A general model of dynamic asset allocation with incomplete information and learning

Carsten Sørensen and Anders B. Trolle
Copenhagen Business School

Abstract
This paper develops a general and flexible multivariate discrete-time model of dynamic asset allocation with incomplete information and learning in the case of time-varying investment opportunity sets. The state variables are described by a vector autoregression and the investor is assumed to have normally distributed and possibly correlated priors on the values of the state variables. We apply the model to an investor who learns about the mean returns on the market and Fama-French SMB and HML portfolios when the size and value premia disappear (possibly stochastically) over time due to trading by other investors. The portfolio implications are shown to be substantial.

JEL Classification: G11

Keywords: Portfolio choice, learning, VAR, predictability, hedging demands

This version: February 2006
First version: December 2004

We thank Michael Brennan, Raman Uppal and seminar participants at Copenhagen Business School, University of Copenhagen and the 3rd Nordic Econometric Meeting in Helsinki for comments. This paper is a heavily revised version of a previous working paper entitled “Dynamic asset allocation with unobservable state variables”. Carsten Sørensen: Department of Finance, Copenhagen Business School, Solbjerg Plads 3, DK-2000 Frederiksberg, Denmark. Phone +45 3815 3605 E-mail: cs.fi@cbs.dk. Anders B. Trolle, temporary address: Department of Finance, Anderson School of Management, 110 Westwood Plaza B504, Los Angeles, CA 90095-1481. Phone: (310) 825 2247. E-mail: anders.trolle@anderson.ucla.edu. Carsten Sørensen gratefully acknowledges financial support from the Danish Social Science Council.
1 Introduction

When investment opportunities are stochastic the solution to a multi-period dynamic portfolio choice problem can differ substantially from the solution to a static or single-period portfolio choice problem, as demonstrated originally by Samuelson (1969) and Merton (1969, 1971, 1973).

Since the mid-1990s a large number of papers have analyzed the case where the state variables determining interest rates (nominal or real) and risk premia follow either a continuous-time or discrete-time vector autoregression (VAR).\(^1\) This setting is particularly tractable when combined with a CRRA utility function since the indirect utility function is exponentially quadratic (in some cases exponentially affine) in the state variables where the coefficient solve a particular set of ODEs (if the model is set in continuous time) or difference equations (if the model is set in discrete time).\(^2\) A fundamental assumption in all of the papers mentioned is complete information about the state variables and the parameters in the return generating model.

The main contribution of this paper is to extend the general setup to the case of incomplete information and learning about the state variables. We assume that the investor has a normally distributed prior on the value of the state vector which is subsequently updated through observations of realized returns and other relevant variables. We work in discrete time and show that the indirect utility function remains exponentially quadratic in the state variables with the coefficient solving a set of difference equations similar, yet different, from the complete information case. Hence, our model retains the tractability offered by the complete information setting.

Our framework is easy to implement and very general as it allows for a flexible specification of the return generating model, any number of assets and state variables as well as correlation between the priors on the state variables.

It contains many papers in the literature on dynamic portfolio choice with incomplete information as special cases. A particularly well-researched case which fits into our framework is

---


\(^2\) This tractability is preserved with the more general Epstein-Zin and Duffie-Epstein utility function provided certain linearizations are performed.
that where investment opportunities are constant but the investor has incomplete information about mean returns. Instead the investor is assumed to have a prior on the mean returns which is updated upon observing realized returns. This case was first analyzed in continuous time by Brennan (1998) who solve the PDE associated with the Hamilton-Jacobi-Bellman (HJB, henceforth) equation numerically. He shows that the prospect of learning about the mean return can have a substantial effect on the optimal portfolio choice. Subsequent work was done by Brennan and Xia (2001) as well as Rogers (2001) and Cvitanic, Lazrak, Martellini and Zapatero (2005) who find closed form solutions. Barberis (2000) analyzes the problem in discrete time but also resorts to solving the HJB equation numerically.

All of these papers assume that the true, but unobservable, mean returns are constant. However, in our framework the mean returns (or, more generally, the state variables driving interest rates, inflation, risk premia and other relevant quantities) can follow a general VAR process and innovations to the mean returns (or state variables) may be correlated with innovations to realized returns. Rodrigues (2002) is able to obtain a closed form solution in a continuous-time setting with only one state variable provided that learning has reached its “steady state” where more information does not reduce the variance of the state variable estimate. An important feature of our framework is that it also works outside of such a “steady state”.

As an application of our framework we revisit the analysis in Brennan and Xia (2001) who analyze the importance of asset pricing anomalies from an investor viewpoint. In particular they consider an investor who allocates wealth between a money market account yielding a constant interest rate and the market portfolio and the two Fama and French (1993) SMB and HML portfolios taking into account that he will learn about the mean returns on these portfolios over time. However, rather than assuming that the true (but unknown) mean returns on these portfolios are constant, we solve the portfolio choice problem when the size and value premia are expected to disappear (either deterministically or stochastically) over time due to trading by other investors. This turns out to have important effects on the investor’s learning.

\(^3\)Naturally there are also papers in the literature on dynamic portfolio choice with incomplete information which fall outside our framework. Examples are Xia (2001) who considers the problem of learning about the degree of mean return predictability and Brandt, Goyal, Santa-Clara and Stroud (2005) who consider learning about all parameters in a VAR return generating model using simulations.
process and portfolio choice.

Our framework is basically an extension and modification of Campbell, Chan and Viceira (2003). In the special case with complete information about the state variables, they investigate the optimal asset allocation and consumption policy of an infinitely-lived investor with Epstein-Zin recursive preferences, and rely on an approximate solution methodology in order to solve for the optimal policies. Campbell et al. (2003) must assume that the return generating VAR model is time-homogenous in order to solve the infinite-horizon investment and consumption problem. Throughout this paper, we consider investors with CRRA utility of terminal wealth which, still, allows us to address the effects of time horizon and risk aversion on optimal asset allocation. However, due to the more simple preference assumption we can relax the time-homogeneity assumption of Campbell et al. (2003) and avoid the log-linear approximation of the consumption to wealth ratio applied in their solution algorithm. With complete information about the state variables, we obtain an explicit solution to the relevant dynamic portfolio problem and a simple recursive solution algorithm for implementing the solution. The recursive solution algorithm solves a particular system of difference equations.

With incomplete information about the state variables, the same solution procedure applies. However, in this case the relevant state variables are now the perceived values of the possibly latent state variables. The perceived values of the state variables can be recursively found by Kalman filtering, which is applicable to any model that has a state space representation. As we demonstrate, the dynamics of the Kalman filtered state variables are also described by a VAR, and obtaining this VAR, which is usually time-inhomogenous, provides a first step in the solution approach. The second step is simply to apply the recursive algorithm from the case with complete information to the derived VAR for the filtered state variables.4

The system of difference equations in our recursive solution algorithm is analogous to the multi-dimensional Ricatti equation that arises in a related continuous-time context when solving the relevant Hamilton-Jacobian-Bellman equation; see e.g. Liu (2001). And, since it is in general possible to translate a multivariate continuous-time VAR model to a discrete-time VAR model our results are in many respects compatible with results in continuous time.5 By letting

---

4This two step solution approach is also relevant for the model and solution algorithm in Campbell et al. (2003), but only applicable in special steady state cases where the filtered state variables follow a time-homogenous VAR.

5Campbell, Chacko, Rodriguez and Viceira (2004) provide an example. Note, however, that it is in general
the rebalancing period shrink to zero in our discrete-time setting, the related continuous-time results will in principle be obtained. However, since data are observed discretely, the discrete-time formulation is directly suitable for econometric purposes.

In a continuous-time framework, Williams (1977), Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986) have previously demonstrated that the dynamic portfolio problem of an investor who cannot directly observe the state variables, separates into a filtering problem, in which the investor estimates the state variable position, and an investment problem, where the filtered estimates are treated as the relevant state variables. Furthermore, in a setting with a single risky asset, Detemple (1986) and Gennotte (1986) show that the uncertainty about the instantaneous excess return on the risky asset does not affect the optimal portfolio choice, which simply takes the form of Merton (1969) by substituting the instantaneous expected excess return with its perceived value. While our discrete-time solution also allows the filtering problem and the investment problem to be handled separately and consecutively, the uncertainty about the exact positions of the unobserved state variables affects the optimal portfolio choice in our discrete-time setting. For example, even in a one-period model with a single risky asset, the relevant variance that must be used along with the perceived value of the expected return in order to determine the optimal portfolio (in the Markowitz (1952), Merton (1969), and/or Samuelson (1969) formulas) is affected by the uncertainty about the true expected excess return. In such a one-period setting, this is equivalent to the Bayesian approach of incorporating estimation uncertainty about mean returns into the portfolio choice problem, as originally carried out by e.g. Klein and Bawa (1976) and Bawa, Brown and Klein (1979).

The paper is organized as follows: We set up the basic multivariate model with complete information about state variables in Section 2. In Section 3 the model and solution are extended to the case with incomplete information and learning about state variables. In order to analyze and pinpoint how incomplete information and learning affect the optimal portfolio allocation, two illustrative examples are presented in Section 4. In Section 5 we extend the analysis.

Consider for example the discrete-time univariate AR(1) model: \( x_{t+1} = \alpha x_t + \epsilon_{t+1} \). The discrete-time solution to the continuous-time counterpart, the Ornstein-Uhlenbeck process, has the restriction \( \alpha > 0 \). This parameter restriction can be a limitation since the process with \( \alpha \leq 0 \) may be reasonable in some contexts since it includes the “white noise” process (\( \alpha = 0 \)) and likely stationary processes (\(-1 < \alpha < 0\)).
in Brennan and Xia (2001), while Section 6 concludes. Two appendices contain details and proofs.

2 The basic multivariate model of portfolio choice

In this section we consider a special case of the general state space model. The discrete-time return dynamics are described by a VAR model in line with Campbell et al. (2003). In many respects, the portfolio solution shares features with the approximate solution obtained by Campbell et al. (2003) for an infinitely long-lived investor with Epstein-Zin recursive preferences. In order to facilitate comparison with the results in Campbell et al. (2003), the notation and model of asset return dynamics in this section are therefore basically adopted from their setting, and then subsequently extended in the following section.

2.1 Preferences

As in Kim and Omberg (1996), Brennan, Schwartz and Lagnado (1997), Sørensen (1999), and Barberis (2000), among others, the investor is endowed with initial wealth, \( W_0 \), which is to be invested to maximize expected CRRA utility of the form

\[
E_0[U(W_T)] \quad \text{where} \quad U(W) = \frac{W^{1-\gamma} - 1}{1-\gamma}
\]

and where \( \gamma > 0 \) is the parameter of constant relative risk aversion, and \( T \) is the investment horizon. For \( \gamma = 1 \), we have the logarithmic utility function, \( U(W) = \log W \), as a limiting special case. The investor is only concerned with maximizing the utility of terminal wealth and is assumed not to use wealth for intermediate consumption nor to accumulate additional wealth due to labor income. Basically, the asset allocation problem may be thought of as the problem faced by an individual who has received a lump sum that must be invested for the purpose of retirement at time \( T \).

2.2 Dynamics of investment returns

The investor can invest in \( n \) assets, and the dynamics of the relevant asset returns and state variables are described by a first-order vector autoregressive process, VAR(1).\(^6\) Following

\(^6\)As noted by Campbell et al. (2003), the assumption of a VAR(1) model is without loss of generality since any vector autoregression can be rewritten as a VAR(1) through expansion of the vector of state variables.
Campbell et al. (2003), let $R_{i,t+1}$ denote the real gross return on asset $i$, and let $r_{i,t+1} = \log(R_{i,t+1})$ denote the similar real log return on asset $i$ ($i = 1, \ldots, n$). The relevant state vector in the analysis is then given in the following stacked form:

$$z_{t+1} = \begin{pmatrix} r_{1,t+1} \\ x_{t+1} \\ s_{t+1} \end{pmatrix}$$

where $x_{t+1} = ((r_{2,t+1} - r_{1,t+1}), (r_{3,t+1} - r_{1,t+1}), \ldots, (r_{n,t+1} - r_{1,t+1}))'$ is an $(n - 1) \times 1$ vector of log excess returns, and $s_{t+1}$ is a vector of other relevant state variables (including e.g., the dividend-price ratio and lagged log excess returns). The state vector, $z_{t+1}$, has dimension $m \times 1$. The excess returns are measured relative to asset 1, which may be thought of as a short risk-free asset. However, the realized return, or ex post return, on asset 1 is more generally allowed to be stochastic. This is especially relevant when the investor does not have access to a risk-free asset in real terms, and $r_{1,t+1}$ instead refers to the realized real return on, say, a short nominal treasury bill, as in our subsequent empirical application of the portfolio choice model.

The VAR(1) model of state variable dynamics is given by

$$z_{t+1} = \Phi_0 + \Phi_1 z_t + v_{t+1}$$

where $\Phi_0$ is an $m$-dimensional vector and $\Phi_1$ is an $m \times m$ matrix. The innovations $v_{t+1}$ are assumed to be serially uncorrelated and identically normally distributed:

$$v_{t+1} \sim \mathcal{N}(0, \Sigma),$$

where

$$\Sigma = \text{Var}_t(v_{t+1}) = \begin{pmatrix} \sigma_1 & \sigma'_{1x} & \sigma'_{1s} \\ \sigma_{1x} & \Sigma_{xx} & \Sigma'_{xs} \\ \sigma_{1s} & \Sigma_{xs} & \Sigma_{ss} \end{pmatrix}.$$  

2.3 Portfolio returns

The solution is based on the following characterization of the log return on the investment portfolio:

$$r_{p,t+1} = r_{1,t+1} + \alpha'_{t}x_{t+1} + \frac{1}{2}\alpha'_{t} \left( \sigma^2_{x} - \Sigma_{xx}\alpha_{t} \right),$$

where

$$\Sigma = \text{Var}_t(v_{t+1}) = \begin{pmatrix} \sigma_1 & \sigma'_{1x} & \sigma'_{1s} \\ \sigma_{1x} & \Sigma_{xx} & \Sigma'_{xs} \\ \sigma_{1s} & \Sigma_{xs} & \Sigma_{ss} \end{pmatrix}.$$
where $\alpha_t$ is an $(n-1)$-dimensional vector of portfolio weights that the investor holds in asset 2 to asset $n$ in the period between $t$ and $t+1$, and where $\sigma^2_x = \text{diag}(\Sigma_{xx})$ is the vector of diagonal elements of the excess return variance-covariance matrix, $\Sigma_{xx}$. The expression for the portfolio log return in (5) can be shown to be exact for small time intervals (i.e. in continuous time), or whenever one interprets $\alpha$ as the constant portfolio weights maintained in the interval between $t$ and $t+1$ by continuously adjusting the portfolio in the interval. However, if the investor is assumed to follow a passive buy-and-hold strategy between $t$ and $t+1$, then (5) is only an approximation of the log portfolio return over the interval. The approximation is also used and discussed in Campbell and Viceira (1999,2001,2002) and Campbell et al. (2003).

2.4 Solving the model

Let $V_t$ denote the value function of the utility maximization problem faced by the investor. Then the value function, $V_t$, and the optimal portfolio policy, $\alpha_t$, must at any discrete time point $t$ satisfy the Bellman equation,

$$V_t = \max_{\alpha_t} \mathbb{E}_t [V_{t+1}] , \ t = 0, \ldots, T - 1,$$

with $V_T = U(W_T)$. In order to solve the model for the optimal portfolio policy and the form of the value function, we make the following conjecture about the functional forms these take:

$$\alpha_t = A_0(t) + A_1(t)z_t$$

where $A_0(t)$ is an $(n-1)$-dimensional vector and $A_1(t)$ is an $(n-1) \times m$ matrix. Furthermore,

$$V_t = \frac{e^{B_0(t) + (1-\gamma)B_1(t)z_t + (1-\gamma)z_t' B_2(t)z_t}}{1-\gamma} W_t^{1-\gamma} - 1$$

where $B_0(t)$ is a scalar, $B_1(t)$ is an $m$-dimensional vector, and $B_2(t)$ is a symmetric $m \times m$ matrix. In particular, at the terminal date it must be the case that $B_0(T) = 0$, $B_1(T) = 0$, and $B_2(T) = 0$ in order to ensure that $V_T = U(W_T)$.

To verify the above conjecture, one can substitute the expression for $V_{t+1}$ into the Bellman equation (6), and make use of the fact that wealth evolves according to $W_{t+1} = W_t e^{r_p,t+1}$, where the log portfolio return, $r_p,t+1$ is given in (5). Then, by evaluating the expectations involved in

---

7Details of the described verification procedure are given in Appendix A.
the Bellman equation and maximizing the resulting expression with respect to $\alpha_t$, one obtains optimal portfolio weights that take the form in (7), where

$$A_0(t) = \Omega(t + 1)^{-1} \left[ H_x \Gamma(t + 1)' \Phi_0 + \frac{1}{2} \sigma_x^2 + (1 - \gamma) H_x \Sigma \Gamma(t + 1) (B_1(t + 1) + H_1') \right],$$

$$A_1(t) = \Omega(t + 1)^{-1} H_x \Gamma(t + 1)' \Phi_1,$$

and where $H_1$ and $H_x$ are selection matrices that select the first element and a vector of the next $n-1$ elements, respectively, from an $m$-dimensional vector. (In particular, $r_{t+1} = H_1 z_{t+1}$ and $x_{t+1} = H_x z_{t+1}$.) Furthermore, in the expressions for $A_0(t)$ and $A_1(t)$ we have made use of the auxiliary matrices,

$$\Omega(t + 1) = \Sigma_{xx} + (\gamma - 1) H_x \Sigma \Gamma(t + 1) H_x',$$

$$\Gamma(t + 1) = (I_m + 2(\gamma - 1) B_2(t + 1) \Sigma)^{-1},$$

where $I_m$ denotes the $m \times m$ dimensional identity matrix.

Now, substituting the optimal portfolio described by (7), (9), and (10) back into the right hand side of the Bellman equation (6) and evaluating the expectations, one obtains $V_t$ in the form conjectured in (8), where

$$B_0(t) = B_0(t + 1) + \frac{1}{2} \ln |\Gamma(t + 1)| + \frac{1}{2} (1 - \gamma) A_0(t)' (\sigma_x^2 - \Sigma_{xx} A_0(t))$$

$$+ \frac{1}{2} (1 - \gamma)^2 \left( B_1(t + 1)' + H_1 + A_0(t)' H_x \right) \Sigma \Gamma(t + 1) \left( B_1(t + 1) + H_1' + H_x' A_0(t) \right)$$

$$+ (1 - \gamma) \Phi_1' \Gamma(t + 1) (B_1(t + 1) + H_1' + H_x' A_0(t) + B_2(t + 1) \Phi_1),$$

$$B_1(t) = A_1(t)' \left( \frac{1}{2} \sigma_x^2 + H_x \Gamma(t + 1)' \Phi_0 + (1 - \gamma) H_x \Sigma \Gamma(t + 1) (B_1(t + 1) + H_1') \right)$$

$$+ \Phi_1' \Gamma(t + 1) (B_1(t + 1) + H_1' + 2 B_2(t + 1) \Phi_1),$$

$$B_2(t) = \frac{1}{2} A_1(t)' \Omega(t + 1) A_1(t) + \Phi_1' \Gamma(t + 1) B_2(t + 1) \Phi_1.$$

This ends the verification proof in the sense that we have proven that the portfolio policy in (7) and the value function in (8) constitute a solution to the Bellman equation (6) when the time-dependent matrices $A_0(t)$, $A_1(t)$, $B_0(t)$, $B_1(t)$, and $B_2(t)$ satisfy the difference equation system (9)–(15). Moreover, the relevant time-dependent matrices involved in the solution are
easily obtained recursively by starting with: (i) setting $B_0(T) = 0$, $B_1(T) = 0$, and $B_2(T) = 0$, (ii) then inserting them into (9) and (10) and (11) and (12) to obtain the optimal portfolio policy at time $T-1$, (iii) then inserting the expressions for $A_0(T-1)$ and $A_1(T-1)$ into (13), (14), and (15) in order to obtain the value function at time $T-1$. The procedure can now be repeated recursively to determine the solution at time $T-2, T-3$, and so on until time $t = 0$.

Note that in order to ensure that the auxiliary matrices $\Omega$ and $\Gamma$ defined in (11) and (12) are indeed invertible, as implicitly assumed in constructing the recursive difference equation system (9)–(15), it suffices to ensure that $B_2(t)$ is positive semi-definite (for all $t = 0, \ldots, T$). Furthermore, since the variance-covariance matrix $\Sigma$ is always positive definite, and $B_2(T) = 0$ is positive semi-definite, it can be inferred by recursive inspection of (15), (11), and (12) that $\gamma \geq 1$ is a sufficient (but not necessary) condition to ensure this.\footnote{In this recursive verification procedure, one can also show (by straightforward application of variants of the matrix result in Lemma 1 in Appendix A) that $\Omega(t)$ is a symmetric positive definite matrix while $\Gamma(t)B_2(t)$ and $\Sigma \Gamma(t)$ are symmetric positive semi-definite matrices for $\gamma \geq 1$.}

For any return parameters, the solution therefore applies to a set of investors that includes the log-utility investor and more risk averse CRRA utility investors. This set of investors is of particular interest and relevance since it is well known that log-utility investors do not hedge changes in the investment opportunity set, while investors that are more risk averse will choose portfolio strategies that reflect a desire to hedge undesirable changes in investment opportunities, as originally described by Merton (1971, 1973) in a continuous-time setting.

3 The general multivariate model

The general version of the return generating model can be written in state space form as

$$z_{t+1} = \Phi_0(t) + \Phi_1(t)z_t + v_{t+1}$$

(16)

and

$$y_t = C_0(t) + C_1(t)z_t + w_t$$

(17)

where $v_t$ and $w_t$ are uncorrelated and normally distributed with variance-covariance matrices $\Sigma(t)$ and $S(t)$. Equation (16) is the transition equation which describes the dynamics of the state variables, as in (3) in the previous section. The state variables, $z_t$, are observed indirectly.
through observations of the process \( y_t \), which is related to \( z_t \) as described by the measurement equation (17). As a special case, the measurement equation can be an identity equation so that the general model collapses to the setup investigated above. However, the state variables are generally not observed directly, and the measurement formulation even allows for “noise” (i.e. \( w_t \)) in the relationship between state variables and observations.

In the following we will suppress the time dependence of the matrices \( \Phi_0(t) \), \( \Phi_1(t) \), \( C_0(t) \), \( C_1(t) \), \( \Sigma(t) \) and \( S(t) \) in (16) and (17) for the sake of notational ease. Also, we will assume that the realized returns, \( r_{1,t} \) and \( x_t \), are observed without error at any time point \( t \), which imposes a special structure on the first rows of the matrices \( C_0 \), \( C_1 \), and \( S \) involved in the observation equation (17). This assumption implies that the realized portfolio return and the wealth of the investor are known when making portfolio decisions at time \( t = 0, 1, \ldots, T - 1 \).

The special structure of the matrices in the measurement equation is of no importance for the form and procedure of the solution, and the model still allows for unobserved state variables in the state vector, \( s_t \) (recall that \( z_t = (r_{1,t}, x_t, s_t)\)’).

Let \( \hat{z}_t = E_t[z_t] \) and \( \hat{z}_{t+1|t} = E_t[z_{t+1}] \). Furthermore, let \( P_t = E_t[(z_t - \hat{z}_t)(z_t - \hat{z}_t)^\prime] \) and \( P_{t+1|t} = E_t[(z_{t+1} - \hat{z}_{t+1|t})(z_{t+1} - \hat{z}_{t+1|t})^\prime] \) denote the corresponding error covariance matrices. The Kalman filter then provides the following updating equations:

\[
\hat{z}_{t+1|t} = \Phi_0 + \Phi_1 \hat{z}_t \tag{18}
\]

\[
\hat{z}_{t+1} = \hat{z}_{t+1|t} + P_{t+1|t}C_1 (C_1 P_{t+1|t}C_1' + S)^{-1} (y_{t+1} - C_0 - C_1 \hat{z}_{t+1|t}) \tag{19}
\]

\[
P_{t+1|t} = \Phi_1 P_t \Phi_1' + \Sigma \tag{20}
\]

\[
P_{t+1} = P_{t+1|t} - P_{t+1|t}C_1 (C_1 P_{t+1|t}C_1' + S)^{-1} C_1 P_{t+1|t}. \tag{21}
\]

The Kalman filter describes how to update the estimate (or perceived value) of the state vector, \( \hat{z}_t \), as new information arrives in the form of observations \( y_t \), and it provides equations for obtaining first- and second order moments of \( \hat{z}_{t+1} \) (as well as \( y_{t+1} \)) conditional on information available at time \( t \). These moments only depend on the information available at time \( t \) through \( \hat{z}_t \) and under normality assumptions, \( \hat{z}_t \) is thus a sufficient statistic with respect to the future

---

\(^9\)Note, furthermore, that general ARMA processes can be represented in state space form; see, e.g., Hamilton (Chapter 13, 1994). This is also possible in our setup.
distribution of, e.g., $z_{t+i}$, $y_{t+i}$ as well as $\hat{z}_{t+i}$ ($i = 1, 2, \ldots$) in the economy. Therefore, the relevant vector of state variables in the following portfolio analysis is the estimate $\hat{z}_t$ of the latent state vector $z_t$.

The dynamics of $\hat{z}_t$ is described by a VAR(1) model of the form

$$\hat{z}_{t+1} = \Phi_0 + \Phi_1 \hat{z}_t + \hat{v}_{t+1}$$

(22)

where the innovations $\hat{v}_{t+1}$ are serially uncorrelated and normally distributed:

$$\hat{v}_{t+1} \sim \mathcal{N}(0, \hat{\Sigma}(t)),$$

and where

$$\hat{\Sigma}(t) = \text{Var}_t(\hat{v}_{t+1}) = P_{t+1|t} C_1' \left( C_1 P_{t+1|t} C_1' + S \right)^{-1} C_1 P_{t+1|t}.$$

(23)

The state variable dynamics in (22) and (23) are analogous to the state variable dynamics in (3) and (38) in the previous VAR(1) model setting, which may be seen as the special case where state variables are directly observed and $\hat{z}_t = z_t$. The VAR(1) in (22) and (23), however, is usually time-inhomogeneous since the variance-covariance matrix $\hat{\Sigma}(t)$ generally is a function of time $t$. However, it may also be noted that the variance-covariance matrix $\hat{\Sigma}(t)$ is updated independently of the state variable realizations by equation (23) and through its relation to $P_{t+1|t}$ and the Kalman filter equations (20) and (21). Hence, since the relevant variance-covariance matrices are predetermined, or deterministic, functions of time, it is possible at time 0 to obtain the relevant last period variance-covariance matrix $\hat{\Sigma}(T - 1)$, by simply recursively updating $\hat{\Sigma}(t)$ using (20) and (21), and starting with the assumed initial variance-covariance estimation error matrix $P_0$ of the unobservable latent state variables. This feature is important when solving investment problems by backward dynamic programming using the Bellman equation.

The following proposition summarizes the above reasoning and procedure, and it provides the optimal portfolio choice in the general state space model setting.

---

10This is seen by inserting the expression for $\hat{z}_{t+1|t}$ in (18) into (19), and using that the prediction error $(y_{t+1} - C_0 - C_1 \hat{z}_{t+1|t})$ is normally distributed with covariance matrix $(C_1 P_{t+1|t} C_1' + S)$. This is often used to establish the likelihood function for parameter estimation of models in state space form (see, e.g., textbook descriptions in Harvey (1989) or Hamilton (1994)).
Proposition 1 Consider the dynamic optimization problem of an investor with constant relative risk aversion and preferences given by (1) who faces investment asset dynamics given by the state space system (16) and (17). Let \( \hat{z}_t \) be the Kalman filtered value of the state variable vector at time \( t \). The optimal portfolio policy and the value function of the problem are then given by

\[
\alpha_t = \hat{A}_0(t) + \hat{A}_1(t)\hat{z}_t \tag{24}
\]

and

\[
V_t = \left( e^{B_0(t)+(1-\gamma)B_1(t)\hat{z}_t+(1-\gamma)\hat{z}_tB_2(t)\hat{z}_t} \right) W_t^{1-\gamma} - 1 \tag{25}
\]

where the matrices \( \hat{A}_0(t), \hat{A}_1(t), \hat{B}_0(t), \hat{B}_1(t), \) and \( \hat{B}_2(t) \) are solutions to the difference equation system (9)-(15) with \( \Sigma \) substituted by \( \hat{\Sigma}(t) \), as defined in (23).

Proof: See Appendix A.

3.1 General optimal dynamic portfolio choice

By inserting the definitions of \( \hat{A}_0(t) \) and \( \hat{A}_1(t) \) (i.e. the difference equation system (9)-(15) with \( \Sigma \) substituted by \( \hat{\Sigma}(t) \)) into (24), the optimal portfolio policy can alternatively be expressed as

\[
\alpha_t = \Omega(t+1)^{-1} \left[ H_x \Gamma(t+1)'(\Phi_0 + \Phi_1\hat{z}_t) + \frac{1}{2} \hat{\sigma}_x^2(t) + (1 - \gamma)H_x\hat{\Sigma}(t)\Gamma(t+1)(\hat{B}_1(t+1) + H_1') \right] \tag{26}
\]

with \( \hat{\sigma}_x^2(t) = \text{diag}(\hat{\Sigma}_{xx}(t)) \), and where \( \hat{\Sigma}_{xx}(t) = H_x\hat{\Sigma}(t)H_x' \) is the excess return variance-covariance matrix given all information available at time \( t \). In order to interpret the optimal portfolio policy in (26), it is constructive to consider two well known special cases: (i) the case of logarithmic utility \( (\gamma = 1) \), and (ii) the case of the myopic investor.

In the special case of a log-investor, the last term in (26) vanishes. Moreover, by inspecting the definitions of \( \Omega(t+1) \) and \( \Gamma(t+1) \) in (11) and (12), it is seen that \( \Omega(t+1) = \Sigma_{xx}(t) \) and that \( \Gamma(t+1) \) is the identity matrix in this special case. Hence, for a log-investor the optimal portfolio policy reduces to

\[
\alpha_t = \hat{\Sigma}_{xx}(t)^{-1} \left[ H_x(\Phi_0 + \Phi_1\hat{z}_t) + \frac{1}{2} \hat{\sigma}_x^2(t) \right] = \hat{\Sigma}_{xx}(t)^{-1} \left[ E_t[x_{t+1}] + \frac{1}{2} \hat{\sigma}_x^2(t) \right]. \tag{27}
\]

In establishing the last equality in (27), we have assumed that the matrix \( H_x \) picks out the excess return elements of the state variable vector, \( z_{t+1} \) (and that \( \hat{x}_{t+1} = x_{t+1} \) since the excess
return elements of the state variable vector are assumed perfectly observed). Since $x_{t+1}$ is the vector of log excess returns, the term $E_t[x_{t+1}] + \frac{1}{2}\hat{\sigma}_x^2(t)$ describes the vector of expected excess returns given the available information at time $t$. If all state variables in $z_t$ were perfectly observed, $E_t[x_{t+1}] = H_x(\Phi_0 + \Phi_1z_t)$. But in the general case, the relevant expectations are based on the filtered state variables, i.e. $E_t[x_{t+1}] = H_x(\Phi_0 + \Phi_1\hat{z}_t)$. The formula for the optimal portfolio of a log-investor (also known as the growth-optimal portfolio) is otherwise identical to formulas in Samuelson (1969) and Merton (1969), except that the relevant variance-covariance matrix $\hat{\Sigma}_{xx}(t)$ incorporates the uncertainty related to the fact that some of the state variables are not being perfectly observed.

A myopic investor has the investment horizon $T = t + 1$ at time $t$, and in this special case we thus have $\hat{B}_1(t+1) = 0$ and $\hat{B}_2(t+1) = 0$. Also, again by inspecting the definitions of $\Omega(t+1)$ and $\Gamma(t+1)$ in (11) and (12), it is seen that $\Omega(t+1) = \gamma\Sigma_{xx}(t)$ and that $\Gamma(t+1)$ is the identity matrix. Hence, for a myopic investor the optimal portfolio policy reduces to

$$\alpha_t = \frac{1}{\gamma} \hat{\Sigma}_{xx}(t)^{-1} \left[ E_t[x_{t+1}] + \frac{1}{2}\hat{\sigma}_x^2(t) + (1 - \gamma)\hat{\sigma}_{1x}(t) \right]$$

(28)

where $\hat{\sigma}_{1x}(t) = H_x\hat{\Sigma}(t)H_x'$ is the covariance vector between the return on benchmark asset 1 and the excess returns on the other $n-1$ risky assets given the information available at time $t$. Whenever asset 1 is risk free (i.e. $\hat{\sigma}_1 = 0$ and $\hat{\sigma}_{1x} = 0$), the portfolio of risky assets coincides with the optimal portfolio of a log-investor. If the benchmark asset 1 is risky, investors with $\gamma \neq 1$ will adjust the allocation by a term $(1 - \gamma)\hat{\sigma}_{1x}$. The myopic portfolio policy in (28) is identical to the myopic portfolio policy in Campbell et al. (2003) in the case where all state variables are observed perfectly. When state variables are not observed perfectly, the portfolio rule in (28) basically takes the same form as in one-period Bayesian portfolio models, as pioneered by Klein and Bawa (1976) and Bawa, Brown and Klein (1979).

The general portfolio policy in (26) can be interpreted as having a myopic component, which is identical to the portfolio in (28), and a hedge term, which is simply defined residually. The term involving $\hat{B}_1(t+1)$ in (26) is a pure hedge term. The remaining terms are closely related, but not identical, to the myopic term.
4 Illustrative examples

To illustrate our framework we consider two relatively simple examples. In both cases the investor allocates wealth between a money market account yielding a constant interest rate, $r$, and a risky stock with an unobservable risk premium, $\mu_t$.

In the first case $\mu_t$ is constant and the investor learns about $\mu_t = \mu$ by observing realized stock returns. We find the optimal investment strategy for an investor who recognizes that he will learn more about $\mu$ during the time he invests in the market. This setup has been analyzed by Brennan (1998) in continuous time.

In the second case we let $\mu_t$ be stochastic and mean-reverting, and innovations in $\mu_t$ may be correlated with innovations in realized stock returns. Again the investor learns about $\mu_t$ by observing realized stock returns. However, while he in the previous case eventually learns everything about $\mu_t$, his learning in this setting is limited since $\mu_t$ is stochastic. We analyze the optimal investment strategy in steady state where no more learning is possible in the sense that the investor does not increase the precision of his estimate of $\mu_t$ over time. This setting has been analyzed by Rodriguez (2002) in continuous time.

Throughout the examples and calibration analysis we use a monthly rebalancing frequency unless otherwise stated.

4.1 Asset allocation with learning about a constant mean

In the first case the dynamics of investment opportunities are given by

$$
\begin{align*}
q_{1,t+1} & = r \\
x_{t+1} & = \mu_t + v_{x,t+1} \\
\mu_{t+1} & = \mu_t,
\end{align*}
$$

where $v_{x,t+1} \sim \mathcal{N}(0, \sigma_x)$. Equations (29), (30), and (31) represent the transition equation (16) in this example. The constant excess return volatility parameter, $\sigma_x$, and the constant risk-free interest rate, $r$, are known by the investor, whereas the constant expected excess return on the stock, $\mu_t = \mu$, is unobservable and unknown to the investor. The measurement equation is not stated explicitly here, but simply describes the situation where the risk-free returns and realized excess stock returns are observed perfectly (i.e. without measurement errors). The
investor has a subjective prior on the risk premium $\mu_t$ given by a normal distribution with mean $m_t$ and variance $p_t$. This distribution is subsequently updated through the Kalman filter recursions as realized excess stock returns are observed. In the continuous-time limit, the above dynamic asset allocation model with learning is equivalent to the model analyzed by Brennan (1998), who assumes a constant interest rate and that the stock price follows a geometric diffusion process with an unknown drift parameter.

In the numerical implementation reported below, the prior mean and standard deviation of the monthly risk premium are given by 0.0048 and 0.0017, respectively. This corresponds to the sample mean and standard deviation of the sample mean using monthly data on the S&P 500 stock index from March 1951 to June 2004. The standard deviation of monthly realized log excess stock returns is set to 0.0418 and the monthly log interest rate is set to 0.0043 – both equal to their sample counterparts.

Table 1 indicates the fraction of wealth allocated to stocks for various risk-aversion coefficients, investment horizons and investor types. $\alpha$ denotes the allocation of a long-term investor who incorporates uncertainty about $\mu_t$ and learning into his investment decision. $\alpha^*$ denotes the allocation of a myopic investor who takes into account uncertainty about $\mu_t$ (but, obviously, does not take into account the effect of future learning). Finally, $\tilde{\alpha}^*$ denotes the allocation of a myopic investor who ignores uncertainty about $\mu_t$ and takes his prior mean as the true value. Uncertainty about $\mu_t$ has two effects. First, the variance of the posterior stock return distribution is increased, which decreases the stock allocation for myopic investors (compare $\alpha^*$ with $\tilde{\alpha}^*$). This effect dates back to Klein and Bawa (1976) and Bawa, Brown and Klein (1979). The effect disappears in continuous time and, hence, is not present in Brennan (1998). Note also that this effect – at least for the parameter values chosen here – tends to be small (risk premium uncertainty adds less than 0.1 percent to the monthly volatility of stock returns).

The second, and more important, effect is associated with learning about the true risk premium over time. Higher-than-expected stock returns will lead to an upward assessment of the risk premium. This creates a (perfect) positive correlation between innovations to realized stock returns and innovations to the risk premium estimate. Hence, although true investment opportunities are constant, the investor perceives them as being time-varying, and in order to hedge these perceived variations the investor will decrease his allocation to stocks. As
indicated in the table – and noted first by Brennan (1998) – this effect is quite significant (compare $\alpha$ with $\alpha^*$).\textsuperscript{11}

### 4.2 Asset allocation with unobservable mean-reverting risk premium

In the second case the dynamics of investment opportunities are given by

\begin{align}
r_{1,t+1} &= r \\
x_{t+1} &= \mu_t + v_{x,t+1} \\
\mu_{t+1} &= \overline{\mu}(1 - \phi) + \phi \mu_t + v_{\mu,t+1},
\end{align}

where $v_{x,t+1} \sim \mathcal{N}(0, \sigma^2_x)$, $v_{\mu,t+1} \sim \mathcal{N}(0, \sigma^2_{\mu})$ and $v_{x,t+1}$ and $v_{\mu,t+1}$ have correlation $\rho$. This discrete-time model of investment opportunities is similar to the infinite horizon asset allocation model considered by Campbell and Viceira (1999). Equations (32), (33), and (34) represent the transition equation (16) in our example. Again, the measurement equation is not stated explicitly for convenience, but it is simply assumed that the risk-free returns and realized excess stock returns are observed perfectly, whereas the equity risk premium, $\mu_t$, is unobservable. The constant risk-free interest rate, $r$, and the excess stock return parameters $\overline{\mu}$, $\phi$, $\sigma_x$, $\sigma_{\mu}$, and $\rho$ are all known by the investor. In particular, $\overline{\mu}$ describes the unconditional (or steady state) mean of the equity risk premium, while $\phi$ describes the degree of mean-reversion of the equity risk premium. In this setting we focus on optimal asset allocation when no more learning is possible (in the sense that more return observations will not decrease the variance of the risk premium estimate) and, hence, the prior distribution of $\mu_t$ is of no importance. To this end, we iterate the Kalman filter recursions until $\hat{\Sigma}(t)$ converges to its steady state value $\Sigma$ (convergence to steady state is guaranteed for any positive semi-definite initial variance-covariance matrix, provided that the system is stationary, see Hamilton (p. 390, 1994)). This in turn provides the basis for calculating the optimal investment strategy. To calibrate the model parameters, we follow Campbell and Viceira (1999) and assume that the risk premium is

\textsuperscript{11}Of course, the quantitative effects of learning in general depend critically on the assumed mean and variance of the prior distribution.
really driven by the dividend-price ratio.\textsuperscript{12} The calibrated parameter values applied below are given in Table 2. Table 2 also contains limiting and equivalent continuous-time model parameter estimates (see Campbell, Chacko, Rodriguez and Viceira (2004) and conversion formulas therein) which are used below to address the question of rebalancing frequency importance in the present context.

Table 3 reports the fraction of wealth allocated to stocks for various risk-aversion coefficients, investment horizons and investor types when the risk premium equals its steady state mean. $\alpha$ and $\alpha^*$ again denote the allocations of a long-term and a myopic investor, respectively, who do not observe the risk premium. As a benchmark we also consider the allocations of a long-term and a myopic investor – denoted $\tilde{\alpha}$ and $\tilde{\alpha}^*$, respectively – who do observe the risk premium. Again the effect of risk-premium uncertainty on the allocation of myopic investors is very small (compare $\alpha^*$ with $\tilde{\alpha}^*$). However, the effect on the allocation of long-term investors is pronounced. In this case the investor who observes the true risk premium has a large positive hedge demand due to the negative correlation between innovations to the risk premium and innovations to realized returns. Since the filtered risk premium is less variable than the true risk premium, the investor who only observes the risk premium indirectly perceives investment opportunities to be less volatile and consequently has a lower hedge demand.

Allocations are often very sensitive to variations in the risk premium. Campbell and Viceira (1999) is a case in point. In Figure 1 we show the fraction of wealth allocated to stocks over time in a simulated sample of 500 observations. We report the allocation for an investor who observes the risk premium and for one who filters it out by observing realized stock returns. The investment horizon is 10 years and $\gamma = 5$. The allocations for the investor who does not observe the risk premium are significantly less volatile than the allocations for the investor who does observe the risk premium.\textsuperscript{13}

\textsuperscript{12} More specifically, we first estimate the following model by maximum likelihood using monthly data on the return and dividend-price ratio of the S&P 500 stock index from March 1951 to June 2004:

$$x_{t+1} = c_x + \phi_x(D/P)_t + v_{x,t+1}$$

$$(D/P)_{t+1} = c_{DP} + \phi_{DP}(D/P)_t + v_{DP,t+1}.$$ 

The parameters of (34) are then obtained as: $\mu = c_x + \phi_x c_{DP}/(1 - \phi_{DP})$, $\phi_x = \phi_{DP}$, and $\sigma^2 = \phi^2 \sigma^2_{DP}$. 

\textsuperscript{13} The average allocation to stocks is lower for the investor who does not observe the risk premium, reflecting primarily the lower average hedge demand.
In Appendix B we illustrate the continuous-time equivalent to the discrete-time setup above. This allows us to get a sense of how fast our discrete-time solution converges to its continuous-time limit as the rebalancing interval goes to zero. In Table 4 we again assume an investment horizon of 10 years and $\gamma = 5$ and compute the optimal fraction of wealth allocated to stocks with discrete rebalancing – at intervals ranging from annually to daily – and continuous rebalancing. Again we report the allocation both for an investor who observes the risk premium and for one who filters it out by observing realized stock returns. The relevant parameter values for the continuous-time return dynamics in Appendix B are provided in Table 2. These are computed from the parameter estimates in the discrete-time return dynamics in the previous section, as described above. Furthermore, the discrete-time return dynamics for various intervals are obtained by appropriately discretizing the continuous-time return dynamics (cf. Campbell et. al. (2004)). We see that the discrete-time solution converges quite fast to the continuous-time solution as the rebalancing interval shrinks toward zero. At daily rebalancing the discrete-time and continuous-time solutions are virtually indistinguishable. At monthly rebalancing the discrete-time solution remains very close to the continuous-time solution.

5 Brennan and Xia (2001) revisited and extended

In a well known paper Brennan and Xia (2001) discuss the importance of asset pricing anomalies from an investor viewpoint. In deciding upon his portfolio allocation an investor must take into account i) the probability that the anomaly is true and not the result of “data mining”, ii) the uncertainty about the estimated mean return associated with the anomaly and iii) the speed with which the anomaly will disappear due to trading by other investors. In order to address i) and ii) they develop a model for the optimal portfolio choice of a long term investor who takes into account that he will learn about the probability of the anomaly being true as well as the mean return of the anomaly. The key is to model the investor’s prior as a mixture of normals where the mixing parameter depends on the investor’s assessment of the anomaly being true. The special case of a pure normal prior is basically a multi-asset version of the Brennan (1998) model, and in this particular case Cvitanic et. al. (2005) derive closed-form solutions to the optimal portfolio choice.
An important limitation of the Brennan and Xia (and Cvitanic et. al.) model, however, is that it implicitly assumes that, if true, the anomaly will persist in the future. In other words, the unobserved risk premia on the anomaly are constant. This makes it impossible to address issue iii) within that model. Our framework is much more general as it allows the unobserved risk premia to follow a general VAR process.\(^\text{14}\) As an illustration of this additional flexibility we extend the Brennan and Xia analysis to the case of optimal portfolio choice under different assumption about how the anomaly disappears due to trading by other investors.

In their application Brennan and Xia (2001) consider an investor who allocates wealth between a money market account yielding a constant interest rate and the market (MKT) portfolio and the two Fama and French (1993) SMB and HML portfolios. Table 5 reproduces Table 1 in Brennan and Xia (2001) showing summary statistics for the returns on these three portfolios for the time period July 1963 to December 1991.\(^\text{15}\)

We model investment opportunity dynamics as

\[
\begin{align*}
    r_{1,t+1} &= r \\
    x_{t+1} &= \mu_t + v_{x,t+1} \\
    \mu_{t+1} &= \Phi \mu_t + v_{\mu,t+1},
\end{align*}
\]

where \(x_t = (x_t^{MKT}, x_t^{SMB}, x_t^{HML})'\) and \(\mu_t = (\mu_t^{MKT}, \mu_t^{SMB}, \mu_t^{HML})'\) denote the realized and expected excess returns, respectively, on the three portfolios. \(v_{x,t+1} \sim \mathcal{N}(0, \Sigma_{xx})\) and \(v_{\mu,t+1} \sim \mathcal{N}(0, \Sigma_{\mu\mu})\) are similarly three-dimensional and assumed uncorrelated. As in Brennan and Xias’ “pure unconstrained prior” example we assume that the investor has normally distributed priors on the mean returns of the three portfolios. The means and variances of the prior distributions are set equal to the sample mean and sample mean variances reported in Table 5. The correlation between the priors is set equal to the sample correlations reported in Table 5. The return covariance matrix, \(\Sigma_{xx}\), is also computed from Table 5 and is assumed known by

\(^{14}\)At the same time our model is less general than the Brennan and Xia model as it relies on a pure normal prior. This makes it impossible to address issue i) within our model.

\(^{15}\)The risk-free rate does not impact the allocations and is set to zero.
the investor. The most general model we consider has

$$\Phi_{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_{\text{SMB}} & 0 \\ 0 & 0 & \phi_{\text{HML}} \end{pmatrix} \quad \text{and} \quad \Sigma_{\mu\mu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\text{SMB}}^2 & 0 \\ 0 & 0 & \sigma_{\text{HML}}^2 \end{pmatrix}. \quad (38)$$

The Brennan and Xia (and Cvitanic et. al.) model, where the size and value premia are constant, is the special case where $\phi_{\text{SMB}} = \phi_{\text{HML}} = 1$ and $\sigma_{\text{SMB}} = \sigma_{\text{HML}} = 0$. When $\phi_{\text{SMB}}, \phi_{\text{HML}} < 1$ and $\sigma_{\text{SMB}} = \sigma_{\text{HML}} = 0$ the size and value premia reverts deterministically towards zero. When $\phi_{\text{SMB}}, \phi_{\text{HML}} < 1$ and $\sigma_{\text{SMB}}, \sigma_{\text{HML}} > 0$ the size and value premia reverts stochastically towards zero.\(^{16}\)

When applying our solution algorithm the transition equation (16) is comprised of equations (35), (36), and (37) while the measurement equation (17), which is not shown explicitly here, simply states that the risk-free rate, $r$, and vector of realized excess portfolio returns, $x_t$, are perfectly observable, whereas the vector of market, size and value premia, $\mu_t$, is unobservable.

First we consider an investor who believes that the true (but unknown) size and value premia decline deterministically over time with a given half-life (while the true (but unknown) market risk premium is believed to be constant). In Table 6 we report the optimal portfolios for half-lives of infinity (corresponding to the case where the value and size premia are believed to be constant), 20 years, 10 years and 5 years. We report both the myopic allocation, which is the same in all four cases and the hedge allocations due to learning. We let the risk aversion take values of 2, 5 and 10 and set the investment horizon to 20 years.\(^{17}\)

When the size and value premia have long half-lives the hedge demands can be substantial. For instance, when the risk aversion is five and the size and value premia are constant the learning effect will reduce the total allocations to SMB and HML by more than a third. When

\(^{16}\)A more general model could allow $\sigma_{\text{MK}}$ and non-diagonal elements in $\Phi_{\mu}$ and $\Sigma_{\mu\mu}$ to be non-zero as well as correlation between $v_{\mu,t+1}$ and $v_{\mu,t+1}$. We leave these extensions for future research.

\(^{17}\)For risk aversions of two and five and infinite half-lives the reader can compare our numbers to Table 2 in Cvitanic et. al. (2005) and Table 2 in Brennan and Xia (2001). Our numbers are extremely close to those in Cvitanic et. al. (2005) based on their closed form solution. The very small discrepancy is due to the fact that they work in continuous time while we work in discrete time with monthly portfolio rebalancings. Our numbers are somewhat further from those in Brennan and Xia (2001) due to inaccuracies in their numerical solution algorithm.
the half-lives shorten the hedge demands also diminish. Intuitively, the investor’s incentive to hedge a decrease in the perceived size and value premia is smaller since the true size and value premia are disappearing faster anyway.

Next we consider an investor who believes that the true (but unknown) size and value premia decline stochastically over time by following univariate mean-zero AR(1) processes. In Table 7 the mean-reversion parameters correspond to a half-lives of 10 years. The variances are set to either zero or are set so that the probability of the true size and value premia becoming negative after 10 years, given the investor’s prior means, is 5 percent, 10 percent or 20 percent.

Increasing the variance of the true size and value premia processes decreases the hedge demand. The reason is that an increasing fraction of the variation in the perceived risk premia become unhedgeable since we have assumed zero correlation between asset returns and innovations to the true premia processes.

So far we have implicitly assumed that the investor makes his portfolio allocation at the end of December 1991 since the time series, which determine the investor’s priors, end in December 1991. It is interesting, therefore, to see how his optimal portfolio allocations would have evolved subsequently. In Figure 2 we show the evolution of the SMB and HML allocations from January 1992 to June 2004 when the risk premia decline deterministically as in Table 6 or stochastically as in Table 7. The risk aversion is set to five while the investment horizon, initially 20 years, becomes shorter as time progresses. The investor priors are updated upon seeing the realized returns on the MKT, SMB and HML portfolios.18

Details behind the allocations in Figure 2 are provided in Figure 3–6. For instance, Panel A in Figure 2 shows the allocations to the SMB portfolio when the size and value risk premia decline deterministically. Figure 3 then decomposes the SMB allocations into their myopic and hedge components and shows the means and standard deviations of the posterior distributions on the size premium.

Consider first the case where the true (but unknown) size and value premia decline deterministically over time (Panel A and B in Figure 2). Shorter half-lives imply higher SMB and HML allocations in the beginning of the sample since the hedge demands are lower as discussed above. However, the shorter the half-lives the faster the SMB and HML allocations tend towards zero. This is due to a decline in the myopic allocation (see Panel A in Figure 3

18 The returns on the MKT, SMB and HML portfolios were obtained from Ken French’ website.
and 4) which is again driven by a decline in the estimated size and value premia (see Panel C in Figure 3 and 4). The hedge allocations in Panel B in Figure 3 and 4 decline (in absolute terms) for a number of reasons. Firstly, the investment horizon decreases. Secondly, the uncertainty about the size and value premia estimates decrease (see Panel D in Figure 3 and 4) which implies that there is a smaller potential for learning. Thirdly, the hedge demand is a function of the myopic allocation and to the extent that the myopic allocation decreases the hedge demand will also decrease.

Consider next the case where the true (but unknown) size and value premia decline stochastically over time (Panel C and D in Figure 2). The SMB and HML allocations tend towards zero but larger variances in the AR(1) processes implies more volatile SMB and HML allocations. The is because the volatility of the estimated size and value premia (see Panel C in Figure 5 and 6) and, therefore, in the myopic allocations (see Panel A in Figure 5 and 6) increase with the variances. The hedge allocations in Panel B in Figure 5 and 6 decline (in absolute terms) because of the shortening of the investment horizon and the downward trend in the myopic allocation. Note, however, that the uncertainty about the size and value premia estimates (see Panel D in Figure 5 and 6) no longer disappears over time. In fact, for large variance in the AR(1) processes the uncertainty may actually increase from its initial level.19

6 Conclusion

In this paper we have developed a general multivariate discrete-time model of dynamic asset allocation with incomplete information and learning about state variables. The state variables are described by a vector autoregression and the investor is assumed to have normally distributed priors on the values of the state variables. The priors can be correlated and are updated through observations of realized returns and other relevant variables. The framework is very flexible and contains many papers in the literature on dynamic portfolio choice with incomplete information as special cases.

19 We have computed Sharpe ratios for the different myopic strategies resulting from the different assumptions about the true processes followed by the value and size premia. All the strategies have Sharpe ratios between 0.70 and 0.80. The strategy of allocating an equal fraction of wealth to the three portfolio, advocated by DeMiguel, Garlappi and Uppal (2005), performs slightly better while the MKT by itself has a Sharpe ratio of 0.48.
As an application of the model we have revisited and extended the “pure unconstrained prior” case in Brennan and Xia (2001) where they analyze the portfolio choice of an investor faced with the market and the two Fama-French SMB and HML portfolios and takes into account that he will learn about the mean returns on these portfolios over time. However, instead of assuming that the true returns are constant we solve the portfolio choice under different assumptions about the rate at which the size and value premia will disappear due to trading by other investors. In the most general case, the premia decline stochastically and can change sign with a certain probability. We show that this realistic extension of the setup has important implications for the optimal portfolios.

In line with Brennan and Xia (2001) we assume that the investor learns about the mean returns on the portfolios by observing their realized returns. However, our framework also allows the investor to incorporate other information such as analysts’ recommendations, earnings reports and macro economic releases into his optimal portfolio choice. Applying our framework to such more general settings is an interesting avenue for future research.
Appendix A

The proof of Proposition 1 as well as the solution forms in (7), (8), (24), and (25) and the recursive solution algorithm in (9)–(15) hinges on Lemma 3 below. Lemma 3 is in turn based on the following Lemma 2, while the matrix result provided in Lemma 1 is relevant for showing that the matrices obtained by the recursive solution algorithm are well defined. The result in Lemma 2 can be obtained by basically rewriting and elaborating on a similar result in Campbell, Chan and Viceira (pp. 10-11, 2002), (under slightly different notation). An explicit proof of Lemma 2, however, is stated for completeness and for direct use and adoption in the subsequent proofs.

Lemma 1 Let $A$ and $B$ be symmetric and positive semi-definite matrices. Then

$$C = (I + BA)^{-1}B$$

is well defined and symmetric and positive semi-definite.

Proof: The following argument shows that $I + BA$ is regular. For an arbitrary vector $v$, assume that

$$v'(I + BA) = 0.$$

By multiplication by $Bv$ from the right, we get

$$v' B v + v' BAB v = 0,$$

and since both $B$ and $BAB$ are symmetric and positive semi-definite matrices, this implies that $v' B v$ and $v' BAB v$ are both zero. But $v' B v = 0$ implies that $v$ belongs to the null space of $B$, i.e. $v' B = 0$ (this follows from the fact that $B$ can always be written as $B = D'D$, see e.g. Hamilton (1994, p. 734), and the following line of implications: $v' B v = 0 \Rightarrow v' D'D v = 0 \Rightarrow (Dv)' Dv = 0 \Rightarrow Dv = 0 \Rightarrow D'Dv = 0 \Rightarrow B v = 0$). Inserting this in our original assumption $v' + v' BA = 0$ implies $v' = 0$. This shows that $I + BA$ is regular. As used below, it can also be noted that $I + AB$ is regular by the same argument.

Since $I + BA$ is regular, $C = (I + BA)^{-1}B$ is well defined. In order to prove that $C$ is symmetric, it can be noted that the symmetry condition $C' = C$ can be written

$$B(I + AB)^{-1} = (I + BA)^{-1}B.$$
But this identity follows by multiplication by \((I + AB)^{-1}\) from the right and \((I + BA)^{-1}\) from the left in the (trivial) identity 

\[(I + BA)B = B(I + AB).\]

Finally, to see that \((I + BA)^{-1}B\) is positive semi-definite, we must show that 

\[v'(I + BA)^{-1}Bv \geq 0\]

for any vector \(v\). But since \(I + BA\) is regular, we may assume \(v' = u'(I + BA)\) for some vector \(u\). Then 

\[v'(I + BA)^{-1}Bv = u'(I + BA)(I + BA)^{-1}B(I + BA)'u \]

\[= u'B(I + AB)u \]

\[= u'Bu + u'BABu \geq 0\]

where the last inequality follows from the fact that both \(B\) and \(BAB\) are positive semi-definite matrices.

**Lemma 2** Let \(\tilde{z}\) have a multivariate normal distribution, \(\tilde{z} \sim \mathcal{N}(\mu_z, \Sigma_{zz})\), and let \(B_0\) be a constant, \(B_1\) an \(n\)-dimensional vector, and \(B_2\) a symmetric and negative semi-definite \(n \times n\) matrix. Then 

\[E_t\left[e^{B_0 + B_1' \tilde{z} + \tilde{z}' B_2 \tilde{z}}\right] = e^{C_0 + C_1' \mu_z + C_2' \mu_z}\]

where 

\[C_0 = B_0 + \frac{1}{2} \ln |\Gamma| + \frac{1}{2} B_1' \Sigma_{zz} \Gamma B_1\]

\[C_1 = \Gamma B_1\]

\[C_2 = \Gamma B_2\]

\[\Gamma = (I - 2B_2 \Sigma_{zz})^{-1}.
\]

Furthermore, \(C_2\) is a symmetric and negative semi-definite matrix.

**Proof:** Let \(\Psi = (\Sigma_{zz}^{-1} - 2B_2)^{-1}\). Since \(\Sigma_{zz}^{-1}\) is symmetric and positive definite and \(-2B_2\) is symmetric and positive semi-definite, \(\Psi\) is symmetric and positive definite (and thus invertible).
Using the definition, invertibility, and symmetry of $\Psi$, we obtain

$$E_t \left[ e^{B_0 + B'_1 \tilde{z} + \tilde{z}' B_2 \tilde{z}} \right] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} (z - \mu_z)' \Sigma_{zz}^{-1} (z - \mu_z)} dz$$

where the term inside the last integral is the density function for an $N(\Psi (B_1 + \Sigma_{zz}^{-1} \mu_z), \Psi)$ distributed random variable which by definition integrates to one. Hence we have

$$E_t \left[ e^{B_0 + B'_1 \tilde{z} + \tilde{z}' B_2 \tilde{z}} \right] = e^{\frac{1}{2} \ln |\Psi| - \frac{1}{2} |\Sigma_{zz}| + \frac{1}{2} \Psi (B_1 + \Sigma_{zz}^{-1} \mu_z)' \Psi (B_1 + \Sigma_{zz}^{-1} \mu_z) + B_0 - \frac{1}{2} \mu_z' \Sigma_{zz}^{-1} \mu_z}$$

where the third equality follows from observing that $\Psi = \Sigma_{zz}^{-1} (I - 2B_2 \Sigma_{zz})^{-1} = \Sigma_{zz} \Gamma$ and that $\Psi = (\Sigma_{zz}^{-1} - 2B_2)^{-1} \Rightarrow \Sigma_{zz}^{-1} = \Psi^{-1} + 2B_2 \Rightarrow \Sigma_{zz}^{-1} \Psi \Sigma_{zz}^{-1} = \Sigma_{zz}^{-1} (I + 2\Psi B_2 - I) = 2\Gamma B_2$.

Finally, since $C_2 = \Gamma B_2 = (I - 2B_2 \Sigma_{zz})^{-1} B_2$, it follows by a straightforward application of Lemma 1 that $C_2$ is a symmetric and negative semi-definite matrix.

**Lemma 3** Let $z_{t+1}$ be described by a VAR(1) model,

$$z_{t+1} = \Phi_0 + \Phi_1 z_t + \nu_{t+1}, \quad \nu_{t+1} \sim \mathcal{N}(0, \Sigma).$$

If $B_2$ is symmetric and negative semi-definite, then

$$E_t \left[ e^{B_0 + B'_1 z_{t+1} + z_{t+1}' B_2 z_{t+1}} \right] = e^{D_0 + D'_1 z_t + \tilde{z}' D_2 \tilde{z}_t}$$

where

$$D_0 = B_0 + \frac{1}{2} \ln |\Gamma| + \frac{1}{2} \Phi_1 \Gamma B_1 + \Phi_0 \Gamma B_2 \Phi_0$$

$$D_1 = \Phi_1 \Gamma B_1 + 2\Phi_1' \Gamma B_2 \Phi_0$$

$$D_2 = \Phi_1 \Gamma B_2 \Phi_1$$

and $D_2$ is symmetric and negative semi-definite.
Proof: Note that \( z_{t+1} \sim \mathcal{N}(\mu_z, \Sigma_{zz}) \) where \( \mu_z = \Phi_0 + \Phi_1 z_t \) and \( \Sigma_{zz} = \Sigma \). By application of Lemma 2, one thus obtains

\[
E_t \left[ e^{B_0 z_{t+1} + B_1' z_{t+1} + z_t' B_2 z_t + 1} \right] = e^{C_0 + C_1' \mu_z + \mu_z' C_2 \mu_z} = e^{D_0 + D_1' z_t + z_t' D_2 z_t}
\]

where \( C_0, C_1, \) and \( C_2 \) are as stated in Lemma 2, and where the last equality is obtained by collection of constant, linear, and quadratic terms of \( z_t \) in the exponential. Thus,

\[
D_0 = C_0 + C_1' \Phi_0 + \Phi_0' C_2 \Phi_0
\]

\[
D_1' = C_1' \Phi_1 + 2 \Phi_0' C_2 \Phi_1
\]

\[
D_2 = \Phi_1' C_2 \Phi_1.
\]

By inserting \( C_0, C_1, \) and \( C_2 \), as stated in Lemma 2, one obtains \( D_0, D_1, \) and \( D_2 \) in the form stated in the lemma and with \( \Gamma = (I - 2B_2 \Sigma)^{-1} \).

Finally, since \( C_2 = \Gamma B_2 \) is symmetric and negative semi-definite (cf. Lemma 2), it follows that \( D_2 = \Phi_1' C_2 \Phi_1 \) is also symmetric and negative semi-definite.

We can now prove Proposition 1 as well as the solution forms in (7), (8), (24), and (25) and the recursive solution algorithm in (9)–(15). (The notation from section 2 is used.)

Proof: (of Proposition 1) Using the conjectured form of the indirect utility function, and the next period wealth as given by \( W_{t+1} = W_t e^{r_{p,t+1}} \) (and with log portfolio return \( r_{p,t+1} \) as given in (5)), we can evaluate the object function in the maximization involved in the relevant Bellman equation (cf. (6)),

\[
G(t, \alpha_t) = \mathbb{E}_t [V_{t+1}]
\]

\[
= \mathbb{E}_t \left[ \left( e^{\alpha_t (t+1) + (1-\gamma)B_1 (t+1)' z_{t+1} + (1-\gamma)z_t' B_2 (t+1) z_t + 1} \right) W_{t+1}^{1-\gamma-1} \right]
\]

\[
= \mathbb{E}_t \left[ \left( e^{\alpha_t (t+1) + (1-\gamma)B_1 (t+1)' z_{t+1} + (1-\gamma)z_t' B_2 (t+1) z_t + (1-\gamma) r_{p,t+1}} \right) W_{t+1}^{1-\gamma-1} \right]
\]

\[
= g(t, \alpha_t) W_{t}^{1-\gamma-1}
\]

\[\text{(39)}\]
where
\[ g(t, \alpha_t) = E_t \left[ e^{B_0(t+1) + (1-\gamma)B_1(t+1)'} z_{t+1} + (1-\gamma)z_{t+1}' B_2(t+1) z_{t+1} + (1-\gamma)r_{p,t+1} \right] \]
\[ = E_t \left[ e^{B_0(t+1) + (1-\gamma)B_1(t+1)'} z_{t+1} + (1-\gamma) \left( r_{1,t+1} + \alpha_t' x_{t+1} + \frac{1}{2} \alpha_t' (\sigma_x^2 - \Sigma_{xx} \alpha_t) \right) \right] \]
\[ = E_t \left[ \exp \left\{ (B_0(t+1) + (1-\gamma) \frac{1}{2} \alpha_t' (\sigma_x^2 - \Sigma_{xx} \alpha_t)) + (1-\gamma) (B_1(t+1)') + H_1 + \alpha_t' H_x \right\} z_{t+1} + z_{t+1}' (1-\gamma) B_2(t+1) z_{t+1} \right] \]
(40)

and where \( H_1 \) and \( H_x \) are selection matrices that select the first element and the excess returns from \( z_{t+1} \), respectively, i.e. \( r_{1,t+1} = H_1 z_{t+1} \) and \( x_{t+1} = H_x z_{t+1} \).

Using Lemma 3, \( g(t, \alpha_t) \) can now be evaluated and written as
\[ g(t, \alpha_t) = e^{D_0(t, \alpha_t) + D_1(t, \alpha_t)'} z_{t+1} + z_{t+1}' D_2(t) z_{t+1} \]
(41)

where
\[ D_0(t, \alpha_t) = B_0(t+1) + \frac{1}{2} \ln |\Gamma(t+1)| + \frac{1}{2} (1-\gamma) \alpha_t' (\sigma_x^2 - \Sigma_{xx} \alpha_t) \]
\[ + \frac{1}{2} (1-\gamma)^2 (B_1(t+1)') + H_1 + \alpha_t' H_x) \Sigma \Gamma(t+1) (B_1(t+1) + H_1' + H_x' \alpha_t) \]
\[ + (1-\gamma) \Phi_0 \Gamma(t+1) (B_1(t+1) + H_1' + H_x' \alpha_t) + (1-\gamma) \Phi_0 \Gamma(t+1) B_2(t+1) \Phi_0 \]
\[ D_1(t, \alpha_t) = (1-\gamma) \Phi_1' \Gamma(t+1) (B_1(t+1) + H_1' + H_x' \alpha_t) + 2(1-\gamma) \Phi_1' \Gamma(t+1) B_2(t+1) \Phi_0 \]
\[ D_2(t) = (1-\gamma) \Phi_1' \Gamma(t+1) B_2(t+1) \Phi_1 \]
\[ \Gamma(t+1) = (I + 2(\gamma-1) B_2(t+1) \Sigma)^{-1} \]

The optimal portfolio weight \( \alpha_t \) must be determined by maximizing the expected next period indirect utility, i.e. by maximizing the expression \( G(t, \alpha_t) \) (and with \( g(t, \alpha_t) \) given in (41)) with respect to \( \alpha_t \). The first order conditions from the optimization problem are given
by

\[ \frac{\partial G}{\partial \alpha_i} = 0 \]

\[ \Downarrow \]

\[ \frac{\partial D_0}{\partial \alpha_i} + \frac{\partial D_1}{\partial \alpha_i} z_t = 0 \]

\[ \Downarrow \]

\[ \frac{1}{2}\sigma_x^2 + (1 - \gamma) H_x \Sigma(t + 1) (B_1(t + 1) + H_1') + H_x \Gamma(t + 1)\Phi_0 + H_x \Gamma(t + 1)\Phi_1 z_t \]

\[ - (\Sigma_{xx} - (1 - \gamma) H_x \Sigma(t + 1) H_x') \alpha_t = 0 \]

\[ \Downarrow \]

\[ \alpha_t = A_0(t) + A_1(t) z_t \]

where \( A_0(t) \) and \( A_1(t) \) are given in (9) and (10). The indirect utility function at time \( t \) can now be determined by substituting the optimal portfolio weights into the expression in (39), i.e. \( V_t = G(t, A_0(t) + A_1(t) z_t) \). By evaluating this expression for \( V_t \), one obtains

\[ V_t = \left( \frac{\epsilon \sigma_0(t, A_0(t) + A_1(t) z_t) + D_1(t, A_0(t) + A_1(t) z_t) z_t + z_t' D_2(t) z_t}{1 - \gamma} \right) W_t^{1 - \gamma} - 1 \]

where the last equality is obtained by the collection of constant, linear, and quadratic terms of \( z_t \), and \( B_0(t), B_1(t), \) and \( B_2(t) \) are given in (13), (14), and (15). The fact that \( B_2(t) \) is symmetric and positive semi-definite follows from the discussion after equations (13), (14), and (15) in Section 2.
Appendix B

In this appendix we present and derive the continuous-time equivalent of the discrete-time setup in Section 4.2. The investment opportunity dynamics are given by

\[
\begin{align*}
\frac{dA_t}{A_t} &= r dt \quad (42) \\
\frac{dS_t}{S_t} &= (r + \eta_t) dt + \sigma_S dZ_{S,t} \quad (43) \\
d\eta_t &= \kappa(\theta - \eta_t) dt + \sigma_{\eta} dZ_{\eta,t}, \quad (44)
\end{align*}
\]

where \(dZ_{S,t}\) and \(dZ_{\eta,t}\) denote Wiener processes with correlation \(\rho\). The risk premium, \(\eta_t\), is unobservable and must be inferred from the realized stock returns. Applying the Kalman-Bucy filter (see e.g. Liptser and Shiryaev (2001), Chapter 12), we can derive the investment opportunity dynamics perceived by the investor:

\[
\begin{align*}
\frac{dA_t}{A_t} &= r dt \quad (45) \\
\frac{dS_t}{S_t} &= (r + \hat{\eta}_t) dt + \sigma_S d\hat{Z}_{S,t} \quad (46) \\
d\hat{\eta}_t &= \kappa(\theta - \hat{\eta}_t) dt + \sigma_{\hat{\eta}} d\hat{Z}_{\eta,t}, \quad (47)
\end{align*}
\]

where \(\hat{\eta}_t\) denotes the estimate of the unobserved risk premium. \(\sigma_{\hat{\eta}}\) and \(d\hat{Z}_{S,t}\) are given by

\[
\sigma_{\hat{\eta}} = \frac{\rho \sigma_{\eta} \sigma_S + v_t}{\sigma_S} \quad (48)
\]

\[
d\hat{Z}_{S,t} = \left( \frac{\eta_t - \hat{\eta}_t}{\sigma_S} \right) dt + dZ_{S,t} \quad (49)
\]

and \(v_t\) – the variance of the risk premium estimate – follows

\[
\frac{dv_t}{v_t} = -2\kappa v_t + \sigma_{\hat{\eta}}^2 - \left( \frac{\rho \sigma_{\eta} \sigma_S + v_t}{\sigma_S} \right)^2. \quad (50)
\]

Note that although the market is incomplete, an investor with the assumed information set will perceive the market as being complete. This “observational completeness” result has been stressed by Rodriguez (2002).

As in Section 4.2 we focus on the steady state where no more learning is possible. In steady state we must have \(\frac{dv_t}{dt} = 0\), which yields a quadratic equation, the non-negative root of which can be shown to equal

\[
v^* = -\kappa \sigma_S^2 - \rho \sigma_{\eta} \sigma_S + \sigma_S \sqrt{\kappa^2 \sigma_S^2 + \sigma_{\eta}^2 + 2\kappa \rho \sigma_{\eta} \sigma_S}. \quad (51)
\]
Hence, in steady state the diffusion term $\sigma_\theta$ is given by (48) with $v_t$ replaced by $v^*$.\footnote{It follows from (51) that $v^* = 0$ when $\rho = 1$ or $\rho = -1$ (the latter only holds true provided that $\kappa \sigma_S^2 - \sigma_y \sigma_S \geq 0$). From (48) it then follows that the true and filtered risk premium coincide in these special cases.}

The closed form solution to the optimal asset allocation strategy of a perfectly informed long-horizon CRRA investor facing investment opportunity dynamics given by (42)-(44) was derived by Kim and Omberg (1996). The same solution can be used to solve the optimal asset allocation strategy of an imperfectly informed investor facing perceived investment opportunity dynamics given by (45)-(47).
\[ \gamma = 2 \quad \gamma = 5 \quad \gamma = 10 \]

\[ T = 2 \quad T = 10 \quad T = 20 \]

<table>
<thead>
<tr>
<th></th>
<th>$\gamma = 2$</th>
<th></th>
<th>$\gamma = 5$</th>
<th></th>
<th>$\gamma = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 2$</td>
<td>$T = 10$</td>
<td>$T = 20$</td>
<td>$T = 2$</td>
<td>$T = 10$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.59</td>
<td>1.48</td>
<td>1.36</td>
<td>0.63</td>
<td>0.56</td>
</tr>
<tr>
<td>$\alpha^*$</td>
<td>1.62</td>
<td>1.62</td>
<td>1.62</td>
<td>0.65</td>
<td>0.65</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>-0.03</td>
<td>-0.14</td>
<td>-0.26</td>
<td>-0.02</td>
<td>-0.09</td>
</tr>
<tr>
<td>$\tilde{\alpha}^*$</td>
<td>1.62</td>
<td>1.62</td>
<td>1.62</td>
<td>0.65</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Notes: Fraction of wealth allocated to stocks for varying risk-aversion coefficients, investment horizons and investor types. $\alpha$ denotes the allocation of a long-term investor who takes account of uncertainty about $\mu_t$ and learning. $\alpha^*$ denotes the allocation of a myopic investor who takes into account uncertainty about $\mu_t$. $\Delta$ is the hedge demand of the long term investor. $\tilde{\alpha}^*$ denotes the allocation of a myopic investor who ignores uncertainty about $\mu_t$.

Table 1: Allocations to stocks with and without learning

32
<table>
<thead>
<tr>
<th>Discrete-time parameters</th>
<th>Continuous-time parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.0043</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.0035</td>
</tr>
<tr>
<td>$\phi_{\mu}$</td>
<td>0.9868</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>0.0416</td>
</tr>
<tr>
<td>$\sigma_{\mu}$</td>
<td>5.28 E-4</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.9184</td>
</tr>
<tr>
<td>$r$</td>
<td>0.0516</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.0531</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.1592</td>
</tr>
<tr>
<td>$\sigma_S$</td>
<td>0.1451</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>0.0222</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.9194</td>
</tr>
</tbody>
</table>

Notes: To the left are parameter estimates of the discrete-time return dynamics (32)-(34). These are in monthly terms. To the right are parameters of the continuous-time return dynamics (42)-(44) in Appendix B. These were obtained from the discrete-time estimates using the conversion formulas in Campbell et. al. (2004) and are in annual terms.

Table 2: Parameters of the discrete-time and continuous-time models presented in Section 4.2.
Notes: Fraction of wealth allocated to stocks for varying risk-aversion coefficients, investment horizons and investor types when the risk-premium equals its steady state mean. $\alpha$ and $\alpha^*$ denote allocations for a long term and a myopic investor, respectively, who filter out the risk premium by observing realized stock returns. $\Delta$ is the hedge demand of the long term investor. $\tilde{\alpha}$ and $\tilde{\alpha}^*$ denote allocations for a long term and myopic investor, respectively, who observe the true risk premium. $\tilde{\Delta}$ is the hedge demand of this long term investor.

Table 3: Allocations to stocks with and without observable risk premium
Notes: Fraction of wealth allocated to stocks for varying rebalancing intervals and investor types when the risk-premium equals its steady state mean. $\alpha$ and $\alpha^*$ denote allocations for a long term and a myopic investor, respectively, who filter out the risk premium by observing realized stock returns. $\Delta$ is the hedge demand of the long term investor. $\tilde{\alpha}$ and $\tilde{\alpha}^*$ denote allocations for a long term and myopic investor, respectively, who observe the true risk premium. $\tilde{\Delta}$ is the hedge demand of this long term investor. Investment horizon is 10 years and $\gamma = 5$.

Table 4: Allocations to stocks for various rebalancing intervals

<table>
<thead>
<tr>
<th></th>
<th>annually</th>
<th>quarterly</th>
<th>monthly</th>
<th>weekly</th>
<th>daily</th>
<th>continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.8486</td>
<td>0.8343</td>
<td>0.8309</td>
<td>0.8296</td>
<td>0.8292</td>
<td>0.8292</td>
</tr>
<tr>
<td>$\alpha^*$</td>
<td>0.5434</td>
<td>0.5139</td>
<td>0.5074</td>
<td>0.5049</td>
<td>0.5043</td>
<td>0.5041</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>0.3052</td>
<td>0.3204</td>
<td>0.3235</td>
<td>0.3247</td>
<td>0.3249</td>
<td>0.3251</td>
</tr>
<tr>
<td>$\tilde{\alpha}$</td>
<td>1.1063</td>
<td>1.0860</td>
<td>1.0810</td>
<td>1.0791</td>
<td>1.0786</td>
<td>1.0785</td>
</tr>
<tr>
<td>$\tilde{\alpha}^*$</td>
<td>0.5627</td>
<td>0.5185</td>
<td>0.5089</td>
<td>0.5052</td>
<td>0.5043</td>
<td>0.5041</td>
</tr>
<tr>
<td>$\tilde{\Delta}$</td>
<td>0.5436</td>
<td>0.5675</td>
<td>0.5721</td>
<td>0.5739</td>
<td>0.5743</td>
<td>0.5744</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>Std. dev.</td>
<td>Std. dev. of mean</td>
<td>Correlations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
<td>-----------</td>
<td>-------------------</td>
<td>--------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>MKT</td>
<td>SMB</td>
<td>HML</td>
</tr>
<tr>
<td>MKT</td>
<td>5.21%</td>
<td>15.7</td>
<td>2.94</td>
<td>1</td>
<td>0.32</td>
<td>-0.38</td>
</tr>
<tr>
<td>SMB</td>
<td>3.25%</td>
<td>10.0</td>
<td>1.89</td>
<td>0.32</td>
<td>1</td>
<td>-0.08</td>
</tr>
<tr>
<td>HML</td>
<td>4.78%</td>
<td>8.8</td>
<td>1.65</td>
<td>-0.38</td>
<td>-0.08</td>
<td>1</td>
</tr>
</tbody>
</table>

Notes: Summary statistics for returns on the market (MKT) portfolio and the two Fama-French SMB and HML portfolios as reported by Brennan and Xia (2001). The time period is July 1963 to December 1991.

Table 5: Summary statistics for the market, SMB and HML portfolios
\[ \gamma = 2 \]
\[ \gamma = 5 \]
\[ \gamma = 10 \]

<table>
<thead>
<tr>
<th>Half-life</th>
<th>MKT</th>
<th>SMB</th>
<th>HML</th>
<th>MKT</th>
<th>SMB</th>
<th>HML</th>
<th>MKT</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Myopic</td>
<td>—</td>
<td>1.78</td>
<td>1.04</td>
<td>4.38</td>
<td>0.71</td>
<td>0.41</td>
<td>1.75</td>
<td>0.36</td>
<td>0.21</td>
</tr>
<tr>
<td>Hedge</td>
<td>\infty</td>
<td>-0.46</td>
<td>-0.27</td>
<td>-1.12</td>
<td>-0.25</td>
<td>-0.15</td>
<td>-0.62</td>
<td>-0.14</td>
<td>-0.08</td>
</tr>
<tr>
<td></td>
<td>20 years</td>
<td>-0.43</td>
<td>-0.16</td>
<td>-0.74</td>
<td>-0.24</td>
<td>-0.09</td>
<td>-0.43</td>
<td>-0.13</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>10 years</td>
<td>-0.41</td>
<td>-0.10</td>
<td>-0.52</td>
<td>-0.23</td>
<td>-0.06</td>
<td>-0.31</td>
<td>-0.12</td>
<td>-0.04</td>
</tr>
<tr>
<td></td>
<td>5 years</td>
<td>-0.38</td>
<td>-0.05</td>
<td>-0.30</td>
<td>-0.21</td>
<td>-0.04</td>
<td>-0.18</td>
<td>-0.11</td>
<td>-0.02</td>
</tr>
</tbody>
</table>

Notes: Fraction of wealth allocated to the market (MKT), SMB and HML portfolios for varying risk-aversion coefficients and assumptions about how fast the size and value premia will disappear due to trading by other investors. The investor is assumed to have normally distributed priors on the mean returns of the three portfolios. The means and variances of the prior distributions are set equal to the sample mean and sample mean variances reported in Table 5. The correlation between the priors is set equal to the sample correlations reported in Table 5. The true (but unknown) market risk premium is assumed to be constant. The true (but unknown) size and value risk premia are assumed to be either constant or decline deterministically with half-lives of 20 years, 10 years or 5 years. We report both the myopic allocation, which is the same in all four cases and the hedge demands due to learning. The investment horizon is 20 years.

Table 6: Allocations when the true (but unobservable) size and value premia decline deterministically over time
\[\gamma = 2 \quad \gamma = 5 \quad \gamma = 10\]

<table>
<thead>
<tr>
<th>Prob.</th>
<th>(\gamma = 2)</th>
<th>(\gamma = 5)</th>
<th>(\gamma = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MKT</td>
<td>SMB</td>
<td>HML</td>
</tr>
<tr>
<td>Myopic</td>
<td>—</td>
<td>1.78</td>
<td>1.04</td>
</tr>
<tr>
<td>Hedge</td>
<td>0 pct.</td>
<td>-0.41</td>
<td>-0.10</td>
</tr>
<tr>
<td></td>
<td>5 pct.</td>
<td>-0.39</td>
<td>-0.10</td>
</tr>
<tr>
<td></td>
<td>10 pct.</td>
<td>-0.39</td>
<td>-0.10</td>
</tr>
<tr>
<td></td>
<td>20 pct.</td>
<td>-0.37</td>
<td>-0.10</td>
</tr>
</tbody>
</table>

Notes: Fraction of wealth allocated to the market (MKT), SMB and HML portfolios for varying risk-aversion coefficients and assumptions about the stochastic variations in the true processes for the size and value premia. The investor is assumed to have normally distributed priors on the mean returns of the three portfolios. The means and variances of the prior distributions are set equal to the sample mean and sample mean variances reported in Table 5. The correlation between the priors is set equal to the sample correlations reported in Table 5. The true (but unknown) market risk premium is assumed to be constant. The true (but unknown) size and value risk premia are assumed to decline stochastically by following univariate mean-zero AR(1) processes. The mean-reversion parameters correspond to a half-lives of 10 years. The variances are set to either zero or are set so that the probability of the true size and value premia becoming negative after 10 years, given the investor’s prior means, is 5 percent, 10 percent or 20 percent. We report both the myopic allocation, which is the same in all four cases and the hedge demands due to learning. The investment horizon is 20 years.

Table 7: Allocations when the true (but unobservable) size and value premia decline stochastically over time
Figure 1: Allocations to stocks with and without observable risk premium

The figure shows the fraction of wealth allocated to stocks when the risk premium is either perfectly observable (⋯) or unobservable (—). In the latter case, the allocation is based on the Kalman filtered estimate of the risk premium. The investment horizon is 10 years and $\gamma = 5$. 
Figure 2: Allocations to SMB and HML portfolio for different assumption about the true processes for the size and value premia

Panel A and B show the allocations to the SMB and HML portfolios over time when the true (but unobservable) size and value premia are assumed to be either constant (· · ·) or decline deterministically with half-lives of 20 years (—), 10 years (—) or 5 years (—). Panel C and D show the allocations to the SMB and HML portfolios over time when the true size and value premia decline stochastically by following univariate mean-zero AR(1) processes. The mean-reversion parameters correspond to a half-lives of 10 years and the variances are set to either zero (—) or are set so that the probability of the true size and value premia becoming negative after 10 years, given the investor’s prior means in December 1991, is 5 percent (—), 10 percent (—) or 20 percent (· · ·).

In all cases the investor is assumed to start out in December 1991 with normally distributed priors on the size and value risk premia (as well as the market risk premia). The means and variances of the prior distributions are set equal to the sample means and sample mean variances reported in Table 5. The correlation between the priors is set equal to the sample correlations reported in Table 5. The priors are subsequently updated upon observing realized returns. The investment horizon is 20 years and $\gamma = 5$. 

Figure 3: Allocations to SMB portfolio when the true (but unobservable) size and value premia decline deterministically over time

Panel A and B decompose the SMB allocations reported in Panel A in Figure 2 into their myopic and hedge components. Panel C and D show the means and standard deviations of the posterior distributions on the size premium. The true (but unobservable) size and value premia are assumed to be either constant (---) or decline deterministically with half-lives of 20 years (---), 10 years (--) or 5 years (--).
Panel A: Myopic allocation
Panel B: Hedge allocation
Panel C: Posterior means of value dist.
Panel D: Posterior std. dev. of value dist.

Figure 4: Allocations to HML portfolio when the true (but unobservable) size and value premia decline deterministically over time

Panel A and B decompose the HML allocations reported in Panel B in Figure 2 into their myopic and hedge components. Panel C and D show the means and standard deviations of the posterior distributions on the value premium. The true (but unobservable) size and value premia are assumed to be either constant (---) or decline deterministically with half-lives of 20 years (--), 10 years (—) or 5 years (---)
Figure 5: Allocations to SMB portfolio when the true (but unobservable) size and value premia decline stochastically over time

Panel A and B decompose the SMB allocations reported in Panel C in Figure 2 into their myopic and hedge components. Panel C and D show the means and standard deviations of the posterior distributions on the size premium. The true size and value premia decline stochastically by following univariate mean-zero AR(1) processes. The mean-reversion parameters correspond to a half-lives of 10 years and the variances are set to either zero (---) or are set so that the probability of the true size and value premia becoming negative after 10 years, given the investor’s prior means in December 1991, is 5 percent (--), 10 percent (---) or 20 percent (----).
Panel A: Myopic allocation

Panel B: Hedge allocation

Panel C: Posterior means of value dist.

Panel D: Posterior std. dev. of value dist.

Figure 6: Allocations to HML portfolio when the true (but unobservable) size and value premia decline stochastically over time

Panel A and B decompose the HML allocations reported in Panel D in Figure 2 into their myopic and hedge components. Panel C and D show the means and standard deviations of the posterior distributions on the value premium. The true size and value premia decline stochastically by following univariate mean-zero AR(1) processes. The mean-reversion parameters correspond to a half-lives of 10 years and the variances are set to either zero (—) or are set so that the probability of the true size and value premia becoming negative after 10 years, given the investor’s prior means in December 1991, is 5 percent (—), 10 percent (— · ·) or 20 percent (···).
References


