer for a discount rate $R$. Firms with projects that fall below the solid line in panel B are firms for which $K > K^{**}$. In our numerical analysis, we use the following parameter values: $\rho = 0.06$, $r = 0.02$, $\mu_0 = 0.10$, $\mu_1 = 0.125$, $\sigma = 0.10$, $\varphi = 0.75$, $\eta = 0.5$, $\lambda = 6$ and $N = 1$. These values imply an expected financing lag of $1/\lambda = 2$ months and a 25% increase in the mean cash flow rate after investment.

Figure 7 shows that only projects with very high internal rates of return (or very low cost of investment) will lead to a double-barrier policy that mirrors those in prior contributions. In our base case environment in which the cost of capital is $\rho = 6\%$, only projects with an internal rate of return above 13.02% will induce the firm to follow barrier policies. As shown by the figure, the cutoff level for the internal rate of return increases with capital supply and the carry cost of cash and decreases with cash flow volatility and asset tangibility.

4.2.2 Determinants of cash holdings

Proposition 1 provides an analytical characterization of the properties of the cash buffer after investment, $C^*_1$. This section examines instead the determinants of target cash holdings
before investment. In our base case, we have $K^{**} = 0.19$. Therefore, we consider two cases: one in which the cost of investment is low in that $K = 0.12$ and the firm follows a barrier strategy and one in which it is high in that $K = 0.32$ and the firm follows a band strategy.

Figure 8, Panel A, plots the target cash buffer $C^*_U$ of the low investment cost case as a function of capital supply $\lambda$, asset tangibility $\varphi$, cash flow volatility $\sigma$, and agency costs $\delta$. Panel B plots the target cash buffers $C^*_W$ and $C^*_H$ as well as the intermediate payout threshold $C^*_L$ of the high investment cost case as functions of these same parameters.

The figure shows that the three target cash buffers $C^*_U$, $C^*_W$, and $C^*_H$ decrease with the arrival rate of investors. That is, as $\lambda$ increases, the likelihood of finding outside investors increases and the need to hoard cash within the firm decreases. This result is consistent with the evidence in OPSW (1999) and BKS (2009), who find that firms hold more cash when their access to external capital markets is more limited. In addition, because cash inside the firm is less valuable, the intermediate payment $C^*_L - C^*_W$ increases with $\lambda$.

Another prediction of the model is that target cash holdings should increase and intermediate dividend payments should decrease with cash flow volatility (since the risk of closure increases with $\sigma$), consistent with the evidence in Harford (1999) and BKS (2009). Interpreting $\delta = \rho - r$ as an agency cost associated with free cash in the firm, the model also predicts that target cash holdings decrease and intermediate dividend payments increase with the severity of agency conflicts, consistent with Harford, Mansi, and Maxwell (2008). Finally, the model predicts that target cash holdings decrease and intermediate dividend payments increase with asset tangibility $\varphi$, consistent with Almeida and Campello (2007).

### 4.2.3 Financing investment

An important question is whether capital supply uncertainty affects investment and the source of funds used by firms when financing investment. To answer this question, we examine the determinants of the probabilities of investment with internal ($P_I(c)$) and external ($P_E(c)$). Appendix J shows how to compute these probabilities.

The top four panels of Figure 9 plot the average probability of investment with internal funds for a cross-section of firms with cash buffers uniformly distributed between 0 and $C^*_U$ ($K < K^{**}$; dashed line) and between 0 and $C^*_H$ ($K > K^{**}$; solid line) as a function of the arrival rate of investors $\lambda$, asset tangibility $\varphi$, cash flow volatility $\sigma$, and the carry cost of cash $\delta$. The two lower panels plot the total probability of investing over a one-year (solid line) and over a three-year (dashed line) horizon as functions of the arrival rate of investors $\lambda$ for a firm with low investment cost (left) and high investment cost (right).
Consider the probability of investment with internal funds (top four panels). In our base case environment, the probability that the average firm invests with internal funds is 15.60% when the cost of investment is low and 2.87% when the cost of investment is high. This suggests that, in most environments, cash holdings will be used mostly to cover operating losses and that firms will wait until external financing arrives before investing, consistent with the large sample studies by OPSW (1999) and BKS (2009) and the survey of Lins, Servaes, and Tufano (2010). Another property of the model illustrated by the figure is that the probability of financing investment with internal funds decreases with the arrival rate of investors and increases with asset tangibility. This last feature follows from the fact that $C_U^*$ decreases with $\varphi$ and implies that the investment-cash flow sensitivity increases with the tangibility of assets, consistent with Almeida and Campello (2007).

Lastly, the lower panels of Figure 9 show that the overall probability of investment decreases as $\lambda$ decreases. In both cases, we assume that $c = K$ so that firms have enough cash to finance investment internally. Thus, our model predicts that a negative shock to the supply of capital may hamper investment even if firms have enough financial slack to fund all investment opportunities internally, consistent with the evidence in Gan (2007), Becker (2007), Lemmon and Roberts (2007), and Campello, Graham, and Harvey (2010).

5 Concluding remarks

We develop a model of investment, financing, and cash management decisions in which investment is lumpy and firms face uncertainty regarding their ability to raise funds in the capital markets. We characterize optimal policies explicitly and demonstrate that the smooth-pasting conditions used to characterize optimal polices in prior contributions are necessary, but may not be sufficient, for an optimum. In the model, firms with high investment costs are qualitatively as well as quantitatively different in their behaviors from firms with low investment costs. In addition, firms may raise outside funds before exhausting internal resources and the optimal payout policy may feature several payout regions, with both smooth and discrete dividend payments. The analysis in the paper also reveals that investment and payout do not always increase with slack for constrained firms and that the choice between internal and external funds does not follow a strict pecking order. Finally, the paper generates a number of new predictions relating the use of inside and outside cash to capital supply and firm characteristics.
Appendix

A    Bargaining with outside investors

In Supplementary Appendix D we show that optimization of the problem in an environment with outside bargaining power $\eta > 0$ and meeting intensity $\lambda$ is equivalent to an auxiliary problem in which there is no bargaining power but a reduced meeting intensity $\lambda^* = \lambda(1 - \eta)$.

On the basis of this result we will assume throughout the appendices that there is no bargaining so that we only need to consider the firm’s optimization problem with $\eta = 0$.

B    Proofs of the results in Section 2.1

B.1    Intuition and road map

To facilitate the proofs, we start by introducing some notation that will be of repeated use throughout the appendix. Let $\mathcal{L}_i$ denote the differential operator defined by

$$\mathcal{L}_i \phi(c) := \phi'(c)(rc + \mu_i) + \frac{\sigma^2}{2} \phi''(c) - \rho \phi(c),$$

set

$$\mathcal{F} \phi(c) := \max_{f \geq 0} \lambda (\phi(c + f) - \phi(c) - f),$$

and denote by $\Theta$ the set of dividend and financing strategies such that

$$E_c \left[ \int_{0}^{\tau_0} e^{-\rho s} (dD_s + f_s dN_s) \right] < \infty$$

for all $c \geq 0$ where $\tau_0$ is the first time that the firm’s cash holdings fall to zero and $E_c[\cdot]$ denotes an expectation conditional on the initial value $C_{0-} = c$.

Let $\hat{V}_i(c)$ denote the value of a firm with no growth option when the mean cash flow rate is $\mu_i$.

In order to apply dynamic programming techniques, we will proceed in four steps.

1. Derive the Hamilton-Jacobi-Bellmann (HJB) equation.
2. Show that any smooth solution $\phi(c)$ to the HJB equation dominates the value function.
3. Conjecture an optimal policy and derive the corresponding firm value.
4. Show that the value of the firm associated with the conjectured optimal policy of Step 3 is indeed a smooth solution to the HJB equation.
In accordance with the theory of singular stochastic control (see Fleming and Soner (1993)), the HJB equation for the value of a firm with no option is given by

$$\max \{L_i \phi(c) + F \phi(c), 1 - \phi'(c), \ell_i(c) - \phi(c)\} = 0, \quad (B.1)$$

where $\ell_i(c) = \ell_i + c$ denotes the liquidation value of the firm. Thus, we are done with Step 1. Lemma B.1 below accomplishes Step 2. In order to proceed to Step 3, we conjecture that the optimal policy is of a threshold form and show that the system defined by (1)-(4) and (8) has a unique smooth solution that is given by $(V_i(c), C^*_i) = (v_i(c; C^*_i), C^*_i)$ for some $C^*_i > 0$. Finally, to complete Step 4, we will show that this function solves (B.1), i.e. that

(a) $V'_i(c) \geq 1$ for all $c \geq 0$,

(b) $\phi(c) = \ell_i(c)$ satisfies $L_i \phi(c) + F \phi(c) \leq 0$ for all $c \leq C^*_i$,

(c) $V_i(c) \geq \ell_i(c)$ for all $c \geq 0$.

As we show below, $V_i(c)$ is concave and therefore items (a) and (c) easily follow. Item (b) follows by direct calculation. Finally, Step 4 is proved in Lemma B.9. Thus, it remains to implement Step 3 and show that a solution exists and that it is concave. To this end, we will introduce another function $w_i(c; b)$ defined as the unique solution to (1) satisfying (4) and (8). As we note in the main text, (1) implies that (8) is equivalent to (9), that is

$$w_i(b; b) = (rb + \mu_i)/\rho.$$

Thus, for any fixed $b$, equation (1) turns into a standard ordinary differential equation that can be explicitly solved via special functions as we show below in Lemma B.2. Lemma B.4 proves the concavity of $w_i(c; b)$. The value function $v_i(c; b)$ satisfies (3)-(4). Thus, in order to determine the optimal threshold it remains to find a $b$ such that $w_i(0; b) = \ell_i(0)$. This is done in Lemma B.8. Obviously, $w_i(c; C^*_i) = v_i(c; C^*_i)$ and the proof is complete.

### B.2 Proofs

**Lemma B.1** If $\phi \in C^2(0, \infty)$ is a solution to (B.1) then $\phi(c; b) \geq \bar{V}_i(c)$.

**Proof.** Let $\phi$ be as in the statement, fix a strategy $(D, f) \in \Theta$ and consider the process

$$Y_i := e^{-\rho t} \phi(C_t) + \int_0^t e^{-\rho s} (dD_s - f_s dN_s).$$

Using the assumption of the statement in conjunction with Itô’s formula for semimartingales (see Dellacherie and Meyer (1980, Theorem VIII.25)), we get that $dY_i = dM_i - e^{-\rho t} dA_t$ where the

---

6Concavity is a rare and useful property for this class of models. As we show below, firm value is no longer globally concave when the firm has a growth option and, without this property, the verification of property (a) above become a lot more difficult.
process $M$ is a local martingale and
\[ dA_t = (\phi(C_t + f_t) - \phi(C_t) - f_t - \mathcal{F}\phi(C_t))dt \]
\[ + (\Delta D_t - \phi(C_t - \Delta D_t) + \phi(C_t)) + (\phi'(C_t) - 1)dD_t^c. \]

where $\Delta D_t$ and $D_t^c$ denote respectively the jump and the continuous components of the dividend policy under consideration, i.e.
\[
\Delta D_t = D_t - \lim_{s \uparrow t} D_s = D_t - D_{t-} \]
\[
D_t^c = D_t - \sum_{s \leq t} \Delta D_s. \]

The definition of $\mathcal{F}$ and the fact that $\phi' \geq 1$ then imply that $A$ is nondecreasing and it follows that $Y$ is a local supermartingale. The liquidation value being nonnegative, we have
\[
Z_t := Y_t \wedge \tau_0 \geq -\int_{\tau_0}^t e^{-\rho s} f_s dN_s
\]
and since the random variable on the right hand side is integrable by definition of the set $\Theta$, we conclude that $Z$ is a supermartingale. In particular,
\[
\phi(C_{0-}) = \phi(C_0) - \Delta \phi(C_0) = Z_0 - \Delta \phi(C_0) \geq E_c[Z_{\tau_0}] - \Delta \phi(C_0)
\]
\[
= E_c \left[ e^{-\rho \tau_0} \phi(C_{\tau_0}) + \int_{0^+}^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s) \right] - \Delta \phi(C_0)
\]
\[
= E_c \left[ e^{-\rho \tau_0} \ell_i(0) + \int_{0^+}^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s) \right] - \Delta D_0 - \Delta \phi(C_0)
\]
\[
\geq E_c \left[ e^{-\rho \tau_0} \ell_i(0) + \int_{0^+}^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s) \right] \tag{B.2}
\]
where the first inequality follows from the optional sampling theorem for supermartingales, the fourth equality follows from $C_{\tau_0} = 0$, and the last inequality follows from
\[
\Delta D_0 + \Delta \phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-} - \Delta D_0}^{C_{0-}} (1 - \phi'(c)) dc \leq 0
\]
The desired result follows by taking supremum over $(D, f) \in \Theta$ on both sides of (B.2). Q.E.D.

**Lemma B.2** Let $b \geq 0$ be fixed. Then, the unique continuously differentiable solution to
\[
\mathcal{L}_i \phi(c; b) + \lambda(\phi(b) - b + c - \phi(c; b)) = 0, \quad c \leq b, \tag{B.3}
\]
\[
\phi(c; b) - \phi(b) + b - c = 0, \quad c \geq b, \tag{B.4}
\]
satisfies
\[ \phi(c; b) = \phi(b)H_i(c; b) + \phi(0)L_i(c; b) \]
\[ + \Pi_i(c; b) - \Pi_i(b; b)H_i(c; b) - \Pi_i(0; b)L_i(c; b). \]

In this equation, we have
\[ \Pi_i(c; b) = \frac{\lambda}{\rho + \lambda} \left( \phi(b) - b + c + \frac{\mu_i + rc}{\rho + \lambda - r} \right), \]
and
\[ L_i(c; b) = E_c \left[ e^{-(\rho + \lambda)\tau_{i,0} - (r\tau_i + \mu_i)\tau_{i,b}} \right] = \frac{G_i(b)F_i(c) - F_i(b)G_i(c)}{G_i(b)F_i(0) - F_i(b)G_i(0)}, \]
\[ H_i(c; b) = E_c \left[ e^{-(\rho + \lambda)\tau_{i,b} - (r\tau_i + \mu_i)\tau_{i,0}} \right] = \frac{F_i(0)G_i(c) - G_i(0)F_i(c)}{G_i(b)F_i(0) - F_i(b)G_i(0)}, \]
with
\[ F_i(x) = M \left( \nu + 1/2; -(rx + \mu_i)^2/(\sigma^2 r) \right), \quad (B.5) \]
\[ G_i(x) = \frac{rx + \mu_i}{\sigma \sqrt{r}} M \left( \nu + 1/2, 23/2; -(rx + \mu_i)^2/(\sigma^2 r) \right), \quad (B.6) \]
where \( \nu = -(\rho + \lambda)/2r \) and \( M(a, b; z) \) is the confluent hypergeometric function defined by (see Dixit and Pindyck, 1994, pp.163):
\[ M(a, b; z) = 1 + a \frac{z}{b} + \frac{a(a+1) z^2}{b(b+1) 2!} + \frac{a(a+1)(a+2) z^3}{b(b+1)(b+2) 3!} + \ldots. \]

The first claim of Lemma B.2 follows by standard arguments. The proof of the second claim is based on the following auxiliary result.

**Lemma B.3** The general solution to the homogenous equation
\[ \lambda \phi_i(c) = L_i \phi_i(c) \quad (B.7) \]
is explicitly given by \( \phi_i(c) = \gamma_1 F_i(c) + \gamma_2 G_i(c) \) for some constants \( \gamma_1, \gamma_2 \) where the functions \( F_i, G_i \) are defined as in equations (B.5) and (B.6).

**Proof.** The change of variable \( \phi(c; b) = g(-(rc + \mu_i)^2/(r\sigma^2)) \) transforms (B.7) for \( \phi \) into Kummer’s ODE for \( g \) and the conclusion now follows from standard results. Q.E.D.

Let \( w_i(c; b) \) be the unique continuously differentiable solution to (B.3) and (B.4) with the boundary conditions \( w_i'(b; b) - 1 = w_i''(b; b) = 0. \)
**Lemma B.4** The function \( w_i(c; b) \) is increasing and concave with respect to \( c \geq 0 \) and strictly monotone decreasing with respect to \( b \).

In order to prove Lemma B.4, we will rely on the following three useful results.

**Lemma B.5** Suppose that \( k \) is a solution to
\[
L_i(k(c) + \phi(c; b) = 0
\]
for some \( \phi \). Then, \( k \) does not have negative local minima if \( \phi(c; b) \geq 0 \) and does not have positive local maxima if \( \phi(c; b) \leq 0 \).

**Proof.** At a local minimum we have \( k'(c) = 0 \), \( k''(c) \geq 0 \) and the claim follows from (B.8) and the nonnegativity of \( \phi \). The case of a non-positive \( \phi \) is analogous. Q.E.D.

**Lemma B.6** Suppose that \( k \) is a solution to (B.8) for some \( \phi(c; b) \leq 0 \) and that \( k'(c_0) \leq 0 \), \( k(c_0) \geq 0 \) and \( |k(c_0)| + |k'(c_0)| + |\phi(c_0)| > 0 \). Then, \( k(c) > 0 \) and \( k'(c) < 0 \) for all \( c < c_0 \).

**Proof.** Suppose on the contrary that \( k'(c) \) is not always negative for \( c < c_0 \) and let \( z \) be the largest value of \( c < c_0 \) at which \( k'(c) \) changes sign. Then, \( z \) is a positive local maximum and the claim follows from Lemma B.5. Q.E.D.

**Lemma B.7** Suppose that \( k \) is a solution to (B.8) for some \( \phi \) such that \( \phi'(c) \leq 0 \) and that \( k'(c_0) \geq 0 \), \( k''(c_0) \leq 0 \) and \( |k'(c_0)| + |k''(c_0)| + |\phi'(c_0)| > 0 \). Then, \( k'(c) > 0 \) and \( k''(c) < 0 \) for all \( c < c_0 \). Furthermore, if \( k''(c_0) = 0 \), then \( k''(c) > 0 \) for \( c > c_0 \) and \( k'(c_0) = \min_{c \geq 0} k'(c) \).

**Proof.** Differentiating (B.8) shows that \( m = k' \) is a solution to \( L_i m(c) + r m(c) + \phi'(c) = 0 \) and the conclusion follows from Lemma B.6. The case \( c > c_0 \) is analogous. Q.E.D.

**Proof of Lemma B.4.** As is easily seen, the function
\[
k(c) = w_i(c; b) - \frac{\lambda}{\lambda + \rho} \left( w_i(b; b) - b + \frac{(\rho + \lambda)c + \mu_i}{\lambda + \rho - r} \right)
\]
is a solution to (B.7) and satisfies \( k'(b) = 1 > 0 \) as well as \( k''(b) = 0 \). Together with Lemma B.7 this implies that \( k(c) \), and hence also \( w_i(c; b) \), is increasing and concave for \( c \leq b \). To establish the required monotonicity, let \( b_1 < b_2 \) be fixed and consider the function \( m(c) = w_i(c; b_1) - w_i(c; b_2) \).

Using the first part of the proof it is easily seen that \( m \) solves
\[
L_i m(c) - \lambda m(c) - \lambda (1 - r/\rho) (b_1 - b_2) = 0
\]
with the boundary conditions \( m'(b_1) = 1 - w'_i(b_1; b_2) < 0 \), \( m''(b_1) = -w''_i(b_1; b_2) \geq 0 \). Thus, it follows by a straightforward modification of Lemma B.7 that \( m \) is monotone decreasing and it only
remains to show that \( m(b_1) > 0 \). To this end, observe that

\[
m(b_1) = w_i(b_1; b_1) - w_i(b_1; b_2) \\
= w_i(b_1; b_1) - w_i(b_2; b_2) + \int_{b_2}^{b_1} w_i'(c; b_2) dc \\
\geq w_i(b_1; b_1) - w_i(b_2; b_2) + b_2 - b_1 = (r/\rho - 1)(b_1 - b_2) > 0
\]

where the first inequality follows from \( w_i'(b; b) = 1 \) and the first part of the proof, and the last inequality follows from the fact that by assumption \( \rho > r \).

Q.E.D.

**Lemma B.8** There exists a unique solution \( C^*_i \) to \( w_i(0; C^*_i) = \ell_i(0) \) and the function \( V_i(c) = w_i(c; C^*_i) = v_i(c; C^*_i) \) is a twice continuously differentiable solution to (B.1).

**Proof.** By Lemma B.2 we have that \( V_i(c) \) is twice continuously differentiable, solves (1) subject to (3), (4) and (8) so we only need to show that

\[
w_i(0; C^*_i) = \ell_i(0)
\]

has a unique solution \( C^*_i \). By Lemma B.4, we have that \( w_i(0; b) \) is monotone decreasing in \( b \). On the other hand, a direct calculation shows that \( w_i(0; 0) = \mu_i/\rho > \ell_i(0) \), \( w_i(0; \infty) < 0 \) and it follows that (B.9) has a unique solution.

To complete the proof, it remains to show that \( V_i \) is a solution to the HJB equation. Using the concavity of \( V_i(c) = w_i(c; C^*_i) \) in conjunction with the smooth pasting condition we obtain that \( 1 - V_i'(c) \) is negative below the threshold \( C^*_i \) and zero otherwise so that

\[
\ell_i(c) - V_i(c) = \int_0^c (1 - V_i'(x)) dx \leq 0.
\]

On the other hand, using the concavity of \( V_i(c) \) in conjunction with Lemma B.2 and the smooth pasting condition we obtain

\[
(L_i + F)V_i(c) = L_i V_i(c) + 1_{\{c < C^*_i\}} \lambda(V_i(C^*_i) - b + V_i(c)) = 1_{\{c \geq C^*_i\}} L_i V_i(c) \\
= (r - \rho)(c - C^*_i)^+ \leq 0
\]

and combining the above results shows that \( V_i(c) \) is a solution to (B.1).

Q.E.D.

**Lemma B.9** We have \( \hat{V}_i(c) \geq V_i(c) \) for all \( c \geq 0 \).

**Proof.** Combining the results of Lemmas B.1 and B.8 shows that \( V_i(c) \geq \hat{V}_i(c) \) for all \( c \geq 0 \). In order to establish the reverse inequality, consider the dividend and financing strategy defined by \( D^*_i = L_i \) and \( f^*_t = (C^*_i - C_{t-})^+ \) where the process \( C \) evolves according to

\[
dC_t = (rC_{t-} + \mu_i) dt + \sigma dB_t - dD^*_t + f^*_t dN_t
\]

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with initial condition $C_{0-} = c \geq 0$ and $L_t = \sup_{s \leq t} (b_t - C^*_t)^+$ where
\[
db_t = (rb_{t-} + \mu_t)dt + \sigma dB_t + (C^*_t - b_{t-})^+dN_t.
\]
In order to show that the strategy $(D^*, f^*)$ is admissible, we start by observing that
\[
E_c \left[ \int_0^\infty e^{-\rho t} f^*_t dN_t \right] \leq E_c \left[ \int_0^\infty e^{-\rho t} C^*_t dN_t \right] = \frac{\lambda C^*_t}{\rho}
\]
where the inequality follows from the definition of $f^*$. Using this bound in conjunction with Itô’s lemma and the assumption that $r < \rho$ we then obtain
\[
E_c \left[ \int_0^t e^{-\rho s} dD^*_s \right] = C_0 + E_c \left[ \int_0^t e^{-\rho s} \left( (r - \rho) C_{s-} + \mu_s \right) ds + \int_0^t e^{-\rho s} f^*_s dN_s \right]
\leq C_0 + E_c \left[ \int_0^\infty e^{-\rho s} \mu_s ds + \int_0^\infty e^{-\rho s} f^*_s dN_s \right] \leq C_0 + \frac{1}{\rho} (\mu_t + \lambda C^*_t)
\]
for any $t < \infty$ and it now follows from Fatou’s lemma that $(D^*, f^*) \in \Theta$. Applying Itô’s formula for semimartingales to
\[
Y_t = e^{-\rho (t \wedge \tau_0)} V_t(C_t \wedge \tau_0) + \int_0^{t \wedge \tau_0} e^{-\rho s} (dD^*_s - f^*_s dN_s)
\]
and using the definition of $(D^*, f^*)$ in conjunction with the fact that $V_t(c)$ solves the HJB equation we obtain that $Y$ is a local martingale. Now, using the fact that $C_t \in [0, C^*_t]$ for all $t \geq 0$ together with the increase of $V_t$ we deduce that
\[
|Y_\theta| \leq |V_t(C^*_t)| + \int_0^\infty e^{-\rho t} (dD^*_t + f^*_t dN_t)
\]
for any stopping time $\theta$ and, since the right hand side is integrable, we conclude that $Y$ is a uniformly integrable martingale. In particular, we have
\[
V_t(c) = Y_{0-} = Y_0 - \Delta Y_0 = Y_0 + \Delta D^*_0 = E_c[Y_{\tau_0}] + \Delta D^*_0
\]
\[
= E_c \left[ e^{-\rho \tau_0} V_t(C_{\tau_0}) + \int_0^{\tau_0} e^{-\rho s} (dD^*_s - f^*_s dN_s) \right] + \Delta D^*_0
\]
\[
= E_c \left[ e^{-\rho \tau_0} \ell_t(0) + \int_0^{\tau_0} e^{-\rho s} (dD^*_s - f^*_s dN_s) \right]
\]
where the third equality follows from the definition of $V_t$ and the fourth from the martingale property of $Y$. This shows that $V_t(c) \leq \tilde{V}_t(c)$ and establishes the desired result.

**Q.E.D.**

**Lemma B.10** The level of cash holdings $C^*_t$ that is optimal for a firm with no growth option is monotone decreasing in $\lambda$ and $\varphi$ and increasing in $\sigma^2$.

**Proof.** Monotonicity in $\varphi$ follows from the definition of $C^*_t$ and the monotonicity of $\ell_t$. To establish the required monotonicity in $\lambda$, it suffices to show that $w_t(0; b, \lambda)$ is monotone decreasing in $\lambda$. 34
Indeed, in this case we have
\[
\ell_i(0) = w_i(0; C_i^*(\lambda_1), \lambda_1) \leq w_i(0; C_i^*(\lambda_1); \lambda_2)
\]
for all \( \lambda_1 < \lambda_2 \) and therefore \( C_i^*(\lambda_2) \leq C_i^*(\lambda_1) \) due to the fact that \( w_i(0; b, \lambda) = \frac{rb+\mu_1}{\rho} \) does not depend on \( \lambda \). As a result, it follows from Lemma B.2 that the function
\[
k(c) = w_i(c; b, \lambda_1) - w_i(c; b, \lambda_2)
\]
for some \( \lambda_1 < \lambda_2 \) satisfies
\[
k(b) = k'(b) = k''(b) = k^{(3)}(b) = k^{(4)}(b) = 0
\]
and solves the ODE
\[
\mathcal{L}_i k(c) - \lambda k(c) = (\lambda_2 - \lambda_1) (w_i(b; b, \lambda_2) - w_i(c; b, \lambda_2) - (b - c)).
\]  
(B.10)
Since, by Lemma B.4, \( w_i(c; b, \lambda_2) \) is concave in \( c \) and \( w_i'(b; b, \lambda_2) = 1 \), the right hand side of (B.10) is nonnegative for all \( c \leq b \) and it follows by a slight modification of Lemma B.5 that \( k(c) \) cannot have a positive local maximum. Since
\[
k^{(5)}(b) = \frac{2}{\sigma^2} (\lambda_1 - \lambda_2) w_i^{(3)}(b; b, \lambda_2) = \frac{2}{\sigma^2} (\lambda_1 - \lambda_2) (\rho - r) < 0,
\]
we conclude that \( k \) is decreasing in a small neighborhood of \( b \). Therefore, it is decreasing for all \( c \leq b \) and hence \( k(c) > k(b) = 0 \) for all \( c \leq b \). Similarly, if \( \sigma_1^2 > \sigma_2^2 \), then \( k(c) = w_i(c; b; \sigma_1^2) - w_i(c; b; \sigma_2^2) \) satisfies
\[
\mathcal{L}_i (\sigma_1^2) k(c) - \lambda k(c) = 0.5 (\sigma_1^2 - \sigma_2^2) w_i''(c; b; \sigma_2^2) > 0
\]
for \( c \leq b \) and the required monotonicity follows by the same arguments as above. Q.E.D.

C Proof of Proposition 2

The proof of Proposition 2 will be based on a series of lemmas. To facilitate the presentation, let \( \hat{V} \) denote the value of the firm and \( \Pi \) denote the set of triples \( \pi = (\tau, D, f) \) where \( \tau \) is a stopping time and \( (D, f) \in \Theta \) is an admissible dividend and financing strategy.

Our first result shows that for any fixed policy \( \pi \), the value of the firm is equal to the present value of all dividends net of issuing costs, up to the time \( \tau \) of investment, plus the present value of the value of the firm at the time of investment.
Lemma C.1 The value of the firm satisfies

\[
\hat{V}(c) = \sup_{\pi \in \Pi} E_c \left[ \int_0^{\tau \wedge \tau_0} e^{-\rho t} \left( dD_t - f_t dN_t \right) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_1(C_\tau) \right].
\]

If \( V_0(c) \geq V_1(c - K) \) for all \( c \geq K \) then it is optimal to abandon the growth option.

Proof. The proof of the first part follows from standard dynamic programming arguments and therefore is omitted. To establish the second part assume that \( V_0(c) \geq V_1(c - K) \) for all \( c \geq K \) and observe that \( \Delta C_\tau = -K + 1_{\{\tau \in \mathcal{N}\}} f_\tau \) where \( \mathcal{N} \) denotes the set of jump times of the Poisson process. Using this identity in conjunction with the first part, we obtain

\[
V(c) \leq \sup_{\pi \in \Pi} E_c \left[ \int_0^{\tau \wedge \tau_0} e^{-\rho t} \left( dD_t - f_t dN_t \right) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_0(C_\tau - 1_{\{\tau \in \mathcal{N}\}} f_\tau) \right],
\]

and the desired result follows since the right hand side of this inequality is equal to \( V_0(c) \) by standard dynamic programming arguments. Q.E.D.

By Lemma C.1, the option has a non-positive net present value if and only if \( V_0(c) \geq V_1(c - K) \) for all \( c \geq 0 \). Thus, in order to establish Proposition 2 it now suffices to show that this condition is equivalent to the inequality \( K \geq K^* \). This is the objective of the following:

Lemma C.2 The constant \( K^* \) is nonnegative and the following are equivalent:

(a) \( K \geq K^* \)

(b) \( K \geq V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*) \)

(c) \( V_0(c) \geq V_1(c - K) \) for all \( c \geq K \).

Proof. The equivalence of (a) and (b) follows from the definition of \( K^* \) and the fact that

\[
V_i(C_i^*) = (rC_i^* + \mu_i)/\rho.
\]

In order to show that the constant \( K^* \) is nonnegative we argue as follows: Since \( \mu_0 < \mu_1 \), the set of feasible strategies for \( V_0 \) is included in the set of feasible strategies for \( V_1 \). It follows that \( V_0 \leq V_1 \) and combining this with the definition of \( C_i^* \) shows that

\[
K^* = V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*) = \max_{C \geq 0} \{ V_1(C) - C \} - \max_{C \geq 0} \{ V_0(C) - C \} \geq 0.
\]

To establish the implication (a)\(\Rightarrow\)(c) it suffices to show that under (a) we have \( V_1(c - K^*) \leq V_0(c) \) for all \( c \geq K^* \). Indeed, if that is the case then (b) also holds since

\[
V_1(c - K) \leq V_1(c - K^*), \quad c \geq K \geq K^*.
\]
due to the increase of the function $V_1$. For $c \geq K^* \vee C_0^*$, the concavity of $V_1$ and the fact that $V_0$ is linear with slope one above the level $C_0^*$ jointly imply that

$$V_1(c - K^*) \leq V_1(C_1^*) + (c - K^* - C_1^*) = V_0(c) + C_0^* - V_0(C_0^*) - K^* - C_1^* = V_0(c)$$

and it remains to prove the result for $c \in [K^*, C_0^*]$. Consider the function $k(c) = V_0(c) - V_1(c-K^*)$. Using Lemma B.2 in conjunction with the fact that $C_0^* < C_1^* + K^*$ by Lemma C.3 below we have that the function $k$ is a solution to

$$L_0 k(c) - \lambda k(c) + (\mu_0 - \mu_1 + rK^*)V_1(c-K^*) = 0$$

on the interval $[K^*, C_0^*]$. Combining Lemma C.3 below with the increase of $V_1$ shows that the last term on the left hand side is positive and since

$$k(C_0^*) = V_0(C_0^*) - V_1(C_0^* - K^*) \geq V_0(C_0^*) - V_1(C^*) - (C_0^* - K^* - C_1^*) = 0,$$

$$k'(C_0^*) = V_0'(C_0^*) - V_1'(C_0^* - K^*) = 1 - V_1'(C_0^* - K^*) \leq 0$$

by the concavity of $V_1$, we can apply Lemma B.6 to conclude that $k(c) \geq 0$ for all $c \leq C_0^*$. Finally, the implication (c)$\Rightarrow$(b) follows by taking $c > C_0^* \vee (C_1^* + K^*)$. Q.E.D.

The optimal cash levels before and after investment are given by $C_0^*$ and $C_1^* + K$ respectively. By Lemma C.2, we have that $V_0(c) \geq V_1(c - K^*)$ for all $c \geq K$. Therefore, it is natural to expect that the better-off firm will have a lower optimal cash level, that is $C_0^* \leq C_1^* + K^*$. Similarly, it is natural to expect that the increase in the mean cash flow rate net of cost is nonnegative for $K \leq K^*$, that is $(\mu_1 - \mu_0)/r \leq K$. It turns out that both of these intuitive results are indeed true, as is shown by the following lemma.

**Lemma C.3** We have $C_0^* < C_1^* + K^*$ and $\mu_1 - \mu_0 - rK^* > 0$.

**Proof.** The definition of $K^*$ implies that the first inequality is equivalent to the second which is in turn equivalent to

$$r(C_1^* - C_0^*) > \mu_0 - \mu_1.$$  \hspace{1cm} (C.1)

Denote by $C^*(\mu)$ the optimal cash holdings for a firm with mean cash flow rate $\mu$ and no growth option. In order to prove the validity of (C.1) it suffices to show that

$$\frac{1}{r} + \frac{\partial C^*(\mu)}{\partial \mu} > 0$$

(C.2)

Let $w(c;b,\mu)$ stand for the function $w_i(c;b)$ with $\mu_i = \mu$ and use a similar notation for the hypergeometric functions $F_i(c), G_i(c)$ and the liquidation value $\ell_i(c)$. Combining the result of
Lemma B.2 with (9) and Abel’s identity
\[ G'(c; \mu)F(c; \mu) - F'(c; \mu)G(c; \mu) = e^{-(rc+\mu)^2} \sqrt{r} \frac{\sigma}{\sigma} \]
we get that
\[ w(c; b, \mu) = q(c; b, \mu) + \frac{\lambda}{\rho + \lambda} \left( \frac{\mu + br}{\rho} - b + c + \frac{\mu_i + rc}{\rho + \lambda - r} \right) \]
where the function on the right hand side is defined by
\[ q(c; b, \mu) = \alpha(b; \mu) F(c; \mu) - \beta(b; \mu) G(c; \mu) \] (C.3)
with
\[ \alpha(c; \mu) = \frac{(r - \rho)\sigma^2 G''(c; \mu)}{2 \sqrt{r} (\rho + \lambda - r) (\rho + \lambda) e^{-\sigma^2 r} (rC + \mu)^2} \]
\[ \beta(c; \mu) = \frac{(r - \rho)\sigma^2 F''(c; \mu)}{2 \sqrt{r} (\rho + \lambda - r) (\rho + \lambda) e^{-\sigma^2 r} (rC + \mu)^2} \]

In this notation, we have that the equation which defines the optimal level of cash holdings for a firm with no growth option is given by
\[ q(0; C^*(\mu), \mu) + \frac{\lambda}{\lambda + \rho} \left( \frac{\mu + (r - \rho)C^*(\mu)}{\rho} + \frac{\mu}{\lambda + \rho - r} \right) = \ell(0). \]

Using equations (B.5) and (B.6) in conjunction with the definition of the functions \( \alpha(c; \mu) \) and \( \beta(c; \mu) \) we obtain
\[ rq_{\mu}(0; b, \mu) = q_c(0; b, \mu) + q_b(0; b, \mu), \]
where a subscript denotes a partial derivative, and it thus follows from the implicit function theorem that
\[ \frac{\partial C^*(\mu)}{\partial \mu} = \frac{\varphi/\rho - q_b(0; C^*(\mu), \mu)/r - q_c(0; C^*(\mu); \mu)/r - B}{q_b(0; C^*(\mu), \mu) - A} \]
where we have set
\[ A = \frac{\lambda}{\lambda + \rho} \left( 1 - \frac{r}{\rho} \right), \quad B = \frac{\lambda}{\lambda + \rho} \left( \frac{1}{\rho} + \frac{1}{\lambda + \rho - r} \right). \]

By Lemma B.4 we have that the function \( q(c; b, \mu) \) is decreasing in \( b \) and since \( A > 0 \) it follows that the validity of inequality (C.2) is equivalent to
\[ -q_b(0; C^*(\mu), \mu) - q_c(0; b; \mu) - r (B - \varphi/\rho) < A - q_b(0; C^*(\mu), \mu), \]
which in turn follows from
\[ q_c(0; b, \mu) + r(B - 1/\rho) > 0. \] (C.4)

Since the difference \( q - w \) is a linear function of \( c \) we have from Lemma B.4 that \( q(c; b, \mu) \) is concave in \( c \) and it follows from the smooth pasting condition that
\[
q_c(0; C^*(\mu), \mu) \geq q_c(C^*(\mu); C^*(\mu), \mu) = w_c(C^*(\mu); C^*(\mu), \mu) - \frac{\lambda}{\lambda + \rho - r} = \frac{\rho - r}{\lambda + \rho - r}.
\]

Combining this with a straightforward calculation shows that (C.4) holds. Q.E.D.

References


Figure 1: Value of a firm with a growth option when $K < K \leq K^{**}$

This figure represents the value of a firm with a growth option as a function of its cash holdings when $K < K \leq K^{**}$. In the shaded area below the investment trigger $C_U^*$, the optimal policy is to retain earnings and search for investors. In the unshaded area between $C_U^*$ and $C_1^* + K$, the optimal policy is to invest in the growth option from internal funds. In the hatched area above $C_1^* + K$, the optimal policy is to invest with internal funds and distribute dividends to decrease cash holdings to the target level $C_1^*$ after investment.
Figure 2: Marginal value of cash under the optimal barrier strategy

This figure plots the marginal value of cash $U'(c)$ under the optimal barrier strategy in an environment where the investment cost is low (dashed line) and in an environment where the costs of investment is high (solid line). In the latter case the marginal value of the cash drops to one at the point $C_\downarrow$ and remains below one over the interval $(C_\downarrow, C_\uparrow)$ indicating that shareholders would rather abandon the option of investing from internal funds and receive dividends than continue hoarding cash inside the firm.
Figure 3: Value of a firm with a growth option when $K^{**} < K \leq K^*$

This figure represents the value of a firm with a growth option as a function of its cash holdings when $K^{**} \leq K < K^*$. In the shaded areas the optimal policy is to retain earnings and search for investors. In the first hatched area, the optimal policy is to distribute dividends to decrease the level of cash holdings to $C^*_W$. In the unshaded area between $C^*_H$ and $C^*_1 + K$ the optimal policy is to invest in the growth option with internal funds. In the second hatched area, the optimal policy is to invest with internal funds and distribute dividends in order to decrease cash holdings to the target level $C^*_1$ after investment.
Figure 4: Marginal value of cash for a firm with a growth option

This figure plots the marginal value of cash $V'(c)$ under the globally optimal strategy (dashed line) and the marginal value of cash $U'(c)$ under the optimal barrier strategy (solid line) in an environment where the investment cost is high. In the latter case the failure of global optimality is due to the fact that the marginal value of the cash drops below one indicating that shareholders would rather abandon the option of investing from internal funds and receive dividends than continue hoarding cash inside the firm.
This figure illustrates the model with multiple options: The firm initially has mean cash flow rate $\mu_0$ and does not any growth option. At the exponentially distributed time $\zeta_1$ the firm receives its first growth option and exercises optimally at the stopping time $\theta_1$. The second growth option then arrives after the exponentially distributed time $\zeta_2 - \theta_1$ has elapsed and is optimally exercised at the stopping time $\theta_2$. This goes on until the optimal exercise of the last growth option at the stopping time $\theta_N$. After that time the mean cash flow rate of the firm remains constant.
**Figure 6:** Value of the firm in the waiting period between growth options

This figure represents the value of a firm as a function of its cash holdings in the waiting period between the exercise of the $i$'th growth option and the arrival of the next one. In this picture the optimal strategy includes two intermediate dividend distribution intervals and three earnings retention intervals whose location are specified by the vectors $(a_i^*, b_i^*)$ and the target $x_i^*$. 
Panel A plots the critical investment costs $K$ (dotted), $K^{**}$ (solid) and $K^*$ (dashed) as functions of the arrival rate of investors $\lambda$, the tangibility of assets $\varphi$, the carry cost of cash $\delta$ and the volatility of cash flows $\sigma$. Panel B plots the internal rate of returns associated with $K^{**}$ (solid) and $K^*$ (dashed) as functions of the same parameters. In each plot the vertical line indicates the base value of the parameter.
Figure 8: Optimal cash holdings for a firm with a growth option

Panel A plots the investment threshold $C_{U}^{*}$ for a firm with a low investment cost as a function of the arrival rate of investors $\lambda$, the tangibility of assets $\varphi$, the carry cost of cash $\delta$ and the volatility of cash flows $\sigma$. Panel B plots the investment threshold $C_{H}^{*}$ (solid) and the payout thresholds $C_{L}^{*}$ (dashed) and $C_{W}^{*}$ (dotted) for a firm with a high investment cost as functions of the same parameters. In each plot the vertical line indicates the base value of the parameter.
The top four panels plots the average probability of investment with internal funds for a firm with a low investment cost (dashed line) and a firm with a high investment cost (solid line) as functions of the arrival rate of investors $\lambda$, the tangibility of assets $\varphi$, the carry cost of cash $\delta$ and the volatility of cash flows $\sigma$. The two lower panels plot the total probability of investment at an horizon of one year (solid line) and three years (dashed line) for a firm with cash holdings $C = K < K^{**}$ (left) and $C = K > K^{**}$ (right) as functions of the arrival rate of investors $\lambda$. In each plot the vertical line indicates the base value of the parameter.
Supplementary Appendix to:
Capital supply uncertainty, cash holdings, and investment
Julien Hugonnier  Semyon Malamud  Erwan Morellec
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This document provides the proofs omitted from the main text.

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This appendix shows that introducing bargaining is equivalent to reducing the arrival rate of outside investors. To simplify the presentation, we start by considering the case of a firm with no growth option before turning to the general case.

If the firm has mean cash flow rate $\mu_i$ and no growth option, its optimization problem can be formulated as

$$V_i(c) = \sup_{\pi \in \Theta} E_c \left[ \int_0^{\tau_0} e^{-\rho s}(dD_s - (f_s + \eta S(C_{s-}^\pi((\pi, V_i))))dN_s) + e^{-\rho \tau_0} \ell_i \right], \quad (D.1)$$

where

$$S(C_{t-}^\pi |(\pi, V_i)) = V_i(C_t^\pi) - V_i(C_{t-}^\pi) - f_t = V_i(C_t^\pi + f_t) - V_i(C_{t-}^\pi) - f_t$$

represents the financing surplus associated with the strategy $\pi = (D, f)$, and $\Theta$ denotes the set of dividend and financing strategies such that

$$E_c \left[ \int_0^{\tau_0} e^{-\rho s}(dD_s + f_s dN_s) \right] < \infty.$$ 

Since the firm value appears in the objective function, the optimization problem in (D.1) is akin to a rational expectations problem: When bargaining over financing, outside investors have to correctly anticipate the strategy that the firm will use in the future. Accordingly, if the function $V_i$ satisfies (D.1) and $\pi^* \in \Theta$ attains the supremum, then we say that $(\pi^*, V_i)$ is a rational expectations equilibrium for the firm.

In order to simplify the construction of such an equilibrium, consider the optimization problem of an auxiliary firm for which investors arrive at rate $\lambda^*$ and do not bargain over the terms of financing.

The following proposition shows that a rational expectations equilibrium can be constructed by considering the optimization problem of an auxiliary firm for which investors arrive at rate $\lambda^*$ and do not bargain over the terms of financing.
Proposition D.1 Consider the optimization problem defined by

\[ V_i^*(c) = \sup_{\pi \in \Theta^*} E^*_c \left[ \int_0^{\tau_0} e^{-\rho s} (dD_s - f_s) dN_s + e^{-\rho \tau_0} \ell_i \right]. \]

and assume that the function \( V_i^* \) is Lipschitz continuous. If there exists a strategy \( \pi^* \in \Theta^* \cap \Theta \) that attains the above supremum then \( (\pi^*, V_i^*) \) constitutes a rational expectations equilibrium.

Proof. Let \( (\pi^*, V_i^*) \) be as in the statement. Using the definition of the optimization problem together with standard dynamic programming arguments, we deduce that

\[ X_t^\pi = e^{-\rho t \wedge \tau_0} V_t^*(C_t^{\pi\tau_0}) + \int_0^{t \wedge \tau_0} e^{-\rho s} (dD_s - f_s) dN_s \]

is a supermartingale under \( P^* \) for any strategy \( \pi \) and a martingale for the optimal strategy \( \pi^* \). By application of the Doob-Meyer decomposition this implies that for every \( \pi \) there exists a predictable process \( \vartheta^\pi \) and a non-decreasing process \( A^\pi \) such that

\[ X_t^\pi = V_t^*(C_0^\pi) + \int_0^t (\vartheta_s^\pi dB_s - dA_s^\pi) + \int_0^t e^{-\rho s} \Delta X_s(dN_s - \lambda^* ds) \]

where \( \Delta Z_t = Z_t - \lim_{s \uparrow t} Z_s \) denotes the jump in the process \( Z \) and the last equality follows from the definition of the process \( X^\pi \). This in turn implies that

\[ Y_t^\pi = e^{-\rho t \wedge \tau_0} V_t^*(C_{t \wedge \tau_0}) + \int_0^{t \wedge \tau_0} e^{-\rho s} (dD_s - (f_s + \eta S(C_s^\pi | (\pi, V_s^*)))) dN_s \]

\[ = X_t^\pi + \int_0^t e^{-\rho s} \eta (\Delta V_t^*(C_s^\pi) - f_s) dN_s \]

\[ = V^*(C_0^\pi) + \int_0^t \vartheta_s^\pi dB_s - A_t^\pi + \int_0^t e^{-\rho s} (1 - \eta)(\Delta V_t^*(C_s^\pi) - f_s) (dN_s - \lambda ds) \]

is a supermartingale under \( P \) for any strategy \( \pi \) and a local martingale under \( P \) for the optimal strategy \( \pi^* \). In particular, we have

\[ V_i^*(c) = Y_0^\pi - \Delta V_i^*(C_0^\pi) \geq E_c [Y_t^\pi] - \Delta V_i^*(C_0^\pi) \]

\[ = E_c \left[ e^{-\rho t \wedge \tau_0} V_t^*(C_{t \wedge \tau_0}) + \int_0^{t \wedge \tau_0} e^{-\rho s} (dD_s - (f_s + \eta S(C_s^\pi | (\pi, V_s^*)))) dN_s \right] \]

\[ - \Delta V_i^*(C_0^\pi) \]

\[ \geq E_c \left[ e^{-\rho t \wedge \tau_0} V_t^*(C_{t \wedge \tau_0}) + \int_0^{t \wedge \tau_0} e^{-\rho s} (dD_s - (f_s + \eta S(C_s^\pi | (\pi, V_s^*)))) dN_s \right] \]
for all \( t \geq 0 \) and every \( \pi \) where the last inequality follows from the fact that

\[
V_i^*(c) \geq \sup_{x \geq 0} (x + V_i^*(c - x)).
\]

Using the assumed Lipschitz continuity of the function \( V_i^* \) together with the dynamics of the cash buffer process it can be shown that

\[
\sup_{t \geq 0} |Y_t^\pi| \leq A_0 + A_1 m^*_\infty + A_2 \int_0^{\tau_0} e^{-\rho s} (dD_s + f_s dN_s)
\]  

\[ \text{(D.2)} \]

for some constants \( (A_i)_{i=0}^2 \). where

\[
m^*_t = \sup_{s \leq t} \left| \int_0^t e^{-\rho s} \sigma dB_s \right|.
\]

Since the right hand side of equation (D.2) is integrable for any \( \pi \in \Theta \) it follows from the dominated convergence theorem and the definition of \( V_i^* \) that

\[
V_i^*(c) \geq \sup_{\pi \in \Theta} E_c \left[ \int_0^{\tau_0} e^{-\rho s} (dD_s - (f_s + \eta_s (C^\pi_s - |(\pi, V_i^*)|))dN_s) + e^{-\rho \tau_0} V_i^*(0) \right] \]  

\[ \text{(D.3)} \]

On the other hand, if \( (\tau_n)_{n=1}^\infty \) denotes a localizing sequence of stopping times for the local martingale \( Y^{\pi^*} \) then

\[
V_i^*(c) = Y_0^{\pi^*} - \Delta V_i^*(C_0^\pi^*) = E_c \left[ Y_{\tau_n}^{\pi^*} + \Delta D_0^* \right] - \Delta V_i^*(C_0^\pi^*) - \Delta D_0^*
\]

\[
= E_c \left[ e^{-\rho \tau_0 \wedge \tau_n} V_i^*(C_{\tau_0 \wedge \tau_n}^\pi^*) + \int_0^{\tau_0 \wedge \tau_n} e^{-\rho s} (dD_s^* - (f_s + \eta_s (C_{s}^\pi^* - |(\pi^*, V_i^*)|))dN_s) \right] - \Delta V_i^*(C_0^\pi^*) - \Delta D_0^*
\]

\[
= E_c \left[ e^{-\rho \tau_0 \wedge \tau_n} V_i^*(C_{\tau_0 \wedge \tau_n}^\pi^*) + \int_0^{\tau_0 \wedge \tau_n} e^{-\rho s} dD_s^* - (f_s + \eta_s (C_{s}^\pi^* - |(\pi^*, V_i^*)|))dN_s) \right]
\]

where the last equality follows from the fact that

\[
V_i^*(c) = \Delta D_0^* + V_i^*(c - \Delta D_0^*) = \Delta D_0^* + V_i^*(C_0^\pi^*)
\]

due to the assumed optimality of \( \pi^* \). Since \( \pi^* \in \Theta \) it follows from equation (D.2), the dominated
convergence theorem and the definition of the function $V^*_i$ that

$$V^*_i(c) = E_c \left[ \int_0^{\tau_0} e^{-\rho s}(dD_s^* - (f_s + \eta S(C_{s-}^\pi((\pi^*, V^*_i))))dN_s) + e^{-\rho \tau_0} \ell_0 \right]$$  \hspace{1cm} (D.4)

Combining equations (D.3) and (D.4) shows that $(\pi^*, V^*_i)$ is a rational expectations equilibrium and completes the proof. Q.E.D.

Having dealt with the case of a firm with no growth option, we now turn to the case of a firm that possesses an option to expand operations. Let $(\pi^*_1, V^*_1)$ be a rational expectations equilibrium after investment that satisfies the conditions of Proposition D.1 and denote by $\Pi$ (resp. $\Pi^*$) the set of triples $\pi = (D, f, T)$ where $T$ is a stopping time and $(D, f) \in \Theta$ (resp. $\Theta^*$) is a dividend and financing strategy. Relying on dynamic programming arguments, we have that the optimization problem of such a firm can be formulated as

$$V(c) = \sup_{\pi \in \Pi} E_c \left[ \int_0^{\tau_0 \wedge T} e^{-\rho s}(dD_s - [f_s + \eta S(C_{s-}^\pi((\pi, V, V^*_1)))]dN_s) + 1_{\{\tau_0 < T\}}e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq T\}}e^{-\rho T} V^*_1(C_{T}^\pi) \right]$$  \hspace{1cm} (D.5)

where

$$S(C_{t-}^\pi((\pi, V, V^*_1))) = 1_{\{t \neq T\}} V(C_{t-}^\pi + f_t) + 1_{\{t = T\}} V^*_1(C_{t-}^\pi + f_t - K) - V(C_{t-}^\pi) - f_t$$

represents the financing and investment surplus. In accordance with our previous definition, $(\pi^*, V, V^*_1)$ forms a rational expectations equilibrium if $(V, V^*_1)$ satisfy (D.5) and $\pi^* \in \Pi$ attains the supremum. The following proposition is the direct counterpart of Proposition D.1 for the case of a firm with a growth option.

**Proposition D.2** Consider the optimization problem defined by

$$V^*(c) = \sup_{\pi \in \Pi^*} E_c^* \left[ \int_0^{\tau_0 \wedge T} e^{-\rho s}(dD_s - f_s dN_s) + 1_{\{\tau_0 < T\}}e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq T\}}e^{-\rho T} V^*_1(C_{T}^\pi) \right]$$

and assume that the function $V^*$ is Lipschitz continuous. If there exists $\pi^* \in \Pi^* \cap \Pi$ that attains the above supremum then the triple $(\pi^*, V^*, V^*_1)$ constitutes a rational expectations equilibrium.

**Proof.** The proof is similar to that of Proposition D.1 and therefore is omitted. Q.E.D.
E Optimaliy of barrier strategies

E.1 Intuition and road map

In this section we construct the function $u(c; C^*_U)$ that fits $V_1(c-K)$ at some cash level $C^*_U$. Whether this fit is smooth will depend on the size of the investment cost $K$.

Let us first consider the function $u(c; K)$ associated to the following policy: Search for external investors to raise funds and invest when $c < K$, otherwise immediately invest and liquidate to receive $\ell_1(c)$. When $c = K$ the firm will just invest and immediately liquidate, so that $u(c; K)$ satisfies the boundary conditions $u(0; K) = \ell_0(0)$ and $u(K; K) = \ell_1(0)$. This policy can be optimal even though the function $u(c; K)$ does not satisfy the smooth fit principle if the marginal value of cash is higher slightly below the investment cost than slightly above, that is if $u'(K; K) > V'_1(0)$. Lemma E.2 confirms this intuition by showing that $u'(K; K) \leq V'_1(0)$ is both necessary and sufficient for the existence and uniqueness of a solution to (11), (12) and (13) with the smooth pasting condition (16). When $u'(K; K) > V'_1(0)$ we let $C^*_U = K$ so that $u(c; C^*_U) = u(c; K)$ and the fit to the post-investment firm value function is non-smooth.

E.2 Proofs

Lemma E.1 Let $F_0, G_0$ be as in (B.5)-(B.6). Let $q$ denote an arbitrary function and define $\hat{q}$ implicitly through

$$q(c) = F_0(c)\hat{q}(Z(c)) = F_0(c)\hat{q}\left(\frac{G_0(c)}{F_0(c)}\right).$$

Then we have:

(a) The function $Z$ is monotone increasing and $\hat{q}(y) = q(Z^{-1}(y))/F_0(Z^{-1}(y))$,

(b) The function $q$ solves (B.7) if and only if the function $\hat{q}$ is linear,

(c) For an arbitrary $c \geq 0$,

$$\min\{\hat{q}'(y)(q(c)/F_0(c))', \hat{q}''(y)(\mathcal{L}_0q(c) - \lambda q(c))\} \geq 0$$

with $y = Z(c)$.

Proof. The first two claims follow by direct calculation using the definition of $\hat{q}$, $F_0$ and $G_0$. The third claim is formula (6.2) in Dayanik and Karatzas (2003). Q.E.D.

Lemma E.2 If the condition $u'(K; K) \leq V'_1(0)$ holds then there exists a unique solution $(C^*_U, u(c; C^*_U))$ to (11), (12), (13), (16) and this solution is such that $u(c; C^*_U) \geq V_1(c-K)$ for all $c \geq K$. 

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Proof. Consider the function defined by
\[ v_1(c - K) = V_1(c - K) - \Phi(c) \]
with \( \Phi(c) \) as in (15). To prove the result, we start by observing that thanks to Lemma E.1, finding a solution to the system (11), (12), (13), (16) is equivalent to finding a linear function \( \phi \) that is tangent to the graph of the function \( \hat{v}_1 \) defined by
\[
v_1(c - K) = F_0(c)\hat{v}_1(Z(c)) = F_0(c)\hat{v}_1 \left( \frac{G_0(c)}{F_0(c)} \right)
\]
and such that \( \phi(Z(0))F_0(0) = \ell_0 - \Phi(0) \). A direct calculation using the results of Lemmas B.8 and C.3 shows that
\[
L_0v_1(c - K) - \lambda v_1(c) = (r - \rho)(c - C^*_1 - K)^+ + (\mu_0 - \mu_1 + rK)V'_1(c - K) \leq 0
\]
for all \( c \geq K \) and it now follows from Lemma E.1 that \( \hat{v}_1(y) \) is concave for all \( y \geq Z(K) \). On the other hand, since \( V_1 \) is concave we obtain
\[
v'_1(c - K) = V_1(c - K) - \frac{\lambda}{\lambda + \rho - r} V'_1(C^*_1) \geq \frac{\lambda}{\lambda + \rho - r} = \frac{\rho - r}{\lambda + \rho - r} > 0
\]
and it follows \( v_1(c - K) \) is positive for sufficiently large values of \( c \). Since \( F_0 \) is nonnegative and decreasing, this implies that the ratio \( v_1(c - K)/F_0(c) \) is increasing for large \( c \) and it now follows from Lemma E.1 that \( \hat{v}_1(y) \) is increasing for large values of \( y \) and is therefore globally increasing in \( y \geq Z(K) \) since it is concave in that region.

Since by assumption \( u'(K; K) \leq V'_1(0) \) we have that the line passing through the points
\[
(Z(0), (\ell_0 - \Phi(0))/F_0(0)) \quad \text{and} \quad (Z(K), (\ell_1 - \Phi(0))/F_0(0))
\]
has a higher slope at \( y = Z(K) \) than \( \hat{v}_1 \). Using the concavity and increase of \( \hat{v}_1 \), it is then immediate that there exists a unique line passing through \( (Z(0), (\ell_0 - \Phi(0))/F_0(0)) \) that is tangent to \( \hat{v}_1 \) at some \( y^* > Z(K) \). Setting \( C^*_U = Z^{-1}(y^*) \) proves the existence of a unique solution to the value matching and smooth pasting conditions. Since \( \hat{v}_1 \) is concave, it lies below its tangent line at \( y^* \) and, transforming back to \( V_1(c - K) \) and \( u(c; C^*_U) \), we get \( u(c; C^*_U) \geq V_1(c - K) \). Q.E.D.

Lemma E.3 We have \( u(c; C^*_U) \leq \hat{V}(c) \) where \( \hat{V}(c) \) denotes the value function of the firm’s optimization problem.

Proof. Consider the investment, dividend and financing strategy \( \pi^U \) defined by \( \tau = \tau_N \land \tau^*_U \),
$D^U = 0$ and
\[ f_t^U = (C_1^* + K - C_t^-)^+ \]  \hspace{1cm} (E.2)

where $\tau_N$ denotes the first jump time of the Poisson process and $\tau_U^*$ denotes the first time that the firm’s cash reserves reach the level $C_U^*$. As is easily seen, we have
\[
E_c \left[ \int_0^{\tau_0} e^{-\rho t} (dD_t^U + f_t^U dN_t) \right] \leq E_c \left[ \int_{\tau_0}^{\infty} e^{-\rho t} (C_1^* + K) dN_t \right] = \frac{\lambda}{\rho} (C_1^* + K)
\]

and it follows that $\pi^U \in \Pi$. On the other hand, using an argument similar to that of the proof of Proposition 1 it can be shown that
\[
Y_t = e^{-\rho t \land \tau_0 \land \tau_U^*} u(C_t \land \tau_0 \land \tau_U^*; C_U^*) + \int_0^{t \land \tau_0 \land \tau_U^*} e^{-\rho t} (dD_s^U - f_s^U dN_s)
\]
is a uniformly integrable martingale. An application of the optional sampling theorem at the finite stopping time $\tau_N$ then implies
\[
u(c; C_U^*) = Y_0 = E[Y_{\tau_N}] = E_c \left[ e^{-\rho t \land \tau_0} u(C_t \land \tau_0; C_U^*) + \int_0^{t \land \tau_0} e^{-\rho t} (dD_s^U - f_s^U dN_s) \right]
\]

and the desired result now follows from Lemma C.1. Q.E.D.

F  The value of a firm that only invests with external funds

F.1 Intuition and road map

The goal of this section is to construct and derive useful properties of the value $W(c)$ of a firm that accumulates cash up to an optimally determined target level $C_W^* = C_W^*(K)$ and invests exclusively from external funds as soon as financing can be secured. The formal existence and basic properties of this function are provided in Lemma F.1.

By Lemma C.2, the option has a positive net present value for $K < K^*$ in which case $V_0(c)$ is smaller than $V_1(c - K)$ for some values of $c > 0$. We expect the same to be true for the function $W(c)$. Clearly, we have that $V_1(c) > W(c)$ and so we expect that $W(c) < V_1(c - K)$ for high levels of cash holdings when the effect of the fixed investment cost is smaller. By definition we have that $W(c) = W(C_W^*) + (C - C_W^*)$ and $V_1(c) = V_1(C_1^*) + (c - C_1^*)$ for $c \geq \max\{C_W^*, C_1^* + K\}$. Therefore,
our conjecture can be confirmed by showing that \( W(C^*_W) - C^*_W < V_1(C^*_1) - (C^*_1 + K) \). This is accomplished by Lemma F.3 whose proof is in turn based on Lemma F.2. Having established this important result, we naturally expect that the inequality \( V_1(c - K) > W(c) \) can be violated only at low cash holdings levels and this intuition is confirmed by Lemma F.4.

### F.2 Proofs

Denote the value of the firm by \( \hat{V}(c) \) as before. The following lemma establishes existence of the function \( W(c) \) and shows that it is concave. In order to state the result let \( w(c; b) \) be the twice continuously differentiable solution to

\[
0 = \mathcal{L}_0 w(c; b) + \lambda (V_1(C^*_1) - C^*_1 - K - c - w(c; b)) = w'(b; b) - 1 = w''(b; b).
\]

The same argument as for the function \( V_i(c) \) implies that

\[
w(b; b) = \frac{rb + \mu + \lambda (V_1(C^*_1) - C^*_1 - K + b)}{\rho + \lambda},
\]

and the optimal target level \( C^*_W \) for the firm’s cash holdings can then be determined by imposing the value matching condition

\[
w(0; C^*_W) = \ell_0(0) \tag{F.1}
\]

at the point where the firm runs out of cash.

**Lemma F.1** There exists a unique solution \( C^*_W \) to (F.1) and the function \( W(c) = w(c; C^*_W) \) is increasing, concave and satisfies \( W(c) \leq \hat{V}(c) \).

**Proof.** The results follow by arguments similar to those we used in the proof of Proposition 1. We omit the details. The fact that \( W(c) \leq \hat{V}(c) \) follows from the fact that the policy corresponding to the value function \( W(c) \) is a priori suboptimal. Q.E.D.

By Lemma C.2, the constant \( K^* \) is the investment cost for which the net present value of the growth option is identically zero. Therefore, as \( K \) increases to \( K^* \) the firm following the strategy associated with \( W(c) \) will gradually reduce its target cash holding level and for \( K = K^* \) the difference between the functions \( V_0(c) \) and \( W(c) \) should vanish at which point we should have \( C^*_W(K^*) = C^*_0 \). The following lemma shows that this intuition is correct.

**Lemma F.2** The threshold \( C^*_W = C^*_W(K) \) is decreasing in \( K \) and satisfies \( C^*_W(K^*) = C^*_0 \).
Proof. Using the same arguments as in the case without growth options we have that solving (F.1) is equivalent to solving for \( b \) in

\[
q(c; b; \mu_0) = \frac{\lambda}{\lambda + \rho} (\frac{r}{\rho} - 1) C_1^* - K + \frac{\mu_1}{\rho} + \frac{\mu_0}{\lambda + \rho - r}) = \ell_0(0)
\]

where the function \( q(c; b; \mu_0) \) is defined as in (C.3). As shown in the proof of Lemma B.4 the function \( q(c; b; \mu_0) \) is monotone decreasing in \( b \), so that \( w(0; b) \) is monotone decreasing. On the other hand, a direct calculation shows that \( w(0; 0) = \mu_0/\rho > \ell_0(0) \) and \( w(0; \infty) < 0 \) and it follows that there exists a unique solution \( C_W^* \) to the value matching condition.

Since the function \( q(c; b; \mu_0) \) is decreasing in \( b \) the monotonicity of \( C_W^* \) with respect to \( K \) follows from the implicit function theorem. To show that the target \( C_W^* \) converges to \( C_0^* \) as the investment cost converges to \( K^* \) we argue as follows. By definition we have

\[
V_1(C_1^*) - C_1^* - K^* = V_0(C_0^*) - C_0^*.
\]

Thus, it follows from Lemma B.8 that the function \( V_0(c) \) solves

\[
0 = \mathcal{L}_0 V_0(c) + \lambda [V_0(C_0^*) - C_0^* + c - V_0(c)] = \mathcal{L}_0 V_0(c) + \lambda [V_1(C_1^*) - C_1^* - K^* + c - V_0(c)]
\]

on the interval \([0, C_0^*]\) with the boundary conditions \( V'_0(C_0^*) = 1, V''(C_0^*) = 0 \) and the desired result follows from the uniqueness part of Lemma F.1.

By definition, we have that \( K < K^* \) if and only if \( V_0(C_0^*) - C_0^* < V_1(C_1^*) - (C_1^* + K) \). The following lemma shows that a similar inequality holds true for the function \( W(c) \).

Lemma F.3 The following are equivalent:

(a) \( K > K^* \),

(b) \( W(C_W^*(K)) - C_W^*(K) > V_1(C_1^*) - (C_1^* + K) \).

Proof. Evaluating the ODE

\[
\mathcal{L}_0 W(c) + \lambda [V_1(C_1^*) - C_1^* - K + c - W(c)] = 0
\]

at the point \( c = C_W^* \) and using the definition of \( K^* \) we obtain that

\[
(\lambda + \rho)(W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K)) = \rho(K - K^*) + (\rho - r)(C_0^* - C_W^*)
\]

and the desired equivalence now follows from Lemma F.2.
Lemma F.4 The following statements hold:

(a) If \( K \geq K^* \) then \( W(c) \geq V_1(c - K) \) for all \( c \geq K \).

(b) If \( K < K^* \) then either \( V_1(c - K) \geq W(c) \) for all \( c \geq K \) or there exists a unique crossing point \( K \leq \bar{c} \leq C_1^* + K \) such that \( V_1(c - K) < W(c) \) if and only if \( c < \bar{c} \).

Proof. We only prove part (b) as both claims follow from similar arguments. Since the function \( W(c) = w(c; C_W^*) \) is concave by Lemma F.1 and \( W'(C_W^*) = 1 \), we have

\[
W(c) \leq W(C_W^*) + c - C_W^*
\]

and it now follows from Lemma F.3 that

\[
k(c) = W(c) - V_1(c - K) \leq W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K) \leq 0.
\]

for all \( c \geq C_1^* + K \). In order to complete the proof of the first part we distinguish three cases depending on the location of the threshold \( C_W^* \).

Case 1: \( C_W^* \leq K \). In this case the function \( W(c) \) is linear for \( c \geq K \). Since \( V_1(c) \) is concave, the functions \( V_1(c - K) \) and \( W(c) \) can have at most two crossing points. But, since \( V_1(c - K) > W(c) \) for large \( c \) as shown above there can be at most one crossing point.

Case 2: \( C_W^* > C_1^* + K \). Suppose towards a contradiction that the function \( k \) has more than one zero and denote by \( z_0 \leq z_1 \) its two largest zeros in the interval \([K, C_1^* + K]\). Then, \( k(c) > 0 \) for \( c \in (z_0, z_1) \) due to the above inequality and it follows that the function \( k \) has a positive local maximum in the open interval \((z_0, z_1)\). Since \( C_W^* > C_1^* + K \) it follows from Lemmas B.8 and F.1 that the function \( k(c) \) satisfies

\[
\mathcal{L}_0 k(c) - \lambda k(c) + (\mu_0 - \mu_1 + rK) V_1'(c - K) = 0 \quad \text{(F.2)}
\]

in the interval \([0, C_1^* + K] \) and the required contradiction now follows from Lemma B.5 and the fact that \( \mu_1 - \mu_0 - rK > 0 \) whenever \( K \leq K^* \) as a result of Lemma C.3.

Case 3: \( K \leq C_W^* \leq C_1^* + K \). If \( z_1 \leq C_W^* \) then the same argument as in Case 2 still applies so assume that \( k(c) \) has zeros in the interval \([C_W^*, C_1^* + K]\). Since \( V_1(c - K) \) is concave and \( k(C_1^* + K) \leq 0 \) we know that it can have at most one zero there. Denote the location of this zero by \( \tilde{c} \) so that \( k(c) > 0 \) for \( c \in [C_W^*, \tilde{c}] \) and \( k(c) \leq 0 \) for \( c \geq \tilde{c} \). Since \( k(c) \) solves (F.2) on \([0, C_W^*]\) and satisfies \( k(C_W^*) > 0 \) as well as \( k'(C_W^*) = 1 - V_1'(C_W^* - K) < 0 \) it follows from Lemma B.6 that \( k(c) > 0 \) for all \( c \leq C_W^* \).

Q.E.D.
G Proof of Theorems 3 and 4

G.1 Intuition and road map

In order to prove Theorems 3 and 4, we need to proceed with the same four steps as in the proof of Proposition 1.

1. The HJB equation now takes the form

\[
\max \{ L_0 \phi(c; b) + F \phi(c; b); 1 - \phi'(c); V_1(c - K) - \phi(c; b), \ell_0(c) - \phi(c; b) \} \leq 0. \tag{G.1}
\]

The only difference between this equation and the HJB equation (B.1) without the option is that since the firm may invest as soon as it has enough cash the value function needs to satisfy the additional condition \( \phi(c; b) \geq V_1(c - K) \) for all \( c \geq K \).

2. This step consists in establishing a verification result for the HJB equation and is accomplished by Lemma G.1 below.

3. To proceed with Step 3 we need to conjecture the form of the optimal policy to (G.1). The first candidate for this is the barrier policy associated with \( u(c; C^*_U) \). For this strategy to be optimal it should be that its value dominates that of any other policy. In particular, it is necessary that \( u(c; C^*_U) \geq W(c) \) for all \( c \geq 0 \) and, since \( u(0; C^*_U) = W(0) = \ell_0(0) \) by construction it is also necessary that \( u'(0; C^*_U) \geq W'(0) \). Lemma G.2 shows this condition is in fact both necessary and sufficient for the function \( u(c; C^*_U) \) to be the value function and for the global optimality of the corresponding barrier policy.

Suppose now that \( W'(0) > u'(0; C^*_U) \) and consider the function \( \tilde{W}(c) \) that solves equation (1) and coincides with the function \( W(c) \) for all \( c \leq C^*_W \). Since the functions \( U(c) \) and \( \tilde{W}(c) \) solve the same equation subject to the same initial value our assumption about the derivatives of these functions at the origin allows us to show that \( \tilde{W}(c) > u(c; C^*_U) \) for all \( c > 0 \). By construction, the function \( \tilde{W}(c) \) touches the linear part of the function \( W(c) \) at the point \( C^*_W \). Pick a cash level \( C^*_L \) that we will vary between \( C^*_W \) and the threshold \( \tilde{C} \) of Lemma F.4 and for each such level denote by \( S(c; C^*_L) \) the unique solution to (1) satisfying value matching and smooth pasting with \( W(c) \) at the point \( C^*_L \). By continuity, there exists a unique \( C^*_L \in (C^*_W, \tilde{C}) \) such that \( S(c; C^*_L) \) touches the graph of the function \( V_1(c - K) \) from above at some point \( C^*_H \) and we will take the function

\[
V(c) = 1_{\{c \leq C^*_L\}} W(c) + 1_{\{C^*_L < c \leq C^*_H\}} S(c; C^*_L) + 1_{\{c > C^*_H\}} V_1(c - K)
\]

as the value of our candidate optimal policy. A rigorous implementation of this construction is provided below.
4. Step 4 consists in proving that the value of our candidate optimal strategy solves the HJB equation (G.1) and is accomplished by Lemma G.5.

Once these four steps are complete, it will remain to show that \( u(c; C_U^*) \) is the value function if and only if the investment cost is below a threshold \( K^{**} \) and that \( C_U^* = K \) if and only if \( K \) is below another threshold \( \underline{K} < K^{**} \). This is done in Lemmas G.6 and G.8.

G.2 Proofs

We start this appendix with a standard verification result for the HJB equation associated with the firm’s problem:

**Lemma G.1** If \( \phi(c) \) is a continuous and piecewise twice continuously differentiable function such that

\[
\max\{L_0\phi(c) + F\phi(c); 1 - \phi'(c); V_1(c - K) - \phi(c), \ell_0(c) - \phi(c)\} \leq 0
\]

and

\[
\phi'(c_-) \geq \phi'(c_+)
\]

at each point of discontinuity of \( \phi'(c) \) then \( \phi(c) \geq \hat{V}(c) \) for all \( c \geq 0 \).

**Proof.** Fix an arbitrary strategy \( \pi \in \Pi \), denote by \( C_t \) the corresponding cash buffer process and consider the process

\[
Y_t = e^{-\rho t \wedge \tau_0} \phi(C_t \wedge \tau_0) + \int_{0+}^{t \wedge \tau_0} e^{-\rho s} (dD_s - f_s dN_s).
\]

Using the Ito-Tanaka formula (see Karatzas and Shreve (1991, Chapter 3.6)) together with arguments similar to those of the proof of Lemma B.1 it can be shown that \( Y_t \) is a local supermartingale and since

\[
Y_t \geq -\int_0^{\tau_0} e^{-\rho s} (dD_s + f_s dN_s)
\]

where the right hand side is integrable by definition of \( \Pi \) we conclude that \( Y_t \) is a supermartingale.
In particular, for any stopping time \( \tau \), we have

\[
\phi(c) = \phi(C_0) - \Delta \phi(C_0) = Y_0 - \Delta \phi(C_0) \geq E_c[Y_\tau] - \Delta \phi(C_0)
\]

\[
= E_c \left[ e^{-\rho T_{\tau_0}} \phi(C_{\tau \wedge \tau_0}) + \int_{0+}^{T_{\tau_0}} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta \phi(C_0)
\]

\[
= E_c \left[ e^{-\rho T_{\tau_0}} \phi(C_{\tau \wedge \tau_0}) + \int_{0}^{T_{\tau_0}} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta D_0 - \Delta \phi(C_0)
\]

\[
\geq E_c \left[ 1_{\{\tau_0 \leq \tau\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau_0 > \tau\}} e^{-\rho \tau} V_1(C_\tau) \right]
\]

\[
+ E_c \left[ \int_{0}^{T_{\tau_0}} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta D_0 - \Delta \phi(C_0)
\]

\[
\geq E_c \left[ 1_{\{\tau_0 \leq \tau\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau_0 > \tau\}} e^{-\rho \tau} V_1(C_\tau) + \int_{0}^{T_{\tau_0}} e^{-\rho s} (dD_s - f_s dN_s) \right]
\] (G.2)

where the first inequality follows from the optional sampling theorem, the second follows from the assumptions of the statement, and the third follows from

\[
\Delta D_0 + \Delta \phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-} - \Delta D_0}^{C_{0-}} (1 - \phi'(c)) dc \leq 0.
\]

Taking the supremum over \( \pi \in \Pi \) on both sides of (G.2) then gives

\[
\phi(c) \geq \sup_{\pi \in \Pi} E_c \left[ 1_{\{\tau_0 \leq \tau\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau_0 > \tau\}} e^{-\rho \tau} V_1(C_\tau) + \int_{0}^{T_{\tau_0}} e^{-\rho s} (dD_s - f_s dN_s) \right]
\]

and the desired result now follows from Lemma C.1. Q.E.D.

**Lemma G.2** If the condition \( u'(0; C_U^*) \geq W'(0) \) holds then the function \( u(c; C_U^*) \) satisfies the conditions of Lemma G.1 and we have \( C_U^* \leq C_1^* + K \).

**Proof.** If \( u'(K; K) \geq V_1'(0) \) then \( C_U^* = K \) and we only need to show that \( u'(c; K) \geq 1 \) for \( c \leq K \). To this end, let the function \( \bar{W}(c) \) be the unique solution to

\[
\mathcal{L}_0 \bar{W}(c) - \lambda \bar{W}(c) + \lambda (V_1(C_1^*) - C_1^* - K + c) = 0, \quad c \geq 0,
\]

which coincides with \( W(c) \) on the interval \([0, C_W^*] \). Since this function satisfies \( \bar{W}''(C_W^*) = 1 \) as well as \( \bar{W}''(C_W^*) = 0 \) it follows from Lemma B.7 that \( \bar{W}'(c) \geq \bar{W}'(C_W^*) = 1 \) for all \( c \geq 0 \). Then, the difference \( m(c) = u(c; K) - \bar{W}(c) \) satisfies

\[
\mathcal{L}_0 m(c) - \lambda m(c) = 0, \quad c \in [0, K].
\] (G.3)

Furthermore, we have \( m(0) = 0 \) as well as \( m'(0) \geq 0 \) since \( u'(0; K) \geq W'(0) \) by assumption and
it now follows from the result of Lemma B.6 that \( m'(c) \geq 0 \) or equivalently \( u'(c; K) \geq \tilde{W}'(c) \geq 1 \), which is what had to be proved.

Now assume that \( u'(K; K) < V_1'(0) \) so that \( C^*_U > K \). In order to show that \( u'(c; C^*_U) \geq 1 \) for all \( c \geq 0 \) consider the function defined by

\[
\phi(c) = u(c; C^*_U) - \tilde{W}(c)
\]

with the function \( \tilde{W}(c) \) as above. By Lemmas F.1 and E.2 we know that \( \phi(c) \) solves (G.3) and, since \( \phi(0) = 0 < \phi'(0) \) by assumption it follows from Lemma B.5 that we have

\[
\begin{align*}
\phi(c) &= u(c; C^*_U) - \tilde{W}(c), \\
\phi'(c) &= u'(c; C^*_U) \geq \tilde{W}'(c) \geq 1, \\
\phi(c) &\geq 0 \quad \text{for} \quad c \leq C^*_U.
\end{align*}
\]

Using equation (G.4) in conjunction with the definition of the liquidation value, it is immediate to show that the function \( u(c; C^*_U) \) satisfies

\[
u'(c; C^*_U) \geq \tilde{W}'(c) \geq 1, \quad c \leq C^*_U.
\]

and, since the inequality \( u(c; C^*_U) \geq V_1(c - K) \) follows from Lemma E.2, the proof that \( u(c; C^*_U) \) satisfies the conditions of Lemma G.1 will be complete once we show that

\[
\mathcal{L}_0 u(c; C^*_U) + \mathcal{F} u(c; C^*_U) \leq 0.
\]

A direct calculation using the definition of the functions \( u(c; C^*_U) \) and \( V_1(c) \) together with the fact that, as shown below, \( C^*_U \leq C^*_1 + K \) gives

\[
\mathcal{L}_0 u(c; C^*_U) + \mathcal{F} u(c; C^*_U) = \begin{cases}
0, & c \leq C^*_U, \\
(rK - \mu_1 + \mu_0)V_1'(c - K), & C^*_U \leq c \leq C^*_1 + K, \\
(r - \rho)(c - (C^*_1 + K)) + \mu_0 - \mu_1 + rK, & c \geq C^*_1 + K.
\end{cases}
\]

and the desired result now follows from the increase of the function \( V_1(c) \) and the fact that we have \( \mu_0 - \mu_1 + rK < 0 \) for all \( K \leq K^* \) by Lemma C.3.

In order show that \( C^*_U \leq C^*_1 + K \) assume that \( u'(K; K) < V_1'(0) \) for otherwise there is nothing to prove and suppose that the desired inequality does not hold. In this case we have that \( u'(C^*_U; C^*_U) = 1 \) and since \( u(c; C^*_U) > V_1(c - K) \) for \( c < C^*_U \) we get that \( u(c; C^*_U) \) is strictly convex in a small neighborhood to the left of the point \( C^*_U \). This implies that \( u'(c; C^*_U) < u'(C^*_U; C^*_U) = 1 \) in this small neighborhood, which is impossible by (G.4).

Q.E.D.
Having dealt with the case in which the firm uses exclusively the barrier strategy associated with the value function \( u(c; C^*_U) \) we now turn to the case in which it combines this strategy with the strategy associated with the function \( W(c) \). In order to state the result let

\[
L_i(c; a, b) = E_c \left[ e^{-(\rho + \lambda) \tau_{i,a} 1_{\{\tau_{i,a} \leq \tau_{i,b}\}}} \right],
\]

\[
H_i(c; a, b) = E_c \left[ e^{-(\rho + \lambda) \tau_{i,a} \wedge \tau_{i,b}} \right] - L_i(c; a, b) = E_c \left[ e^{-(\rho + \lambda) \tau_{i,a} 1_{\{\tau_{i,b} \leq \tau_{i,a}\}}} \right],
\]

where the stopping time \( \tau_{i,x} \) denotes the first time that the uncontrolled cash buffer process of a firm with mean cash flow rate \( \mu_i \) reaches the level \( x \). Closed form expressions for these two functions are provided in Appendix I.

**Lemma G.3** Assume that \( u'(0; C^*_U) < W'(0) \). Then the unique piecewise twice continuously differentiable solution to the free boundary problem (17)–(21) is given by

\[
V(c) = \begin{cases}
W(c), & c \leq C^*_L, \\
S(c), & C^*_L \leq c \leq C^*_H, \\
V_1(c - K), & c \geq C^*_H,
\end{cases}
\]

for some constants \( C^*_W \leq C^*_L \leq C^*_H \) with \( C^*_L > C^*_U \) where

\[
S(c) = \Phi(c) + (W(C^*_L) - \Phi(C^*_L))L_0(c; C^*_L, C^*_H) + (V_1(C^*_H - K) - \Phi(C^*_H))L_0(c; C^*_L, C^*_H)
\]

Furthermore, \( \max\{W(c), V_1(c - K)\} \leq V(c) \) for all \( c \geq 0 \).

**Proof.** By Lemma E.1, finding a solution to (17)–(21) is equivalent to finding a linear function that is tangent to the graph of the functions \( \hat{p}(c) \) and \( \hat{v}_1(c) \) defined by

\[
p(c) = W(c) - \phi(c; b) = F_0(c)\hat{p}(Z(c)) = F_0(c)\hat{p}\left(\frac{G_0(c)}{F_0(c)}\right),
\]

and (E.1). A direct calculation using the results of Lemma F.1 shows that

\[
\mathcal{L}_0 p(c) - \lambda p(c) = (r - \rho)(c - C^*_W)^+
\]

and it now follows from Lemma E.1 that the function \( \hat{p}(c) \) is linear for \( y \leq Z(C^*_W) \) and strictly concave otherwise. Since \( W(c) \) is concave by Lemma F.1, we get

\[
p'(c) = W'(c) - \frac{\lambda}{\lambda + \rho - r} \geq W'(C^*_W) - \frac{\lambda}{\lambda + \rho - r} = \frac{\rho - r}{\lambda + \rho - r} > 0.
\]

Since \( F_0 \) is nonnegative and decreasing, the ratio \( p(c)/F_0(c) \) is positive and strictly increasing for
sufficiently large $c$. Therefore, Lemma E.1 implies that $\hat{w}$ is increasing for sufficiently large values of $y$ and, since $\hat{p}(y)$ is concave, it is globally increasing.

Since $u(0; C_U^*) = W(0) = \ell_0$, we have that $\hat{w}(c)$ and the function

$$\hat{u}(y) = (u(Z^{-1}(y); C_U^*) - \Phi(Z^{-1}(y)))/F_0(Z^{-1}(y))$$

are both linear on $[Z(0), Z(C_W^*) \land Z(C_U^*)]$ and coincide at the point $Z(0)$. On the other hand, the inequality $u'(0; C_U^*) < W'(0)$ implies that $\hat{u}(y) \leq \hat{p}(y)$ for all $y \in [Z(0), Z(C_W^*) \land Z(C_U^*)]$. It follows that $C_W^* \leq \tilde{C} < C_U^*$ because $\hat{p}$ is a linear function that crosses the graph of the concave function $\hat{v}_1(y)$ at $Z(\tilde{C})$ and, by the definition of $\tilde{C}$, we have $\hat{p}(y) < \hat{v}_1(y)$ for all $y > Z(\tilde{C})$.

Since we know that $u(c; C_U^*) \geq V_1(c - K)$ for all $c \leq C_U^*$ by Lemma E.2, we get that the linear function defined by

$$\bar{p}(y) = \frac{\hat{p}(Z(C_W^*)) - \hat{p}(Z(0))}{Z(C_W^*) - Z(0)} y + \frac{\hat{p}(Z(0)) Z(C_W^*) - \hat{p}(Z(C_U^*)) Z(0)}{Z(C_U^*) - Z(0)}$$

is tangent to the concave function $\hat{p}(y)$ and lies strictly above the concave function $\hat{v}_1(y)$ for all $y \geq Z(\tilde{C})$. On the other hand, since

$$\hat{v}_1(Z(\tilde{C})) = \hat{p}(Z(\tilde{C})) \quad \text{and} \quad \hat{v}_1'(Z(\tilde{C})) > \hat{p}'(Z(\tilde{C}))$$

as a result of Lemma F.4, we have that the tangent line to $\hat{p}$ at the point $y = Z(\tilde{C})$ lies strictly below $\hat{v}_1$ for $y > Z(\tilde{C})$. By continuity, this implies that there exists a unique $y^*_L \in (Z(C_W^*), Z(\tilde{C}))$ such that the tangent line to $\hat{p}$ at $y^*_L$ is also tangent to $\hat{v}_1$ at some $y^*_H > y^*_L$. Setting

$$C_W^* = Z^{-1}(Z(C_W^*)) \leq C_L^* = Z^{-1}(y^*_L) < Z^{-1}(y^*_H) = C_H^*$$

produces the unique solution to value matching and smooth pasting conditions. Since $\hat{p}(c)$ is increasing and concave its tangent line at the point $y^*_L$ crosses the vertical axis above the level $\hat{p}(Z(0))$. Therefore, if the target $C_U^*$ were greater or equal to $C_H^*$ then this tangent would have to cross the vertical axis below $\hat{u}(Z(0)) = \hat{p}(Z(0))$ thus leading to a contradiction.

To complete the proof it now only remains to show that $V(c) \geq \max\{W(c), V_1(c - K)\}$ but this follows immediately from the fact that since the functions $\hat{p}(c)$ and $\hat{v}_1(c)$ are both concave we have $\hat{v}(c) \geq \max\{\hat{p}(c), \hat{v}_1(c)\}$ for all $c \geq 0$.

Q.E.D.

**Lemma G.4** If $u'(0; C_U^*) < W'(0)$, then the function $V(c)$ of Lemma G.3 satisfies the conditions of Lemma G.1 and we have $C_H^* < C_1^* + K$.

**Proof.** To show that $V'(c) \geq 1$ for all $c \geq 0$ we start by observing that this inequality holds in the region $[0, C_L^*] \cup [C_H^*, \infty)$ as a result of the definition and Lemmas B.8, F.1, E.2. On the other

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hand, since we know that \( C^*_H \geq C^*_L \geq C^*_W \), we have \( V'(C^*_L) = W'(C^*_L) = 1 \) and

\[
V(c) \geq W(c) = W(C^*_W) + c - C^*_W, \quad C^*_L \leq c \leq C^*_H.
\]

This immediately implies that \( V''(c) \geq 0 \) and since \( J(c) = V'(c) \) is a solution to

\[
\mathcal{L}_0 J(c) + \lambda - (\lambda - r) J(c) = 0,
\]

it follows from Lemma B.5 that \( J'(c) = V''(c) \) can have at most one zero in the interval \( \mathcal{I} = [C^*_L, C^*_H] \). If no such zero exists then \( V''(c) \geq 0 \) in \( \mathcal{I} \) and consequently \( V'(c) \geq V'(C^*_L) = 1 \) for all \( c \in \mathcal{I} \). If on the contrary \( V''(c) \) has one zero located at some \( c^* \in \mathcal{I} \) then we have that \( V'(c) \) reaches a global maximum over \( \mathcal{I} \) at the point \( c^* \) and since \( V'(C^*_H) = V'(C^*_H - K) \geq 1 \) due to the concavity of \( V_1(c) \) we conclude that \( V'(c) \geq 1 \) holds for all \( c \in \mathcal{I} \).

Let us now show that \( C^*_H < C^*_i + K \). If not then we have \( V'(C^*_H) = 1 \) and since \( V(c) > V_1(c - K) \) for all \( c \geq K \) we have that \( V(c) \) is convex in a neighborhood of \( C^*_H \). This in turn implies that \( V'(c) < V'(C^*_H) = 1 \) in this neighborhood, which is impossible. Using the fact that \( V'(c) \geq 1 \) in conjunction with the definition of the liquidation value, we obtain

\[
V(c) = V(0) + \int_0^c V'(x) dx = \ell_0(0) + \int_0^c \ell_0'(0) dx + c = \ell_0(c).
\]

Finally, since \( C^*_W \leq C^*_L \leq C^*_H \leq C^*_i + K \) by the first part of the proof it follows from the definition and concavity of the functions \( W(c) \) and \( V_1(c) \) that

\[
\mathcal{L}_0 V(c) + \mathcal{F} V(c) = \begin{cases} 
0, & c \leq C^*_W, \\
(r - \rho)(C - C^*_W), & C^*_W \leq c \leq C^*_L, \\
0, & C^*_L \leq c \leq C^*_H, \\
AV_1^*(c - K) + (r - \rho)(c - C^*_i - K)^+, & c \geq C^*_H,
\end{cases}
\]

where we have set \( A = \mu_0 - \mu_1 + rK \). By Lemma C.2 we know that \( A < 0 \) whenever \( K \leq K^* \). Therefore it follows from the increase of the function \( V_1(c) \) that we have

\[
\mathcal{L}_0 V(c) + \mathcal{F} V(c) \leq 0, \quad c \geq 0.
\]

and the proof is complete. Q.E.D.

**Lemma G.5** We have \( V(c) \leq \hat{V}(c) \) for all \( c \geq 0 \) where \( \hat{V}(c) \) denotes the value function of the firm’s optimization problem.

**Proof.** Let \( \tau^*_L = \tau_{0, C^*_L} \) (resp. \( \tau^*_H = \tau_{0, C^*_H} \)) denote the first time that the uncontrolled cash buffer of a firm with mean cash flow rate \( \mu_0 \) falls below \( C^*_L \) (resp. increases above \( C^*_H \)). Using arguments
similar to those of the proof of Lemma G.2 it can be shown that

\[
V(c) = E_c \left[ 1_{\{\tau_H < \tau_N \wedge \tau_L \}} e^{-\rho \tau_H} V_1(C^*_H - K) + 1_{\{\tau_L < \tau_N \wedge \tau_H \}} e^{-\rho \tau_L} W(C^*_L) 
+ 1_{\{\tau_N < \tau_L \wedge \tau_H \}} e^{-\rho \tau_N} (V_1(C^*_1) - C^*_1 - K + C_{\tau_N}) \right].
\]

On the other hand, using arguments similar to those of the proof of Proposition 1 it can be shown that the function \( W(c) \) satisfies

\[
W(c) = E_c \left[ 1_{\{\tau_0 < \tau_N \}} e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq \tau_N \}} e^{-\rho \tau_N} V_1(C^*_1) + \int_{0}^{\tau_0 \wedge \tau_N} e^{-\rho s} (dL_s - f^U_s dN_s) \right]
\]
where \( L_t = \sup_{s \leq t} (b_t - C^*_W)^+ \) with
\[
\frac{db_t}{dt} = (rb_t + \mu_i) dt + \sigma dB_t + (C^*_W - b_t^-)^+ dN_t,
\]
and the process \( f^U_t \) is defined as in equation (E.2). Combining these two equalities and using the law of iterated expectations then gives

\[
V(c) = E_c \left[ 1_{\{\tau_0 < \tau_N \}} e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq \tau_N \}} e^{-\rho \tau_N} V_1(C^*_1) + \int_{0}^{\tau_0 \wedge \tau_N} e^{-\rho s} (dD_s - f^U_s dN_s) \right]
\]
where we have set \( \tau = \tau_N \wedge \tau_H^* \) and the cumulative dividend process is defined by
\[
D_t = \int_{0}^{t} 1_{\{C_s \leq C^*_L \}} dL_s.
\]
The same arguments as in the proof of Lemma E.3 then show that strategy \((\tau, D, f^U)\) is admissible and the desired result now follows from the result of Lemma C.1. Q.E.D.

**Lemma G.6** There exists a unique \( K^{**} \in (0, K^*) \) such that \( u'(0; C_U^*) > W'(0) \) if and only if \( K < K^{**} \).

**Proof.** We will use the notation \( P(c; K) = u(c; C_U^*(K); K) \) and \( W(c; K) \) to show the dependence of these functions on the investment cost. From the proof of Lemma G.8 we know that \( P'(0; K) > W'(0; K) \) for sufficiently small values of \( K \). Similarly, we know that the function \( V_1(c - K^*) \) touches the function \( W(c; K^*) \) from below at \( C_1^* + K^* \) so that \( C_1^*(K^*) = C_H^*(K^*) = C_1^* + K^* \) and \( P'(0; K^*) < W'(0; K^*) \). Therefore, it suffices to show that there exists a unique critical investment cost such that we have

\[
P'(0; K^{**}) = W'(0; K^{**}).
\]
Assume towards a contradiction that this is not the case so that there exist $K_1 < K_2$ such that $P'(0; K_i) = W'(0; K_i)$. Let the function $\bar{W}_i(c)$ denote the unique solution to

$$\mathcal{L}_0 \bar{W}_i(c) - \lambda \bar{W}_i(c) + \lambda V_1(C^*_1) - c_1 - K_i + c = 0, \quad c \geq 0,$$

which coincides with the function $W(c; K_i)$ on the interval $[0, C^*_W(K_i)]$. From the proof of Lemmas G.2 and B.7 we know that this function is concave for $c \leq c^*_i = C^*_W(K_i)$ and convex for $c \geq c^*_i$ so that $\bar{W}'_i(c) > W'(c^*_i) = 1$ for all $c \neq c^*_i$. Since $P(0; K_i) = \bar{W}_i(0)$ by definition, the equality $U'(0; K_i) = W'(0; K_i)$ implies that the two functions coincide for $c \leq C^*_U(K_i)$. Consider the function defined by

$$m(c) = \bar{W}_i(c) - V_1(c - K_i)$$

and which satisfies (F.2). If $C^*_U(K_i) > K_i$, then $m(C^*_U(K_i)) = m'(C^*_U(K_i)) = 0$. On the other hand, it follows from the proof of Lemma G.2 that $m(c) \geq 0$ for all $c \geq K_i$. If $C^*_U(K_i) = K_i$ then $\bar{W}'_i(C^*_U(K_i)) \geq V'_1(0)$ implies that $m(C^*_U(K_i)) = 0$ as well as $m'(C^*_U(K_i)) \geq 0$ and therefore $m(c) \geq 0$ for $c \geq K_i$ by Lemma B.6. Now consider the function

$$k(c) = \bar{W}'_2(c) - \bar{W}'_1(c)$$

which is a solution to

$$\mathcal{L}_0 k(c) - (\lambda - r)k(c) = 0, \quad c \geq 0,$$

and satisfies $k(c^*_2) < 0, k(c^*_1) > 0$. Since $k(c)$ cannot have negative local minima by Lemma B.5 we have that there exists a unique point $c_x \in (c^*_2, c^*_1)$ such that $k(c_x) = 0, k'(c_x) > 0$ and $k(c) > 0$ for all $c > c_x$ and $k(c) < 0$ for $c < c_x$. That is, $\bar{W}_2(c) - \bar{W}_1(c)$ attains a global minimum at the point $c_x$ and $(\bar{W}_2 - \bar{W}_1)''(c_x) > 0$. Evaluating the differential equation

$$\mathcal{L}_0 (\bar{W}_2 - \bar{W}_1) - \lambda (\bar{W}_2 - \bar{W}_1)(c) + \lambda (K_1 - K_2) = 0$$

at the point $c = c_x$ we get

$$\bar{W}_2(c_x) - \bar{W}_1(c_x) > \frac{\lambda}{\rho + \lambda} (K_1 - K_2),$$

and therefore

$$\bar{W}_1(c) - \bar{W}_2(c) < \frac{\lambda}{\rho + \lambda} (K_2 - K_1)$$
for all \( c \geq 0 \). However, since \( \tilde{W}_1(c) \geq V_1(c - K_1) \) for \( c \geq K_1 \) and \( V'_1(c) \geq 1 \) for all \( c \geq 0 \) we finally conclude that
\[
\frac{\lambda}{\rho + \lambda} (K_2 - K_1) \geq W_1(C^*_H(K_2)) - W_2(C^*_H(K_2)) = W_1(C^*_H(K_2)) - V_1(C^*_H(K_2) - K_2) \\
\geq V_1(C^*_H(K_2) - K_1) - V_1(C^*_H - K_2) \geq K_2 - K_1,
\]
which establishes a contradiction. Q.E.D.

**Lemma G.7** We have
\[
u(c; K) = \Phi(c, K) + \frac{G_0(K)(\ell_0 - \Phi(0; K)) - G_0(0)(\ell_1 - \Phi(K; K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}F_0(c) \\
- \frac{F_0(K)(\ell_0 - \Phi(0; K)) - F_0(0)(\ell_1 - \Phi(K; K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}G_0(c)
\]
where the function \( \Phi(c) = \Phi(c; K) \) is defined as in equation (15).

**Proof.** The proof follows by direct calculation and thus is omitted. Q.E.D.

**Lemma G.8** There exists a unique \( K \in (0, K^**) \) such that, for \( K \in (0, K^**) \), we have \( u'(K, K) < V'_1(0) \) if and only if \( K > K \).

**Proof.** First of all, we claim that \( \lim_{K \downarrow 0} u'(K; K) = \infty \). Indeed, up to a first order approximation, we have that
\[
G_0(K)F_0(0) - F_0(K)G_0(0) \approx (G'_0(0)F_0(0) - F'_0(0)G_0(0))K = \alpha K
\]
with \( \alpha > 0 \). Therefore, it follows from Lemma G.7 that
\[
u'(K; K) \approx \frac{(G_0(0)(\ell_0 - \ell_1)F'_0(0)) - F'_0(0)(\ell_0 - \ell_1)G'_0(0))}{\alpha K} = \frac{\ell_1 - \ell_0}{K}
\]
and the required assertion follows from the fact that \( \ell_1 \geq \ell_0 \). Since \( u'(K; K) \) is continuous in \( K \) it remains to show that \( u'(K; K) = V'_1(0) \) can have at most one solution. Suppose to the contrary that \( K_1 < K_2 \leq K^** \) are two solutions, let \( g_i(c) = u(c; K_i) \) so that
\[
L_0g_i(c) - \lambda g_i(c) + \lambda(V_1(C^*_i) - C^*_1 - K_i + c) = 0, \quad c \geq 0.
\]
and observe that since \( K_i \leq K^** \) we have that \( g'_i(c) \geq 1 \) for all \( c \leq K_i \) By Lemma G.2. Now, a direct calculation using the above differential equation shows that the functions \( h_i(y) = g_i(y + K_i) \) satisfy
\[
\sigma^2 \frac{y^2}{2}h''_i(y) + (ry + rK_i + \mu_0)h'_i(y) - (\rho + \lambda)h_i(y) + \lambda(V_1(C^*_i) - C^*_1 + y) = 0
\]
for all \( c \geq 0 \) and it follows that the function \( m(c) = h_1(c) - h_2(c) \) solves

\[
\frac{\sigma^2}{2} m''(y) + (ry + rK_1 + \mu_0)m'(y) - (\rho + \lambda)m(y) + r(K_1 - K_2)h_2'(y) = 0
\]

subject to \( m(0) = m'(0) = 0 \). Since \( h_2'(y) > 0 \), we know from Lemma B.6 that the function \( m(c) \) is positive and monotone decreasing and it follows that \( h_1(c) > h_2(c) \) for \( c < 0 \). Since the function \( h_2(c) \) is monotone increasing this in turn implies that we have

\[
\ell_0 = h_1(-K_1) > h_2(-K_1) > h_2(-K_2) = \ell_0,
\]

which provides the required contradiction. Q.E.D.

**H Many options with issuance and search costs**

In this appendix, we consider the extension of the model to finitely many growth options outlined in section 3 with the additional feature that upon raising outside funds the firm incurs not only the bargaining cost \( \eta S_f V(c) \) but also a fixed cost \( \kappa \). In such a model, the firm will look for outside funds only when the financing surplus \( S_f V(c) \) exceeds the fixed cost and we will show below that this occurs precisely when the firm’s cash reserve are below a constant trigger level.

**Remark 10 (Search costs)** Since financing opportunities arrive at the jumps times of a Poisson point process, the presence of a fixed cost of financing \( \kappa \) is equivalent to that of a search cost \( \kappa_s = \kappa / \lambda \) that the firm incurs continuously over time when searching for investors.

In order to solve the firm’s optimization problem in the presence of multiple growth options we start by formulating two auxiliary problems, whose solutions will serve as building blocks in the construction of the value of the firm and the optimal strategy.

**Problem 1.** Let \( V_o \) be a nonnegative function, denote by \( \tau_o \) a random time distributed according to an exponential distribution with parameter \( \lambda_o > 0 \), and consider

\[
V_{n,i}(c; V_o) = \sup_{(f,D) \in \Theta} \mathbb{E}_c \left[ \int_{0}^{\tau_o \wedge \tau_0} e^{-\rho t} \left( dD_t - a(f_t) dN_t + 1_{\{\tau_0 < \tau_o\}} e^{-\rho \tau_0} \ell_i + 1_{\{\tau_o < \tau_0\}} e^{-\rho \tau_0} V_o(C_{\tau_0}) \right) dt \right],
\]

subject to

\[
dC_t = (rC_t + \mu_t) dt + \sigma dB_t - dD_t + f_t dN_t,
\]  

(H.1)

where the stopping time \( \tau_0 \) stands for the firm’s liquidation time, the set \( \Theta \) is defined as in Appendix B.1 of the main text and we have set \( a(x) = x + \kappa 1_{\{x > 0\}} \).
The goal of Problem 1 is to determine the optimal financing and payout strategy in the waiting period between growth options, given that the next option arrive at the random time $\tau_0$ at which point the value of the firm will be some given function $V_0(C_{\tau_0})$. Following the same logic as in previous appendices, we have that the HJB equation associated with Problem 1 is

$$\max\{L_i V_{n,i}(c) + F V_{n,i}(c) + \lambda_0 (V_0(c) - V_{n,i}(c)), 1 - V_{n,i}'(c), \ell_i(c) - V_{n,i}(c)\} = 0,$$

where the operator $F$ now takes the form

$$F V(c) = \lambda \max_{f \geq 0} (V(c + f) - a(f) - V(c)).$$

to take into account the presence of the fixed cost. Using the methods developed above for the case without the investment option, it is possible to show that the optimal policy for Problem 1 is of barrier type as soon as the function $V_0(c)$ is concave. However, in the many options case studied here the function $V_0(c)$ will coincide with the value function $V_{n,i}(c)$ of a firm with a growth option and we know from the analysis of the single option case in the main body of the paper that this function generally fails to be concave. This non-concavity significantly alters the optimal policy and leads to the multiple dividend distribution intervals reported by Theorem 5.

In order to deal with the optimization problem of the firm in the phase where it already holds an option, we now introduce a second auxiliary problem:

**Problem 2.** Given a nonnegative and piecewise $C^2$ function $V_n$, consider the optimal dividend, financing, and investment problem defined by

$$V_{o,i}(c; V_n) = \sup_{\pi \in \Pi} E_c \left[ \int_0^{\tau \wedge \tau_0} e^{-\rho t} (dD_t - a(f_t)dN_t) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_i + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_n(C_{\tau}) \right].$$

subject to (H.1) where $\Pi$ denotes the set of $\pi = (\tau, D, f)$ such that $(D, f) \in \Theta$ is an admissible financing and payout strategy, and $\tau$ is a stopping time.

The goal of Problem 2 is to determine the optimal financing, payout, and investment strategy of a firm holding a growth option, given that upon investing the value of the firm will be $V_n(c)$. The corresponding HJB equation takes the form

$$\max\{L_i V_{o,i}(c) + F V_{o,i}(c), 1 - V_{o,i}'(c), \ell_i(c) - V_{o,i}(c), V_n(c) - V_{o,i}(c)\} = 0.$$

Having solved Problems 1 and 2 for arbitrary functions $V_0$ and $V_n$, we will construct the value function of a firm with multiple growth options recursively. Using the result of Proposition 1, we know the value $V_N(c) = V_{n,N}(c)$ of a firm that has exhausted its growth potential. Taking this function as given, we solve Problem 2 with $V_0(c) = V_N(c - K_N)$ to determine the value $V_{o,N}(c)$ of
the firm in the period where it holds its last growth option. Then, we solve Problem 1 with the function $V_n(c) = V_{o,N}(c)$ to determine the value $V_{n,N-1}(c)$ of the firm in the period where it awaits the arrival of the last growth option, and continuing this process allow to compute the value of the firm in all phases.

**H.1 Solution to Problem 1**

We start by solving Problem 1 for a fixed function $V_o(c)$. Since the algorithm for constructing the value function of this problem is quite involved, we briefly describe the main idea. As in the case without growth options, we use a shooting-based construction that starts from a target cash level and shoots backward towards the value matching condition at zero.

- We start with a conjectured target cash level $b$ and require that the value function be $C^2$ at that point. This requirement together with the ODE (H.2) below for the conjectured value function uniquely pins down the value at that cash level. Then, we define the conjectured value function $\psi(c; b)$ as the unique solution to the corresponding ODE

- If this solution satisfies $\psi'(c; b) > 1$ for all $c \geq 0$, we are done. Otherwise, let $\xi_1(b)$ be the first level of cash from the right at which $\psi'(c; b) = 1$. This defines the upper bound of the first interior dividend distribution region and we define $\psi$ to be linear in $c$ below that cash level, until it hits the lower bound. At the lower bound, the value function has to satisfy the $C^2$-condition, which, as we show below, means that $\psi$ has to hit the graph of an explicitly given function $\phi(c; b)$.

- At that point, the algorithm restarts: we again define $\psi$ to be the solution to the ODE until the derivative $\psi'$ hits 1 again, in which case we define it to be linear until it hits the graph of $\phi$ again, etc.

Once the function $\psi(c; b)$ has been constructed for any target cash level $b$, the optimal target level $x^*_i$ is determined to match the value matching $\psi(0; x^*_i) = \ell_i$ at the origin.

By construction, the function $\psi(c; b)$ defined above always satisfies $\psi'(c; b) \geq 1$, however, in order to complete the verification argument, we need to check the supermartingale condition inside all the dividend distribution intervals. This is technically non-trivial. Throughout this process, we will always make the following technical assumption:

**Assumption 1** The function $V_o(c)$ is such that $\max_c V_o(c) > \ell_i$, $V'_o(c) \geq 1$ for all $c \geq 0$ and $V'_o(c) = 1$ for sufficiently large $c$.

This assumption will always be fulfilled in the problems under consideration because, as we show below, it is always optimal for the firm to distribute dividends when its cash buffer is sufficiently
large. Denote by $Y(c) = Y(c; b)$ the unique twice continuously differentiable solution to
\[ \mathcal{L}_i Y(c) + \lambda_o (V_o(c) - Y(c)) + \lambda (Y(b) - (b - c) - Y(c) - \kappa)^+ = Y'(b) - 1 = Y''(b) = 0 \quad (H.2) \]
for $c \leq b$ and satisfying
\[ Y(c; b) = Y(b; b) + (c - b) \]
otherwise. The fact that such a function exists follows from the results in Appendix F and we note that the smoothness of the function $Y(c; b)$ and (H.2) jointly imply that
\[ Y(b; b) = rb + \mu + \lambda_o V_o(b) \]
Having computed the function $Y(c; b)$ we let
\[ \theta(b) = \arg \max_{x \geq b} \left\{ \frac{rx + \mu + \lambda_o V_o(x)}{\rho + \lambda_o} - x = \frac{rb + \mu + \lambda_o V_o(b)}{\rho + \lambda_o} - b \right\}, \]
and define
\[ \phi(x; b) = \max \left\{ \frac{rx + \mu + \lambda_o V_o(x)}{\rho + \lambda_o}, \frac{rx + \mu + \lambda_o V_o(x) + \lambda(Y(b; b) - (\theta(b) - x) - \kappa)}{\rho + \lambda_o + \lambda} \right\} . \quad (H.3) \]
Note that we have $Y(b; b) = Y(\theta(b); \theta(b))$ by construction and that since $\phi'(x; b) < 1$ for sufficiently large $x$ we know that the function $\phi(x; b) - x$ is monotone decreasing for sufficiently large $x$. Then, we define the function
\[ \psi(c; b) = \psi(c; b; V_o(\cdot)) = Y(c; \theta(b)) \]
for $c \geq \zeta_1(b)$ where
\[ \zeta_1(b) = \arg \max_{c \in [0, \theta(b)]} \{ Y(c; \theta(b)) = \phi(c; b) \}. \]
Here, and everywhere in the sequel, we use the convention that the maximum of an empty set is zero, i.e., $\max\{\emptyset\} = 0$. We then have the following result.

**Lemma H.1** The function $\psi(c; b)$ is concave on $[\zeta_1(b), \theta(b)]$ and satisfies $\psi'(c; b) \geq 1$ in that interval.

**Proof.** Since $Y'(\theta(b); \theta(b)) = 1$, it suffices to prove concavity. Let $\tilde{Y}(c; \theta(b)) = Y(c; \theta(b)) - c.$
Differentiating (H.2) and evaluating the result at the point $c = \theta(b)$, we get

$$\frac{1}{2} \sigma^2 \tilde{Y}''(\theta(b); \theta(b)) = \rho + \lambda_o - r - \lambda_o V_o'(\theta(b)) \geq 0$$  \hspace{1cm} \text{(H.4)}$$

because $\phi'(\theta(b); b) - 1 \leq 0$ by the definition of $\theta(b)$. Since $Y''(\theta(b); \theta(b)) = 0$, this in turn implies that $\tilde{Y}(c; \theta(b))$ is concave in a small neighbourhood of $\theta(b)$. Now assume that (H.4) is strict (the general case follows by a small modification of the arguments), suppose that the function is not concave on $[\zeta_1(b), \theta(b)]$ and let

$$c_* = \text{arg max}\{c \leq \theta(b) : \tilde{Y}''(c; \theta(b)) = 0\}$$

Since the function $\tilde{Y}(c; \theta(b))$ is concave on $[c_*, \theta(b)]$, and $\tilde{Y}(c; \theta(b)) \leq \phi(c; b)$ on $[\zeta_1(b); \theta(b)]$ by definition of $\zeta_1(b)$, we get that $\tilde{Y}'(c_*; \theta(b)) \geq 0$ and therefore

$$0 = \frac{1}{2} \sigma^2 \tilde{Y}''(c_*; \theta(b)) = (\rho + \lambda)(\tilde{Y}(c_*; \theta(b)) - \phi(c_*; b) + c_*) - (rc_* + \mu)\tilde{Y}'(c_*; \theta(b)) < 0,$$

which is a contradiction. Q.E.D.

In order to proceed further in the construction, let $\kappa_1(b)$ denote the first point below $\zeta_1(b)$ where the functions $Y(c; \theta(b))$ and $\phi(c; b)$ coincide, that is

$$\kappa_1(b) = \text{arg max}_{c \in [0, \zeta_1(b))} \{Y(c; \theta(b)) = \phi(c; b)\},$$

and consider the following algorithm: If

$$\delta_1(b) = \min_{c \in [\kappa_1(b), \zeta_1(b)]} Y'(c; \theta(b)) > 1$$

then we continue the function $\psi(c; b)$ further for lower values of $c$ as the solution to the above ODE. If on the contrary $\delta_1(b) \leq 1$ then we let

$$\xi_1(b) = \text{arg max}_{c \in [\kappa_1(b), \zeta_1(b)]} \{c : Y'(c; \theta(b)) = 1\}$$

$$\theta_1(b) = \text{arg max}_{c \in [0, \xi_1(b)]} \{c : \phi(c; b) = c - \xi_1(b) + Y(\xi_1(b); \theta(b))\}$$

and continue the function for lower values of $c$ by letting

$$\psi(c; b) = c - \xi_1(b) + Y(\xi_1(b); \theta(b))$$

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for $c \in [\theta_1(b), \zeta_1(b)]$ and $\psi(c; b) = H(c; \theta_1(b))$ for $c \in [\zeta_2(b), \theta_1(b)]$ where $H(c) = H(c; \theta_1(b))$ is the unique twice continuously differentiable solution to

$$
0 = \mathcal{L}_i H(c) + \lambda_o (V_o(c) - H(c)) + \lambda(Y(b; b) - (\theta(b) - c) - H(c) - \kappa)^+ \tag{H.5}
$$

and we have set

$$
\zeta_2(b) = \arg \max_{c \in [0, \theta_1(b)]} \{H(c; \theta_1(b)) = \phi(c; b)\}.
$$

Continuing this process, we arrive at a function $\psi(c; b)$ defined on $[0, \theta(b)]$ that is linear on the finite union of intervals given by $\bigcup_{j=1}^k [\theta_j(b), \zeta_j(b)]$ for some $k \in \mathbb{N}$ and satisfies the differential equation (H.5) on the complement of these intervals.

**Lemma H.2** The function $\psi(c; b)$ satisfies both $\psi'(c; b) \geq 1$ and the supermartingale property

$$
\mathcal{L}_i \psi(c; b) + \lambda (Y(\theta(b); \theta(b)) - (\theta(b) - c) - \psi(c; b) - \kappa)^+ + \lambda_o (V_o(c) - \psi(c; b)) \leq 0 \tag{H.6}
$$

for all $c \geq 0$.

**Proof.** First, we need to show that $\psi'(c; b) \geq 1$. We will only consider the first step in the above construction of the function $\psi(c; b)$ (i.e., up to the boundary $\theta_1(b)$). Since the construction of $\psi(c; b)$ follows the same steps for $c \in [0, \theta_1(b)]$, the general claim follows by induction.

By Lemma H.1, the function $\psi(c; b)$ is concave for $c \geq \zeta_1(b)$. Thus, if $Y'(c; \theta(b))$ hits 1 before $\psi(c; b)$ hits the graph of $\phi(c; b)$, we have that the desired inequality $\psi'(c; b) \geq 1$ holds for all $c$ above the first interior candidate dividend distribution region. Suppose now that $Y(c; \theta(b))$ hits the graph of $\phi(c; b)$ again before $Y'(c; \theta(b))$ hits 1 and let

$$
\tilde{Y}(c; \theta(b)) = Y(c; \theta(b)) - c
$$

as before. If $\tilde{Y}''(\theta_1(b); \theta(b)) \leq 0$ then the same argument as in the proof of Lemma H.1 implies that $\psi(c; b)$ stays concave as long as $\psi(c; b) \leq \phi(c; b)$. Assume now that $\tilde{Y}''(\theta_1(b); \theta(b)) > 0$ so that the function $\tilde{Y}'(c; \theta(b))$ is increasing in a small neighbourhood of $\theta_1(b)$ and suppose that the required assertion is not true. In this case, $\tilde{Y}'(c; \theta(b))$ is increasing in a right neighbourhood of

$$
c_* = \arg \max \{c \leq \theta_1(b) : \tilde{Y}''(c; \theta(b)) = 0\}.
$$
In conjunction with the differential equation this implies that

\[ \frac{1}{2} \sigma^2 \dddot{Y}(c_s; \theta(b)) = (\rho + \lambda) (\dddot{Y}(c_s; \theta(b)) - \phi(c_s; b) + c_s) < 0 \]

which is a contradiction. Continuing by induction, we get that \(\psi'(c; b) \geq 1\) for all \(c \geq 0\) and it only remains to prove the supermartingale property.

In the regions in which \(\psi'(c; b) \neq 1\) we have that \(\psi(c; b)\) solves (H.6) by construction. Therefore, the supermartingale property only has to be shown in the regions where the function is linear. This follows by direct calculation because in those regions \(\psi(c; b) \geq \phi(c; b)\). Q.E.D.

**Theorem H.3** There exists a unique constant \(x_i^*\) such that \(\psi(0; x_i^*) = \ell_i\) and \(\psi(c; x_i^*) = V_{n,i}(c; V_o)\) gives the value function of Problem 1. Furthermore, the region in which the firm optimally searches for outside investors is given by \(\{c \leq C\}\) for some \(C \geq 0\).

**Proof.** Let \(b^* = \arg \max_{c \geq 0} \{\phi(c; b) - c\}\). For this choice we clearly have that \(\psi(c; b^*) = \phi(b^*; b^*)\) for all \(c \geq 0\) and it follows from Assumption 1 that \(\psi(0; b^*) > \ell_i\). On the other hand, we have

\[ \lim_{b \to \infty} \{\psi(0; b) - b\} \leq \lim_{b \to \infty} \{\phi(b; b) - b\} = -\infty \]

because \(V_o'(b) = 1\) for sufficiently large \(b\) by Assumption 1 and the existence claim follows by continuity. Uniqueness follows from the verification argument. Indeed, by the same verification argument as in the case with no investment options, \(V_{n,i}(c; V_o)\) is the value function of the firm. Therefore, the target cash level is unique. To complete the proof it remains to establish that the search region

\[ \{c \geq 0 : V_{n,i}(x_i^*; V_o) - x_i^* + c - \kappa \geq V_{n,i}(c; V_o)\} \]

is an interval but this immediately follows from the fact that \(V''_{n,i}(c; V_o) = \psi'(c; x_i^*) \geq 1\). Q.E.D.

An immediate consequence of our algorithm is

**Lemma 11** The number of dividend distribution intervals does not exceed the number of local minima of \(\phi(c; x_i^*) - c\) plus one.

**Proof of Theorem 5.** The proof of Theorem 5 follows directly from Theorem H.3, Proposition 11, the characterisation of the firm value \(V_{o,i}(c)\) provided below and Lemma H.13. Q.E.D.

**Proof of Proposition 6.** The proof is based on the observation that \(\psi(c; b)\) and \(\psi'(c; b)\) are, respectively, decreasing and increasing in \(b\). The proof of this claim is analogous to that of Lemma B.4 and is omitted. It immediately follows that \(x_i^*\) is decreasing in \(\varphi_i\). Similarly, the claim about

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the dividend distribution region follows because \( \psi'(c; b) \) is monotone increasing in \( b \) and therefore the region \( \{ c \geq 0 : \psi'(c; x_i^*) = 1 \} \) is expanding as the target level \( x_i^* \) decreases. The proof of monotonicity in \( \lambda_i \) follows by similar arguments, based on Lemma B.6. Q.E.D.

**Lemma H.4** We have

\[
V_{n,i}(x_i^*; V_o) \geq \frac{\mu_{i-1} + r(x_i^* + K_i)}{\rho}
\]

**Proof.** By construction we have the value function \( V_{n,i}(c; V_o) \) is twice continuously differentiable and satisfies \( 0 = 1 - V_{n,i}'(x_i^*; V_o) = V_{n,i}''(x_i^*; V_o) \). Combining this with (H.5) shows that

\[
V_{n,i}(x_i^*; V_o) = \frac{rx_i^* + \mu_i + \lambda_o V_o(x_i^*)}{\rho}.
\]

and the desired claim follows since, by assumption, \( \mu_i - \mu_{i-1} \geq rK_i \) and \( V_o(c) \geq 0 \). Q.E.D.

The following observation allows to determine the critical value \( \kappa_{\text{max}} \) of the fixed cost above which the firm optimally decides to never raise outside funds and concludes our discussion of the solution to Problem 1.

**Proposition H.5**

\[
\kappa_{\text{max}} = \left( V_{n,i}(x_i^*; V_o) - V_{n,i}(0; V_o) - x_i^* \right) \bigg|_{\kappa=0}.
\]

**H.2 Solution to Problem 2**

Having constructed the solution to Problem 1, we now present a general algorithm for solving Problem 2. Proceeding as in the previous cases, we start by picking a candidate option exercise threshold \( b \) that we will later vary to obtain value matching at the origin. In order to construct the associated value, we start by defining an auxiliary function \( Y_{o,i}(c; b) \) that is set to coincide with \( V_{n,i}(c - K_i) \) for \( c \geq b \) and is constructed as follows on the interval \([0, b] \):

1. If \( b - K_i \geq x_i^* \) then we let

\[
\zeta_0 = \max \left\{ c < x_i^* + K_i : Y_{o,i}(c; b) > \frac{rc + \mu_i}{\rho} \right\}
\]

and define the auxiliary function by setting

\[
Y_{o,i}(c; b) = V_{n,i}(c - K_i) = V_{n,i}(x_i^*) + (c - K_i - x_i^*)
\]

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for $c \in [\zeta_0, b]$ and $Y_{o,i}(c; b) = H(c)$ for $c \in [0, \zeta_0]$ where the function $H(c)$ is the unique twice continuously differentiable solution to

$$L_{i-1}H(c) + \lambda(V_{n,i}(x_i^*) - (c - K_i - x_i^*) - H(c) - \kappa)^+ = 0$$  \hspace{1cm} (H.7)

subject to the value matching and smooth-pasting conditions

$$0 = H(\zeta_0) - V_{n,i}(\zeta_0 - K_i) = 1 - H'(\zeta_0)$$

at the point $\zeta_0$ (recall the convention that the supremum of an empty set is zero).

2. If $b - K_i < x_i^*$ then we let $Y_{o,i}(c; b) = H(c)$ for all $c \in [0, b]$ where the function $H(c)$ is the unique twice continuously differentiable solution to the differential equation (H.7) subject to the value matching and smooth-pasting conditions

$$0 = H(b) - V_{n,i}(b - K_i) = H'(b) - V'_{n,i}(b - K_i)$$

at the point $b$.

Given the auxiliary function $Y_{o,i}(c; b)$ we let

$$\zeta_1(b) = \max\{c \in [0, b] : Y'_{o,i}(c; b) = 1\}$$

denote the first point at which the derivative reaches one (and zero if such a point does not exist), and define the function

$$\phi(x) = \max\left\{\frac{rx + \mu}{\rho} ; \frac{rx + \mu + \lambda(V_{n,i}(x_i^*) - (x_i^* + K_i - c) - \kappa)}{\rho + \lambda}\right\}$$

in complete analogy with (H.3). The same arguments as the study of Problem 1 imply that we necessarily have $Y_{o,i}(\zeta_1(b); b) > \phi(\zeta_1(b))$. Consequently,

$$\theta_1(b) = \max\{c \in (0, \zeta_1(b)) : \phi(c) = Y_{o,i}(\zeta_1(b)) + c - \zeta_1(b)\}$$

is well-defined. Finally, we define the value function $w_{o,i}(c; b)$ associated with the given candidate investment trigger by setting

$$w_{o,i}(c; b) = 1_{\{c \geq \theta_1(b)\}} [Y_{o,i}(c \vee \zeta_1(b); b) + (c - \zeta_1(b))^+] + 1_{\{c \leq \theta_1(b)\}} H(c)$$
where the function $H(c)$ is the unique twice continuously differentiable solution to (H.7) subject to the value matching and smooth-pasting conditions

$$0 = H(\theta_1(b)) - Y_{o,i}(\theta_1(b)) = H'(\theta_1(b)) - 1.$$ 

Note that due to the definition of $\phi(x)$ we have that $w_{o,i}(c; b)$ is twice continuously differentiable at the point $\theta_1(b)$ where it satisfies the high contact condition $w''_{o,i}(\theta_1(b); b) = 0$.

**Proposition H.6** The function $w_{o,i}(c; b)$ satisfies the HJB equation

$$\max\{\mathcal{L}_{i-1} w_{o,i}(c; b) + \lambda \mathcal{F} w_{o,i}(c; b), V_{n,i}(c - K_i) - w_{o,i}(c; b), 1 - w'_{o,i}(c; b)\} = 0.$$ 

**Proof.** By construction we have that $w'_{o,i}(c; b) \geq 1$ for $c \geq \theta_1(b)$ and the same arguments as in the study of Problem 1 imply that $w_{o,i}(c; b)$ is concave on the interval $[0, \theta_1(b)]$ and therefore satisfies $w'_{o,i}(c; b) \geq 1$ on this interval. This immediately implies that there exists a threshold $C_{i,o}(b) \geq 0$ such that we have

$$\mathcal{F} w_{o,i}(c; b) = (V_{n,i}(x^*_i) - (x^*_i + K_i - c) - w_{o,i}(c; b) - \kappa)^+$$

$$= 1_{\{c \leq C_{i,o}(b)\}}(V_{n,i}(x^*_i) - (x^*_i + K_i - c) - w_{o,i}(c; b) - \kappa).$$

On the other hand, the inequality

$$\mathcal{L}_{i-1} w_{o,i}(c; b) - \rho w_{o,i}(c; b) + \mathcal{F} w_{o,i}(c; b) \leq 0$$

holds as an identity for $c \in [0, \theta_1(b)] \cup [\zeta_1(b), b]$ and as an inequality in $[\theta_1(b), \zeta_1(b)]$ because $w_{o,i}(c; b) \geq \phi(c)$ in this interval. To prove the inequality for $c \geq b$, we will need to show that the search thresholds satisfy $C_{o,i}(b) \leq C_{n,i} + K_i$ and to this end it clearly suffices to show that $w'_{o,i}(c; b) \leq V'_{n,i}(c - K_i)$ for all $c \geq K_i$. Since the two other possible cases are completely analogous we only consider the case where the trigger satisfies $b \in (C_{n,i} + K_i, x^*_i + K_i)$. Let

$$k(c) = w_{o,i}(c; b) - V_{n,i}(c - K_i)$$

and suppose for the moment that we have both $V'_{n,i}(c) > 1$ and $w'_{o,i}(c; b) > 1$ for all $c \geq 0$. Then, for $c \geq \max\{C_{o,i}(b), C_{n,i} + K_i\}$, we have

$$\mathcal{L}_{i-1} k(c) + (rK_i - (\mu_i - \mu_{i-1})) V'_{n,i}(c - K_i) = 0$$

(H.8)

and it follows from Lemma B.6 that the function $k(c)$ is decreasing. This immediately implies that
we have $C_{o,i}(b) \leq C_{n,i} + K_i$ and it follows that the function $k(c)$ satisfies

$$0 = L_{i-1}k(c) + (rK_i - \mu_i + \mu_{i-1})V'_{n,i}(c - K_i) - (V_{n,i}(x_i^*) - (x_i^* + K_i - c) - V_{n,i}(c) - \kappa)$$

on the interval $[C_{o,i}(b), C_{n,i} + K_i]$ and (H.8) on the interval $[0, C_{o,i}(b)]$. If there are intervals where either of the derivatives is equal to one then we only need to show that

$$V'_{o,i}(c - K_i) = 1 \implies w'_{o,i}(c; b) = 1.$$

Let $(\theta_1(x_i^*), \xi_1(x_i^*)), \ldots, (\theta_k(x_i^*), \xi_k(x_i^*))$ with $\theta_1 > \ldots > \theta_k$ give the dividend distribution intervals for the function $V_{n,i}(c)$. We only consider the first interval. The general case follows by induction. By definition, we have that $V'_{n,i}(\zeta_1(b)) = 1$. Therefore, the same argument as above implies that we have $w'_{o,i}(\zeta_1(b) + K_i; b) \leq 1$ and therefore $w'_{o,i}(\zeta_1(b) + K_i; b) = 1$ since $w'_{o,i}(c; b) \geq 1$ by construction. Now, consider the function

$$\phi_{n,i}(c) = \frac{rc + \mu_i + \lambda_o(V_{n,i}(c) - V_{n,i}(c)) + \lambda(V_{n,i}(x_i^*) - x_i^* + c - \kappa)^+}{\rho + \lambda}.$$

By the construction of the interval $[\theta_1(b), \zeta_1(b)]$ we have

$$V_{n,i}(c - K_i) \geq \phi_{n,i}(c - K_i), \quad c \in [\theta_1(b) + K_i, \zeta_1(b) + K_i]$$

Furthermore, the same argument as above based on Lemma B.6 shows that $w_{o,i}(c; b) \geq V_{n,i}(c - K_i)$ for all $c \in [\zeta_1(b), \theta(b)]$. Let

$$\phi_{o,i}(c) = \frac{rc + \mu_i - 1 + \lambda(V_{n,i}(x_i^*) - x_i^* + c - \kappa)^+}{\rho + \lambda}.$$

Using the fact that $V_{o,i}(c) - V_{n,i}(c) > 0$ for all $c > 0$ and $\mu_i - \mu_{i-1} > rK_i$ by assumption we deduce that the inequality

$$\phi_{n,i}(c - K_i) > \phi_{o,i}(c) \quad \text{(H.9)}$$

holds if either $c > \max\{C_{n,i} + K_i, C_{o,i}\}$ or $C_{n,i} + K_i > C_{o,i}$. Since, as shown above, the latter inequality necessarily holds whenever $\zeta_1(b) + K_i \leq \max\{C_{n,i} + K_i, C_{o,i}\}$ we have that (H.9) holds in a neighbourhood of $\zeta_1(b)$. By continuity this in turn implies that

$$w_{o,i}(c; b) > V_{n,i}(c - K_i) \geq \phi_{n,i}(c - K_i) > \phi_{o,i}(c) \quad \text{(H.10)}$$

in a neighbourhood of $\zeta_1(b)$ and it follows that $w_{o,i}(c; b)$ is linear with slope equal to one in that neighbourhood. By definition, $w_{o,i}(c; b)$ remains linear with slope one until it hits the graph of $\phi_{o,i}(c)$
at some level $\xi_1(b)$ of the cash buffer and it only remains to show that $\xi_1(b) < \theta_1(b)$. Suppose to the contrary. By definition, we cannot have $c_{o,i} \in [\xi_1(b), \xi_1(b)]$. Therefore $\phi_{n,i}(c - K_i) > \phi_{o,i}(c)$ holds also for $c > \xi_1(b)$ and hence (H.10) holds true over the interval $[\xi_1(b), \xi_1(b)]$. But this is impossible because $w_{o,i}(\xi_1(b); b) = \phi_{o,i}(\xi_1(b))$ by definition of $\xi_1(b)$. Thus, we conclude that $w'_{o,i}(c; b) \leq V_{n,i}'(c - K_i)$ and therefore $w_{o,i}(c; b) \geq V_{n,i}(c - K_i)$ for all $c \geq K_i$. Q.E.D.

**Lemma H.7** The function $w_{o,i}(c; b)$ is monotone increasing in $b$

**Proof.** Let $b_1 > b_2$. Since, by the above $w'_{o,i}(c; b) \leq V'_{n,i}(c - K_i)$, we have

$$w'_{o,i}(b_2; b_1) \leq V'_{n,i}(b_2 - K_i) = w'_{o,i}(b_2; b_2).$$

Thus defining $k(c) = w_{o,i}(c; b_1) - w_{o,i}(c; b_2)$, we get that $k(b_2) > 0, k'(b_2) \leq 0$ and the required assertion now follows from Lemma B.6. Q.E.D.

In order to state the next results consider the twice continuously differentiable function defined by $W_i(c; b) = W_i(c \wedge b; b) + (c - b)^+$ where the function $W_i(c; b)$ is the unique twice continuously differentiable solution to

$$0 = \mathcal{L}_{i-1} \tilde{W}_i(c; b) + \lambda(V_{n,i}(x_i) - x_i - K_i + c - W_i(c; b) - \kappa)^+$$

subject to the boundary conditions $1 - \tilde{W}_i(b; b) = \tilde{W}_i''(b; b) = 0$.

**Lemma H.8** The function $W_i(c; b)$ is monotone increasing with respect to $b \geq 0$ there exists a unique threshold $C_{i,W}^*$ such that $W_i(0; C_{i,W}^*) = \ell_i$. Furthermore, for any $b \geq 0$ there exists a threshold $C_{i,W}(b)$ such that

$$\{c \leq b : V_{n,i}(x_i) - x_i - K_i + c - W_i(c; b) - \kappa > 0\} = [0, C_{i,W}(b)]$$

and the function $\tilde{W}_i(c; C_{i,W}^*)$ satisfies $\tilde{W}_i''(c; C_{i,W}^*) \geq 1$ for all $c \geq 0$.

**Proof.** The proof is similar to that of Lemma F.1 and therefore is omitted. Q.E.D.

To simplify the notation in what follows we let $\tilde{W}_i(c) = \tilde{W}_i(c; C_{i,W}^*)$ and $W_i(c) = W_i(c; C_{i,W}^*)$ unless there a risk a confusion.

**Lemma H.9** If $K_i < C_{i,W}^*$ then

$$W_i(C_{i,W}^*) < V_{n,i}(x_i) - x_i - K_i + C_{i,W}^*$$

and either $W_i(c) < V_{n,i}(c - K_i)$ for all $c \geq K_i$ or there exists a unique crossing point $\tilde{C}_i$ with the property that $W_i(c) < V_{n,i}(c - K_i)$ if and only if $c > \tilde{C}_i$. 33
Proof. The proof is analogous to that of Lemmas F.3 and F.4. Q.E.D.

Proof of Proposition 7. The proof follows directly from Lemma H.9. Q.E.D.

The following lemma follows by the same arguments as Lemma G.8. For simplicity, we assume that $\kappa < \kappa_{\text{max}}$ with $\kappa_{\text{max}}$ defined as in Proposition H.5.

Lemma H.10 Let

$$\tilde{\ell}_i = V_{n,i}(x_i^*) - x_i^* - K_i - \kappa.$$ 

and for any $z > 0$ define the function $g_i(c; z)$ to be the unique twice continuously differentiable solution to

$$L_i g_i(c; z) + \lambda 1_{c \leq z} (\tilde{\ell}_i + c - g_i(c; z)) = 0$$

with the boundary conditions

$$0 = \ell_i - g_i(0; z) = z + \tilde{\ell}_i - g_i(z; z)$$

Then we have $\lim_{z \to 0} g_i'(z; z) = \infty$.

Lemma H.11 Suppose that $K < K_i^*$ and $\min_{c \geq K_i}(\tilde{W}_i(c) - V_{n,i}(c - K_i)) \leq 0$. Then, there exists a threshold $z_\ast > 0$ such that

(a) $g_i(c; z_\ast) \geq V_{n,i}(c - K_i)$ for all $c \geq K_i$,

(b) There exists a unique point $C_{U,i}^* \geq K_i$ such that $g_i(C_{U,i}^*; z_\ast) = V_{n,i}(C_{U,i}^* - K_i)$.

If $C_{U,i}^* = K_i$ then the optimal policy is to exercise at $c = K_i$ and liquidate. Otherwise, the optimal policy is to exercise the growth option at $C_{U,i}^*$.

Proof. Let us subtract from both $g_i(c; z)$ and $V_{n,i}(c)$ the function

$$\Phi_i(c) = \frac{\lambda}{\rho + \lambda} \left( V_{n,i}(x_i^*) + c - x_i^* - K_i - \kappa + \frac{\mu_i - 1 + rc}{\rho + \lambda - r} \right)$$

and denote the new functions by $\tilde{g}_i(c; z)$ and $\tilde{V}_{n,i}(c)$. Then, let us apply to these functions the transformation of Lemma E.1 with meeting intensity $\lambda = 0$, i.e. consider the equation $L_i f = 0$, and apply the transformation using the solutions $F_i,0(c)$ and $G_i,0(c)$ to this equation. Denote by $\tilde{g}_i(c; z)$ and $\tilde{V}_{n,i}(c)$ the resulting functions. Lemma E.1 immediately yields that $\tilde{g}_i(c; z)$ is concave on the interval $[0, F_i(z)/G_i(z)]$ and linear afterwards, whereas $\tilde{V}_{n,i}(c)$ is globally concave. Furthermore,
it follows from Lemma H.10 that on \([F_i(z)/G_i(z), +\infty]\) the slope of the function \(\hat{g}_i(c; z)\) converges to infinity as \(z\) decreases to zero and this implies that \(g_i(c; z) > V_{n,i}(c - K_i)\) for any sufficiently small values of \(z\) and all \(c \geq K_i\).

The next important observation is that \(\bar{W}_i(c) = g_i(c; C_{i,W})\) for all \(c \geq 0\). Indeed, both functions satisfy the same ODE with the same boundary conditions at the origin and the point \(C_{i,W}\) so the claim follows by the uniqueness of the solution to a second order equation. Let us now show that \(g_i(c; z) > \bar{W}_i(c)\) for \(z < C_{i,W}\) and all \(c > 0\). Indeed, by Lemma B.6 we have that the function \(k(c) = g_i(c; z) - \bar{W}_i(c)\)
is either monotone increasing or monotone decreasing and the claim for \(c \in [0, z]\) follows because \(g_i(z; z) > \bar{W}_i(z)\) for sufficiently small values of \(z\) by Lemma H.10. The claim for \(c \geq z\) follows directly from the result of Lemma B.6.

Since \(\min_{c \geq K_i}(\bar{W}_i(c) - V_{n,i}(c - K_i)) \leq 0\) by assumption, the existence of a threshold \(z_\ast\) satisfying the conditions of the statement follows by continuity. It is also clear that if \(C_{U,i}^* > K_i\), then \(g_i(c; z_\ast)\) satisfies the smooth pasting condition at the point \(C_{U,i}^*\) and therefore

\[
g_i(c; z_\ast) = w_{a,i}(c; C_{U,i}^*), \quad c \leq C_{U,i}^*.
\]

In particular, this implies that the function \(g_i(c; z_\ast)\) touches the graph of the function \(V_{n,i}(c)\) at a single point and the proof is complete.

**Q.E.D.**

**Lemma H.12** Suppose that \(K < K_i^*\) and \(\min_{c \geq K_i}(\bar{W}_i(c) - V_{n,i}(c - K_i)) > 0\). Then, there are thresholds \(C_{i,H}^* > C_{i,L}^* > C_{i,W}^*\) such that

\[
w_{a,i}(c; C_{i,H}^*) = W_i(c)
\]

for all \(c \leq C_{i,L}^*\).

**Proof.** Denote by \(\delta_i(c; b)\) the unique twice continuously differentiable solution to the equation

\[
\mathcal{L}_i \delta_i(c; b) + \lambda(V_{n,i}(x_i^*) - x_i^* - K_i + c - \kappa - \delta_i(c; b))^+ = 0
\]

with the boundary conditions

\[
\delta_i(b; b) - W_i(b) = \delta_i'(b; b) - W_i'(b).
\]

The same argument as in the proof of Lemma G.4 implies that \(\delta_i'(c; b)\) can have at most a single turning point. If \(\delta_i'(c; b)\) is non-increasing, then there will be at most a single \(C_{\delta,i}\) at which \(\tilde{\ell}_i - \delta_i(c; b)\)
changes sign. Otherwise, there will be at most two such points. By construction we have

$$\delta_i(c; C^*_i,W) = \bar{W}_i(c) > V_{n,i}(c - K_i)$$

for all \(c \geq K_i\) and it follows from Lemma H.9

$$\delta_i(\tilde{C}_i; \tilde{C}_i) = V_{n,i}(\tilde{C}_i - K_i).$$

Therefore, by continuity, there exists a threshold \(b = C^*_{i,L}\) such that the function \(\delta_i(c; C^*_{i,L})\) touches the graph of the function \(V_i(c - K_i)\) from above at some point \(C^*_{i,H} > C^*_{i,L}\) and the same argument as in the proof of Lemma G.4 implies that \(\delta'_i(c; C^*_{i,L}) \geq 1\). Q.E.D.

**Proof of Theorem 8.** By Proposition H.6, we only need to show that there exists a solution to the equation \(w_{o,i}(0; b) = \ell_{i-1}\) but this follows directly from Lemmas H.11 and H.12. Verification follows by the same arguments as above. Q.E.D.

**Lemma H.13** The function \(V_{n,i}(c)\) has at most \(N - n\) intervals of convexity.

**Proof.** We prove the claim by induction. The case \(i = N\) follows from Lemma B.4. Suppose now that the claim is proved for \(V_{n,i}(c)\), and let us prove it for \(V_{n,i-1}(c)\). First, we note that by construction the function \(V_{o,i}(c)\) can have at most one interval of convexity in the no-investment region and combining this with the induction hypothesis implies that the function \(V_{o,i}(c)\) has altogether at most \(N - i + 1\) intervals of convexity. Furthermore, it follows from Lemma B.7 that the second derivative of \(V_{o,i}(c)\) inside a cash retention interval can change sign at most once. It follows immediately from the construction of the function \(V_{n,i-1}(c)\) that it cannot have more intervals of convexity that the function \(V_{o,i}(c)\) and this completes the induction step. Q.E.D.

**Proof of Proposition 9.** We only prove monotonicity in \(\lambda_{i-1}\). The other claims are established similarly. Let \(\lambda_1 < \lambda_2\) and define

$$k(c; b) = w_{o,i}(0; b; \lambda_2) - w_{o,i}(0; b; \lambda_1).$$

Then we have \(k(b) = k'(b) = 0\) as well as \(k''(b) < 0\) and it follows from Lemma B.6 that \(k(c)\) cannot have negative local minima. Thus, \(k(c) < 0\) and it follows that \(w_{o,i}(0; b; \lambda)\) is monotone decreasing in \(\lambda\). Monotonicity in \(\varphi_i\) is a direct consequence of Proposition H.14 below. Q.E.D.

**Proposition H.14** Suppose that an increase in a parameter \(\alpha\) increases \(V_{n,i}(c)\) and simultaneously decreases \(V'_{n,i}(c)\). Then, the threshold for investment from internal funds is decreasing in \(\alpha\).
Proof of Proposition H.14. For simplicity, we only consider the case without issuance costs. Let $\alpha_1 > \alpha_2$. By continuity, it suffices to consider the cases where both parameter values $\alpha_1$, $\alpha_2$ correspond to either the $C_{U,i}^*$ or the $C_{H,i}^*$ regimes. For simplicity, we omit the index $i$ for the various threshold and simply denote them by $C_{U}^*$, $C_{H,i}^*$, $C_{L,i}^*$ and $C_{W}^*$ to denote them.

Consider first the case of a barrier policy and suppose that the desired monotonicity does not hold so that there exist $\alpha_1 > \alpha_2$ with $C_{U,i}^*(\alpha_1) = C_{U}^*(\alpha_2) = C_{U}^*$. Let $A_j = V_{n,i}(C_{U,i}^* - K_i(\alpha_1); \alpha_j)$ and consider the function defined by

$$R_j(c) = V_{o,i}(c; \alpha_j) - A_j.$$  \hspace{1cm} (H.11)

By assumption, we have that

$$0 = R_1(C_{U}^*) = R_2(C_{U}^*) = R'_1(C_{U}^*) \leq R'_2(C_{U}^*)$$  \hspace{1cm} (H.12)

and $A_1 > A_2$. Furthermore the function $R_j(c)$ satisfies

$$\mathcal{L}_{i-1}R_j(c) - \rho A_j + \lambda(V_{n,i}(x_i^*(\alpha_2); \alpha_j) - x_i^*(\alpha_j) - K_i(\alpha_j) - A_j + c - R_j(c)) = 0.$$  \hspace{1cm} (H.13)

on the interval $[0, C_{U}^*]$ it follows that the function $k = R_1 - R_2$ satisfies

$$0 = \mathcal{L}_{i-1}k(c) + \lambda k(c) + \rho(A_2 - A_1) + \lambda Z = k(C_{U}^*) \geq k'(C_{U}^*)$$

where the constant $Z$ is defined by

$$Z = V_{n,i}(x_i^*(\alpha_1); \alpha_1) - x_i^*(\alpha_1) - K_i(\alpha_1) - A_1 - (V_{n,i}(x_i^*(\alpha_2); \alpha_2) - x_i^*(\alpha_2) - K_i(\alpha_2) - A_2)$$

and the notation $K_i(\alpha_j)$ indicates the possible dependence of the investment cost on $\alpha$. We claim that $Z \leq 0$. Indeed, since $1 \leq V_{n,i}'(c - K_i(\alpha_1); \alpha_1) \leq V_{n,i}'(c - K_i(\alpha_1); \alpha_2)$ by assumption we get

$$y_i(\alpha_1) = x_i^*(\alpha_1) + K_i(\alpha_1) \leq x_i^*(\alpha_2) + K_i(\alpha_1) = y_i(\alpha_1, \alpha_2)$$

and it follows that

$$Z = \int_{C_{U}^*}^{y_i(\alpha_1)} (V_{n,i}'(c - K_i(\alpha_1); \alpha_1) - 1) \, dc - \int_{C_{U}^*}^{y_i(\alpha_1, \alpha_2)} (V_{n,i}'(c - K_i(\alpha_1); \alpha_2) - 1) \, dc \leq 0.$$  \hspace{1cm} (H.14)

By Lemma B.6 this in turn implies that the function $k(c)$ is monotone decreasing and it follows that we have $0 < k(0) = A_2 - A_1 < 0$ which is a contradiction.
Suppose now that both parameters correspond to a band strategy and \( C^*_L(\alpha_1) = C^*_L(\alpha_2) = C^*_H \) for some \( \alpha_1 > \alpha_2 \). On the interval, \([\max\{C^*_L(\alpha_1), C^*_L(\alpha_2)\}, C^*_H]\), the functions \( R_j(c), j = 1, 2 \) defined in (H.11) satisfy (H.12)-(H.13) and therefore the same argument as above implies that the function \( k = R_1 - R_2 \) is monotone decreasing on \([\max\{C^*_L(\alpha_1), C^*_L(\alpha_2)\}, C^*_H]\). Consequently, \( R'_1 \leq R'_2 \) and hence \( R'_1 \) hits the value of 1 earlier (from the right) than \( R'_2 \). That is, \( C^*_L(\alpha_1) > C^*_L(\alpha_2) \) and hence \( k(c) \) is decreasing on \([C^*_L(\alpha_1), C^*_H]\).

It follows that the function \( k(c) \) is also decreasing on \( I \). Since \( k(C^*_L) = 0 \), we have that \( R_1(c) \geq R_2(c) \) on \([C^*_W(\alpha_1), C^*_H]\). We will now show that \( C^*_W(\alpha_1) \leq C^*_W(\alpha_2) \). Indeed, the algorithm for the construction of the value function \( w_{\alpha_1}(c; b) \) implies that the threshold \( C^*_W \) is the first point below \( C^*_L(\alpha_j) \) where \( R_j(c) \) hits the graph of the function defined by

\[
\phi_j(c) = \frac{(r + \lambda)c + \mu_{i-1} + Z_j}{\rho + \lambda}
\]

with the constant

\[
Z_j = -\rho A_j + \lambda(V_1(C^*_1(\alpha_j); \alpha_j) - C^*_1(\alpha_j) - K_j(\alpha_j) - A_j).
\]

By (H.14) and the inequality \( A_1 > A_2 \), we have \( Z_2 > Z_1 \). Therefore, since \( R_2(c) \leq R_1(c) \) on the interval \([C^*_L(\alpha_1), C^*_H]\) we have that the function \( R_2(c) \) hits the graph of the function \( \phi_2(c) \) at a cash level that is higher than the cash level at which the function \( R_1(c) \) hits the graph of the function \( \phi_1(c) \) and it follows that we have both

\[
R_2(C^*_W(\alpha_1)) < R_1(C^*_W(\alpha_1)) \quad \text{and} \quad R'_2(C^*_W(\alpha_1)) > R'_1(C^*_W(\alpha_1)).
\]

The same argument as in the first part of the proof now implies that \( k(0) > 0 \) which provides the required contradiction because \( k(0) = A_2 - A_1 < 0 \) by assumption.

Q.E.D.

I  Probabilities of investment

The probabilities of investment from internal and external funds can be computed as

\[
P_I(c) = f(c; (0, C^*_U)), \quad P_E(c) = 1 - f(c; (0, C^*_U)) - g(c; (0, C^*_U))
\]
for $K < K^{**}$, and

$$
P_I(c) = 1_{c > C_L^*} f(c; (C_L^*, C_H^*)) ,
$$
$$
P_E(c) = 1_{c \leq C_L^*} (1 - h(c \land C_W^*; (0, C_W^*)))
+ 1_{c > C_L^*} (1 - f(c; (C_L^*, C_H^*)) - g(c; (C_L^*, C_H^*)) h(C_W^*; (0, C_W^*)) )
$$

otherwise. In these equations, the bounded functions $f$, $g$ and $h$ are defined by

$$
f(c; (A, B)) = E_c \left[ e^{-\lambda \tau_{0b}} 1_{\{\tau_{0b} \leq \tau_{0a}\}} \right] ,
$$
$$
g(c; (A, B)) = E_c \left[ e^{-\lambda \tau_{0A}} 1_{\{\tau_{0A} \leq \tau_{0b}\}} \right] ,
$$
$$
h(c; (A, B)) = E_c \left[ e^{-\lambda \hat{\tau}_{0A}(B)} \right] ,
$$

for some $A \leq B$ where $\tau_{0b}$ denotes the first time that the uncontrolled cash buffer process with mean cash flow rate $\mu_0$ reaches the nonnegative level $b$ and $\hat{\tau}_{A}(B)$ denotes the first time that the cash buffer process with mean cash flow rate $\mu_0$ reaches the level $A \leq B$ given that it is reflected from above at the level $B$.

The following proposition relies on standard methods to provide closed form expressions for the functions $f$, $g$ and $h$ and thereby allows to compute the probabilities of investment.

**Proposition I.1** The functions $f$, $g$ and $h$ solve

$$(rc + \mu_0) \phi'(c) + \frac{1}{2} \sigma^2 \phi''(c) - \lambda \phi(c; b) = 0 , \quad c \in (A, B) ,$$

subject to the boundary conditions

$$
f(A; (A, B)) = g(B; (A, B)) = 0 ,
$$
$$
g(A; (A, B)) = f(B; (A, B)) = 1 ,
$$
$$
h(A; (A, B)) = 1 - h'(B; (A, B)) = 1 ,
$$

and are explicitly given by

$$
f(c; (A, B)) = \frac{G_0(A) F_0(c)}{G_0(A) F_0(B) - F_0(A) G_0(B)} + \frac{F_0(A) G_0(c)}{F_0(A) G_0(B) - G_0(A) F_0(B)} ,
$$
$$
g(c; (A, B)) = \frac{G_0(B) F_0(c)}{F_0(A) G_0(B) - G_0(A) F_0(B)} + \frac{G_0(A) F_0(B)}{G_0(A) F_0(B) - F_0(A) G_0(B)} ,
$$
$$
h(c; (A; B)) = \frac{G_0'(B) F_0(c)}{F_0(A) G_0'(B) - G_0(A) F_0'(B)} + \frac{G_0(A) F_0'(B)}{G_0(A) F_0'(B) - F_0(A) G_0'(B)} ,
$$

where the functions $F_0$ and $G_0$ are defined in equations (B.5) and (B.6).