Optimal Risk Sharing With Limited Liability

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Abstract

We solve the general problem of optimal risk sharing among a finite number of agents with limited liability. We show that the optimal allocation is characterized by endogenously determined ranks assigned to the participating agents and a hierarchical structure of risk sharing, where all agents take on risks only above the agent-specific thresholds determined by their ranks. When all agents have CARA utilities, linear risk sharing is optimal between two adjacent thresholds.

We use our general characterization of optimal risk sharing with limited liability to solve the problem of optimal insurance design with multiple insurers. We show that the optimal thresholds, or deductibles, can be efficiently calculated through the fixed point of a contraction mapping. We then use this contraction mapping technique to derive a number of comparative statics results for optimal insurance design and its dependence on microeconomic characteristics.

Keywords: optimal risk sharing, limited liability, optimal insurance design

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1 Introduction

The modern theory of efficient risk sharing goes back to the fundamental papers by Borch (1962) and Wilson (1968), who characterized efficient risk sharing among several agents (typically more than two) with heterogeneous preferences. Based on that, Wilson (1968) further developed the theory of syndicates. Both Borch and Wilson based their analysis on an important assumption that a complete set of state-contingent contracts is available for risk allocation. In many real-life situations, however, available contracts are characterized by limited liability: Agents are willing to take only risks that do not exceed a certain (possibly state-contingent) level. This situation is particularly true for insurance contracts, for which the corresponding indemnity schedules (coverage functions) are always assumed to be non-negative and lower than the total loss. As has been shown by Arrow (1971, 1973) and Raviv (1979), limited liability may significantly alter the structure of optimal allocation. Namely, the efficient risk-sharing rule between two agents (i.e., the insured and the single insurer) is generally characterized by the presence of a deductible. To the best of our knowledge though, nothing is known about the structure of optimal risk sharing with limited liability in the presence of more than two agents.

In this paper we first solve the general problem of optimal risk sharing among a finite number of agents with limited liability. We show that the optimal allocation is characterized by endogenously determined ranks assigned to the participating agents and agent-specific thresholds. These thresholds lead to a hierarchical structure of risk sharing, with every agent taking on only risks above the threshold determined by his rank. The ranks assigned to the agents are determined by the marginal contribution of the agent’s utility to the total welfare function, i.e., by the product of the corresponding social utility weight and the agent’s marginal utility, evaluated at the minimal risk level.

We use our general characterization of optimal risk sharing to solve the problem
of optimal insurance design for an insured facing multiple insurers with heterogeneous preferences and discount factors.\footnote{The approach of allocating the risk across multiple entities to achieve more efficient risk sharing is known as the subscription model and is common in the insurance industry. In fact, at Lloyd’s of London, the world’s leading insurance market providing specialist insurance services to business, almost any single risk is insured (underwritten) by multiple insurers (underwriters). In his speech on the future of the insurance industry, Lord Levene, the former chairman of Lloyd’s of London, said that “The first point which I want to make about the future of insurance is that the subscription model is not just alive and well – it is thriving. Lloyd’s made record profits in 2009. Throughout the financial crisis, it maintained A+ ratings. Over three hundred years, it has never failed to pay a valid claim.” Source: http://www.lloyds.com/Lloyds/Press-Centre/Speeches/2011/03/The-Future-of-the-Insurance-Industry. Another practical reason of having multiple insurers for one risk is that an insurance company could avoid refusing customers by distributing large risk across many insurers and thus provide a complete service.} Our results imply that limited liability of both the insurance buyer and sellers together with insurers’ heterogeneity naturally gives rise to optimal claims splitting through a tranche structure, with different tranches characterized as the regions for which limited liability constraints are binding for different groups of insurers. The total uncertain loss is divided into several tranches, whose boundaries are the insurer-specific deductibles. Different insurers provide partial coverage for losses inside multiple tranches.

In the general risk-sharing problem, the social utility weights (Pareto weights) corresponding to different agents are specified exogenously. In contrast, Pareto weights in the optimal insurance design problem are determined endogenously by the insurers’ participation constraints. Another important and realistic aspect of the optimal insurance design problem is its intertemporal nature. Indeed, there is always a (sometimes significant) delay between the insurance premium payment and the arrival of an insurance event. We endogenize this important aspect of insurance markets by introducing intertemporal utility maximization for all agents. This allows us to express the Pareto weights via the marginal rates of intertemporal substitution and link them to various agents’ characteristics.

We next provide an overview of the problem of optimal insurance design with mul-
tiple insurers. The insurance buyer faces the risk of a random loss $X$ at time one, with maximal value $\bar{X}$. To protect against the loss, the buyer designs a basket of $N$ nonnegative indemnity schedules (also referred to as coverage functions) at time zero, with $N$ being less than or equal to the number of types of different insurers in the market. The indemnity schedule for insurer $i$ is characterized by a nonnegative payment $F_i = F_i(X)$ which is contingent on the realization of $X$ and is transferred from the insurer $i$ to the insured. Furthermore, the total insurance reimbursement cannot exceed the size of the loss $X$. To focus on the role of heterogeneity in preferences and endowments, we ignore the effects of asymmetric information and moral hazard and assume that the probability distribution of $X$ is exogenously given and all market participants agree on it. Given the insurers’ heterogeneous preferences, endowments and reservation utilities, the insured knows the lowest price, the premium, that each insurer requires for providing insurance against any given risk.

Following the characterization of the solution to the general risk-sharing problem, the optimal indemnities can be described in terms of an endogenously determined optimal tranching structure. The insurance premia determine the insurers’ time zero consumption and, consequently, their minimal marginal rate(s) of intertemporal substitution (MMRIS), which captures the minimal marginal premium they are willing to accept at time zero for providing insurance against an infinitesimal part of $X$ at time one. Then, the insurers are ranked according to their MMRIS and the optimal tranching of $X$ is

\[2\] In particular, the insurance buyer optimally selects insurers, so $N$ may happen to be strictly smaller than the number of available insurer types.

\[3\] Note that we do not need to require that market participants know the true distribution of $X$, but rather that they have the same beliefs about it. In particular, the risk premium they require for providing insurance could also be interpreted as an uncertainty premium the insurers charge for not knowing the exact distribution of $X$.

\[4\] We elaborate on this important concept in Section 5.
determined, \( X = \sum_j \text{Tranche}(Z_j, Z_{j+1}) \). Here, \( \text{Tranche}(a, b) = \max(a, \min(X, b)) - a \) and the tranche thresholds, \( 0 = Z_{N+1} \leq Z_N \leq \cdots \leq Z_1 \leq Z_0 = \bar{X} \), are defined in equilibrium given the insurers’ MMRIS.

By construction, one of the agents takes on all the risks below the threshold \( Z_N \) and this agent can be either the (i) insured or (ii) the insurer with the highest rank. We show that case (ii) takes place if and only if there exist \( K \geq 1 \) insurers whose MMRIS is smaller than the MMRIS of the insured. In this case, the minimal deductible is equal to zero and the insured chooses to buy full insurance coverage up to the level \( Z_{N-K+1} \) of losses. Furthermore, the \( \text{Tranche}(0, Z_N) \) is fully insured by the insurer with the smallest MMRIS, and \( Z_N \) serves as the deductible for the insurer with the second lowest MMRIS, who fully co-insurers the tranche \((Z_N, Z_{N-1})\) with the first insurer. The third insurance tranche \((Z_{N-1}, Z_{N-2})\) is fully co-insured by the three insurers with the lowest MMRIS, and so on. Finally the large losses (corresponding to \( X \geq \text{Tranche}(Z_{N-K+1}) \)) are never fully insured. Specifically, for every subsequent insurance tranche inside this high loss range, the insured adds the next highest ranked insurer to those who already provide insurance against smaller losses and buys additional partial insurance against this tranche from all these insurers. In case (i), the insured retains full exposure to small losses below the minimal deductible (i.e., \( X \leq Z_N \)), but otherwise, the insurance design is similar to that in case (ii): The insured gradually buys insurance in subsequent tranches from the insurers in the order of their increasing MMRIS, so that the insurer with the lowest MMRIS provides co-insurance against all tranches above the minimal deductible \( Z_N \), the insurer with the second lowest MMRIS co-insures all tranches above the deductible level \( Z_{N-1} \), etc.

This prioritized tranche-sharing structure is very intriguing. It arises because of insurers’ risk aversion and the heterogeneity of their marginal valuations. The insured optimally insures the first tranche above the minimal deductible with the insurer re-
quiring the lowest marginal premium. Because this insurer is risk averse, however, the marginal premium increases with the level of losses. Just as the level of losses reaches the next deductible level, the first insurer’s marginal premium reaches that of the second-highest ranked insurer, and it becomes optimal for the insured to buy co-insurance of the subsequent tranche from this second-highest ranked insurer. Continuing the process gradually, as the level of losses increases, insurers with higher marginal premia start participating in the trade, until the whole range of \( X \) is exhausted.

For efficient allocation corresponding to the optimal insurance design problem, the endogenously determined Pareto weights are proportional to the MMRIS and determine both the deductible levels and the whole structure of the optimal allocation. For this reason, investigating their dependence on the model parameters is an important and challenging problem. Our second main result is that the insurers’ MMRIS can be calculated through the fixed point of an explicitly constructed contraction mapping. This result is crucial, both for numerical calculation of optimal indemnities and for studying the dependence of deductibles on microeconomic characteristics. In particular, we use this result to derive several interesting comparative statics results.

2 Related Literature

As we have explained above, our general characterization of efficient allocations under limited liability constraints extends the classical results of Borch (1962) and Wilson (1968) and can therefore be applied to a large variety of economic problems such as Walrasian equilibrium allocations in complete markets under constraints. In particular, since we allow for heterogeneous discount factors, our results are related to those of Gollier and Zeckhauser (2005), who studied the effect of such a heterogeneity on effi-
cient intertemporal allocations.\footnote{See also a recent paper by Kazumori and Wilson (2009) that studied general efficient intertemporal allocations and extended Wilson (1968) to a dynamic setting.} We show that limited liability constraints together with heterogeneity in discount factors may lead to the failure of classical aggregation results.

In the literature on optimal insurance design, the most closely related to ours is the paper by Raviv (1979). He considered the same optimal insurance problem as ours, but with a single insurer and provided necessary and sufficient conditions for the optimality of a deductible. Thus, our results on the optimal insurance design can be viewed as an extension of Raviv (1979) to the case of multiple insurers. In addition, in contrast to Arrow (1971) and Raviv (1979), we also study the intertemporal aspect of optimal insurance design. This allows us to express the optimal allocation in terms of the marginal rates of intertemporal substitution and to link them to various agents’ characteristics.

Numerous papers studied optimality of deductibles in optimal insurance design in various settings, extending the original model of Raviv. See, for example, Doherty and Schlesinger (1983), Huberman, Mayers and Smith (1983), Blazenko (1985), Gollier (1987), Gollier (1996), Gollier and Schlesinger (1995,1996), Gollier (2004), Dana and Scarsini (2007). Eeckhoudt, Gollier, and Schlesinger (1991) studied the dependence of the optimal deductible on the distribution of losses. All these papers assumed that there is a single insurer. The only class of models with multiple insurers that has been extensively studied in the insurance literature corresponds to perfect risk sharing between insurers though a secondary complete capital market, which is unrealistic in many actual situations. See Aase (2008) for an overview.

Cohen and Einav (2007) and Cutler, Finkelstein and McGarry (2008) found empirical support for the importance of preferences heterogeneity in insurance design and its impact on the optimal deductible choice.

Finally, our paper is also related to the literature on optimal risk sharing via security design. See Allen and Gale (1994) and Duffie and Rahi (1995) for overviews of this
The remainder of the paper is organized as follows. Section 3 is devoted to the general risk sharing problem with limited liability. In Section 4 we formulate the optimal insurance design problem and characterize optimal indemnities for a finite number of insurers. Section 5 shows how optimal deductibles can be computed using the fixed point of a contraction mapping and provides several important comparative statics results. Section 6 presents a special case when all insurers have CARA utilities, in which we show the optimality of linear co-insurance inside each tranche. Section 7 investigates the effects of savings, fixed costs of insurance and reinsurance on insurance design. Section 8 concludes the paper and points out some future research directions.

3 Optimal Risk Sharing with Limited Liability: A General Result

Consider a group of agents indexed by $i = 1, \cdots, N+1$. All agents share identical beliefs and differ only in their wealth $w_i$ and their von Neumann-Morgenstern utility functions $u_i(c_i)$ satisfying the standard Inada conditions on their domain of definition. A social planner assigns Pareto weight $\mu_i$ to each agent $i$, $i = 1, \cdots, N + 1$ and wants to allocate either (1) a random loss or (2) a random gain of size $X \geq 0$ across the agents. The state contingent loss (or gain) function for agent $i$ is denoted by $F_i(X)$. We assume that the social planner has access to a complete set of state-contingent contracts, satisfying the limited liability constraint: $F_i(X) \geq 0$.\footnote{In case (2) (a random gain), the meaning of limited liability is clear: No agent is liable for an undesired outcome and will never suffer losses because of that. In case (1) (a random loss), assumption $F_i(X) \geq 0$ means that no sub-group of agents is liable for more than the total loss size $X$. Indeed, if $F_i(X) \geq 0$ is violated, agents $j \neq i$ in aggregate are effectively subsidizing agent $i$, taking on losses $X - F_i(X)$ that exceed their maximal liability $X$.} Then in the case of random loss, agent $i$’s wealth
is given by
\[ c_i(X) \equiv w_i - F_i(X) \] (1)
and the social planner’s problem is
\[
\max \left\{ \sum_i \mu_i E[u_i(w_i - F_i(X))] : \sum_i F_i(X) = X, F_i(X) \geq 0 \right\}.
\] (2)

We will also frequently use the inverse marginal utility \( q_i \) of agent \( i \), satisfying \( q_i(u'_i(x)) = x \). To describe the optimal allocation, we will need several definitions.

**Definition 3.1**

- **In case (1) (random loss):** For an agent \( i \), we denote by \( \text{rank}(i) \) the number the agent will have when all agents are reordered so that smaller \( \text{rank}(i) \) implies larger \( \mu_i u'_i(w_i) \). That is, \( \text{rank}(i) = N + 1 \) if agent \( i \) has the smallest \( \mu_i u'_i(w_i) \), \( \text{rank}(i) = N - 1 \) if insurer \( i \) has the second smallest \( \mu_i u'_i(w_i) \), etc., and \( \text{rank}(i) = 1 \) if agent \( i \) has the largest \( \mu_i u'_i(w_i) \).\(^7\)

- **In case (2) (random gain):** For an agent \( i \), we denote by \( \text{rank}(i) \) the number the agent will have when all agents are reordered so that larger \( \text{rank}(i) \) implies larger \( \mu_i u'_i(w_i) \).

Having defined the ranking order, we can define the deductible levels.

**Definition 3.2** Let \( Z_0 = \bar{X}, Z_{N+1} = 0 \). For any \( k \in \{1, \cdots, N\} \), let \( K = \text{rank}^{-1}(k) \) be the agent whose rank is equal to \( k \). Then,

- **In case (1) (a random loss),** let
\[
\tilde{Z}_k = \sum_{i : \text{rank}(i) \geq k} \left( w_i - q_i \left( \mu_i u'_i(w_i) \right) \right)
\] (3)

\(^7\)If two agents have the same \( Y_i \), we give them subsequent rankings in any order. However, it is important to do it so that \( \text{rank}(i) \neq \text{rank}(j) \) for \( i \neq j \).
• In case (2) (a random gain), let
\[
\tilde{Z}_k = \sum_{i: \text{rank}(i) \geq k} \left( q_i \left( \mu_i^{-1} \mu_K u'_K(w_K) \right) - w_i \right) .
\]

(4)

Then, we let
\[
Z_k = \min\{X, \tilde{Z}_k\}.
\]

A direct calculation (see Appendix) implies that:
\[
Z_0 \geq Z_1 \geq \cdots \geq Z_N \geq Z_{N+1} = 0.
\]

The indemnity schedule,
\[
\text{Tranche}(a, b) = \begin{cases} 
0 & , x < a \\
 x - a & , x \in (a, b) \\
b - a & , x > b 
\end{cases}
\]

will be referred to as a tranche. For simplicity, we denote:
\[
\text{Tranche}_j = \text{Tranche}(Z_{j+1}, Z_j).
\]

Note that Tranche\(_j\) will be empty if \(Z_j = Z_{j+1}\). We say that an agent \(i\) participates in the tranche, Tranche\(_j\), if \(F_i(x) \not\equiv 0\) for \(x \in (Z_{j+1}, Z_j)\).

The main result of this section is the following theorem.

**Theorem 3.3** There always exists a unique optimal allocation \(\{F_i\}_{i=1}^{N+1}\) with the following being true:

(1) Optimal contracts \(F_i\) are continuous and (weakly) monotone increasing in \(X\);
(2) For each $i$, $Z_{\text{rank}(i)}$ is the threshold corresponding to agent $i$, that is $F_i(X) > 0$ if and only if $X > Z_i$.

(3) For each Tranche $j$, there exists a function $\xi_j(X)$ such that

$$
\mu_i u'(c_i(X)) = \xi_j(X)
$$

(5) for all agents $i$ with $\text{rank}(i) \geq j + 1$.

We discuss only the case of a random loss here. The intuition in the case of a random gain is analogous. Theorem 3.3 shows that the social planner allocates the risks across agents according to the agents’ marginal utility losses. First, the risk is allocated to the agent with the smallest marginal utility loss. Then, as the level of $X$ increases, the marginal utility loss also increases and hits the level of the agent with the second smallest marginal loss when $X = Z_N$, making it optimal to allocate some of the risk to this “second best” agent. Continuing this process, the social planner gradually allocates all the loss among certain insurers. In between two successive thresholds (i.e., inside a given tranche), constraints are not binding and the allocation satisfies the well-known Borch (1962) rule (5): All marginal utilities are equalized across agents who share the risk.

Recall that

$$
R_i(x) = -\frac{u_i'(x)}{u_i''(x)}
$$

is the absolute risk tolerance of agent $i$. Wilson (1968) showed that the slopes of the sharing rules in a Pareto-efficient allocation can be characterized in terms of agents’ absolute risk tolerances. The following result is an extension of Wilson’s characterization for the constrained Pareto-efficient allocation in our model.
Proposition 3.4

\[
\frac{d}{dx} F_i(x) = \begin{cases} 
0, & x \leq Z_{\text{rank}(i)} \\
\frac{R_i(c_i)}{\sum_{j: \text{rank}(j) \geq k+1} R_j(c_j)}, & x \in (Z_{k+1}, Z_k), 0 \leq k \leq \text{rank}(i) - 1
\end{cases}
\]

The intuition behind the formula for the slope is the same as in Wilson (1968): The fraction of the aggregate risk agent \( i \) ends up taking is proportional to agent \( i \)'s risk tolerance. The set of agents with whom agent \( i \) is sharing risks, however, depends on the level of \( X \) and changes from tranche to tranche.

4 Optimal Insurance Design with Multiple Insurers

4.1 Model Setup

The model’s participants consist of an insurance buyer (the insured) and a set of \( N \) insurance sellers (the insurers). The insurance buyer faces the risk of a random loss, described by a nonnegative bounded random variable \( X \) with the largest potential loss \( \text{esssup} X = \bar{X} \). In addition, the insurance buyer is endowed with other (not explicitly modeled) assets, generating a cash flow \((w_0, w_1)\). The insurance buyer is an intertemporal expected utility maximizer, with von Neumann-Morgenstern utility \( U \) and a discount factor \( \delta \).

Insurer \( i, i = 1, \cdots, N \), is endowed with an income flow \((w_{0i}, w_{1i})\).\(^8\) Each insurer is an intertemporal expected utility maximizer, with a von Neumann-Morgenstern utility \( u_i \) and a discount factor \( \delta_i \). All utilities are assumed to satisfy standard Inada conditions on their domain of definition.

\(^8\)For simplicity, we assume that the cash flows \((w_0, w_1)\) and \((w_{0i}, w_{1i}), i = 1, \cdots, N\), are deterministic and exogenously given. Our results directly extend to the case when endowments are stochastic; however, the expressions become more complicated, and we omit it for the reader’s convenience. Our techniques can also be directly extended to allow for hedging and raising cash using bonds or other securities. See Section 7 below for details.
It is important to note that the preference parameters \((\delta, U)\) and \((\delta_i, u_i)\) should not be interpreted directly as the “true” preferences of the insured and the insurers. Rather, this is a stylized way to model the insured’s and the insurers’ subjective attitudes to the particular sources of risk in \(X\).\(^9\) Consequently, the insurer’s risk aversion, determined by \(u_i\), can be interpreted as the size of the risk premium the insurer requires for taking that particular type of risk, inherent in \(X\).

To (partially) insure against potential random loss \(X\), the insured designs a basket \(F_i(X), i = 1, \cdots, N\) of insurance contracts (also known as indemnity schedules, or, coverage functions), contingent on the realization of the loss \(X\). Because we are interested in the risk sharing problem, we assume that there is no asymmetric information and therefore the true probability distribution of \(X\) is known to all market participants. A basket of coverage functions is called admissible if, for all \(i\), \(F_i(X) \geq 0\) for all values of \(X\) (limited liability for the insured) and

\[
F = \sum_{i=1}^{N} F_i \leq X
\]

(limited liability for the insurers). That is, we assume that insurance reimbursement is always nonnegative and the total reimbursement cannot exceed the size of the loss. Given an insurance contracts design \((F_i)\), the insured retains exposure to the residual loss \(X - F\).

We assume that the insured can choose any basket satisfying the above admissibility conditions. The price, paid by the insured to insurer \(i\) (i.e., the insurance premium for the coverage function \(F_i\)) is denoted by \(P_i = P_i(F_i)\). We assume that both insurance provision and insurance design is potentially costly due to administrative expenses, search costs for finding insurers, etc. These costs are a deadweight loss to both insured and

\(^9\)For example, this attitude could depend on capital and other regulatory constraints.
insurers. For simplicity, we assume in our model that a variable cost of insurance coverage $F_i(X)$ is proportional to the premium, both for the insured and the insurer, with coefficients $\tau$ and $\theta$ respectively. Then, insurer $i$ is only getting a fraction of $(1 - \theta)P_i$, whereas the insured is actually paying $(1 + \tau)P_i$. Therefore, this model is equivalent to a model where insurance provision costs are zero (i.e., $\theta = 0$), whereas insurance coverage costs are given by $\alpha \equiv \frac{1 + \tau}{1 - \theta} - 1$. For notational convenience, we will from now on assume that this is the case and call $\alpha$ the proportional insurance cost.\footnote{For example, $\alpha$ may be related to administration expenses of the insurer at time zero, such as the underwriting cost, background checking cost, and so on. Arrow (1971, p. 204) writes: “It is very striking to observe that among health insurance policies of insurance companies in 1958, expenses of one sort or another constitute 51.6 percent of total premium income for individual policies, and only 9.5 percent for group policies.” We interpret this observation as a strong support for our assumption that the cost is proportional to the premium size, and suggests that $\alpha \in [0.1, 0.6]$ may be a reasonable interval of values for the proportional cost, depending on the precise circumstances. Another component of the cost is the insurance broker commission which is a fixed percentage of the premium quoted by an insurer. This also gives direct evidence of our way of modeling insurance cost. Source: http://www.willis.com/documents/publications/General_Publications/How_We_Get_Paid.pdf}

In Section 7.2, we also introduce fixed insurance coverage costs on top of the proportional costs.

As is common in the literature on optimal insurance design (see, e.g., Raviv (1979)), we assume that an insurer is willing to provide insurance coverage for $F_i(X)$ if and only if the premium $P_i$ satisfies the insurer’s participation constraint

$$u_i(c_{0i}) + \delta_i E[u_i(c_{1i})] \geq L_i,$$  \hspace{1cm} (6)

where

$$c_{0i} = w_0 + P_i, \quad c_{1i} = w_{1i} - F_i(X)$$  \hspace{1cm} (7)

is the insurer’s consumption after entering the contract and where

$$L_i = u_i(w_{0i}) + \delta_i u_i(w_{1i})$$
is the insurer’s reservation utility. Given the contracts \((P_i, F_i), i = 1, \cdots, N\), the insured’s consumption is given by:

\[
c_0 = w_0 - (1 + \alpha) \sum_{i=1}^{N} P_i, \quad c_1 = w_1 - X + F(X).
\] (8)

The problem of the insured is thus to design an admissible basket \((F_i)\) so as to maximize his expected utility,

\[
U(c_0) + \delta E[U(c_1)],
\]

under the budget constraints (8) and participation constraints (6).

Clearly, the insured will always optimally choose the premium to bind participation constraints (6) for the insurers, and therefore the insurance premium satisfies

\[
P_i(F_i) = -w_{0i} + v_i(L_i - \delta_i E[u_i(w_{1i} - F_i(X))]),
\] (9)

where \(v_i\) is the inverse of the insurer’s utility:

\[
v_i(u_i(x)) = x.
\]

Here, it should once again be pointed out that the preference parameters \((\delta_i, u_i)\) should not be interpreted directly as the “true” preferences of the insurers. Rather, it is a simple (and necessarily stylized) way of incorporating intertemporal substitution attitudes and risk aversion into insurance pricing. For example, if insurer \(i\) is risk neutral, we get \(P_i(F_i) = \delta_i E[F_i]\). This result is the classical actuarial fair value premium rule (see,

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11The assumption that the reservation utility coincides with the utility before entering the contract is made for technical purposes, to avoid discontinuities in the price \(P_i\). It is possible to relax this assumption at the cost of getting messier results.
e.g., Borch (1962)), and the difference $\ell_i \equiv \delta_i - 1$ is commonly referred to as the fixed percentage loading. In particular, it is always assumed that $\ell_i > 0$, that is $\delta_i > 1$. Thus, in our setting, we will allow for discount factors $\delta_i > 1$ and assume that $\delta_i$ incorporates both time discounting and fixed percentage loadings.

### 4.2 Single Insurer

In this section we present a solution to the optimal insurance design problem for the case of a single insurer. Since there is only one insurer $i = 1$, we use index $I$ as the subscript for the latter.

If the constraints $0 \leq F(X) \leq X$ are not binding, risk-sharing between the insured and the insurer is Pareto-efficient and maximizes social welfare:

$$
\max_F \left( U(w_1 - X + F) + a u_I(w_{1I} - F) \right)
$$

for some welfare weight $a > 0$. Writing down the first-order condition, we get $F(x) = g(a, x)$, where the function $g(a, x)$ is the unique solution to:

$$
a u'_I(w_{1I} - g) - U'(w_1 - x + g) = 0. \tag{10}
$$

By definition, our allocation is constrained Pareto-efficient, and therefore the function (10) describes the optimal allocation for those values of $X$ for which the constraints are not binding, and $F(X) = 0$ or $X$ when the corresponding constraint is binding. Below, we provide necessary and sufficient conditions for the constraints to be binding and describe the corresponding regions of values of $X$.

To this end, we need several definitions. Let

$$P_{\text{max}} = -w_{0I} + v_I (L_I - \delta_I E[u_I(\omega_{1I} - X)])$$
be the minimal premium for which the insurer is willing to provide full insurance against the whole loss $X$ (i.e., when the constraint $F(X) \leq X$ is binding for all values of $X$), and let

$$P_{\text{mid}} = -w_{0I} + v_I \left( L_I - \delta_I E \left[ u_I \left( w_{1I} + g \left( \frac{U'(w_1)}{u'_I(w_{1I})}, X \right) \right) \right] \right)$$

be the premium for a Pareto-optimal contract $F(X) = g(a, X)$ for which the constraints $0 \leq F \leq X$ are not binding and $a = U'(w_1)/u'_I(w_{1I})$ is simply the ratio of marginal utilities of time-one endowments. Then, define

$$K_{\text{min}} = \log \frac{U'(w_1)/U'(w_0 - (1 + \alpha)P_{\text{max}})}{(1 + \alpha)u'_I(w_{1I} - X)/u'_I(w_{0I} + P_{\text{max}})},$$

$$K_{\text{mid}} = \log \frac{U'(w_1)/U'(w_0 - (1 + \alpha)P_{\text{mid}})}{(1 + \alpha)u'_I(w_{1I})/u'_I(w_{0I} + P_{\text{mid}})},$$

and

$$K_{\text{max}} = \log \frac{U'(w_1 - \bar{X})/U'(w_0)}{(1 + \alpha)u'_I(w_{1I})/u'_I(w_{0I})}.$$

All three numbers, $K_{\text{max}}, K_{\text{mid}}, K_{\text{min}}$, are given by differences between the growth rates of the undiscounted marginal values of consumption of the insured and the insurer. $K_{\text{max}}$ corresponds to the case when the insurer provides full insurance against $X$ and is calculated at the maximal level $\bar{X}$. Similarly, $K_{\text{mid}}$ corresponds to the case when the constraints $0 \leq F \leq X$ are not binding and is calculated at $X = 0$. Finally, $K_{\text{min}}$ corresponds to the case when the constraint $F \geq 0$ is binding for all values of $X$ so that there is no trade ($F = 0$), and the constraint is calculated at the maximal level $X = \bar{X}$.

It follows directly from the definition that

$$K_{\text{max}} > K_{\text{mid}} > K_{\text{min}}.$$

We also need the inverse of the insurer’s marginal utility $q_I$ and the inverse of the insured’s
marginal utility $Q$:

\[ q_I(u'_I(x)) = x \]
\[ Q(U''(x)) = x. \]

Under the conditions described, we can state:

**Theorem 4.1** (1) **If**

\[ \log \frac{\delta_I}{\delta} < K_{\min}, \]

*then full insurance is optimal,*

\[ F(X) = X; \]

(2) **if**

\[ K_{\min} < \log \frac{\delta_I}{\delta} < K_{\text{mid}}, \]

*then there exists a threshold $Z(a) \in (0, \bar{X})$ such that $F(X)$ is a combination of full insurance coverage for $X < Z(a)$ and Pareto-optimal sharing (10) for $X \geq Z(a)$,*

\[ F_a(X) = \begin{cases} 
X & , X \leq Z(a) \\
\frac{g(a,X)}{X} & , X > Z(a)
\end{cases} \]

(3) **if**

\[ K_{\text{mid}} < \log \frac{\delta_I}{\delta} < K_{\max}, \]

*then the optimal policy is characterized by a deductible $Z(a) \in (0, \bar{X})$ and optimal*
risk sharing according to (10) for $X \geq Z(a)$,

$$F_{a}(X) = \begin{cases} 0, & X \leq Z(a) \\ g(a, X), & X > Z(a) \end{cases}$$

and

(4) if

$$K_{\text{max}} < \log \frac{\delta}{\delta},$$

then there is no trade; that is, $F(X) = 0$.

Furthermore,

$$Z(a) = \begin{cases} w_{11} - q_{I}(a^{-1}U'(w_{1})) & \text{in case (2)} \\ w_{1} - Q(a u_{I}'(w_{11})) & \text{in case (3)} \end{cases},$$

and $a$ is the unique solution to

$$a = \frac{(1 + \alpha) \delta^{-1} U'(w_{0} - (1 + \alpha) P_{I}(F_{a}(X)))}{\delta^{-1} u_{I}'(w_{0I} + P_{I}(F_{a}(X)))},$$

(12)

where $P_{I}$ is given by (9).

To gain a better understanding of the intuition behind the optimal risk sharing of Theorem 4.1, we note that the minimal marginal premium ($\pi_{I}$) the insurer is willing to accept at time zero for an additional unit of insurance coverage at time one is given by the marginal rate of intertemporal substitution (MRIS) of the insurer,

$$\pi_{I} = \frac{\delta_{I} u_{I}'(c_{1B})}{u_{I}'(c_{0B})}.$$  (13)

Similarly, the maximal marginal premium $\pi$ that the insured is willing to pay at time
zero is given by:
\[ \pi = \frac{\delta U'(c_1)}{(1 + \alpha) U'(c_0)}, \]  

which is equal to the MRIS of the insured.\(^{12}\) Therefore, the trade will take place only if
\[ \pi_I \leq \pi. \]  

In general, the inequality \(\pi_I \leq \pi\) will be violated for some levels of \(X\). If the quotient \(\delta_I/\delta\) of the discount factors is sufficiently small, insurance is cheap and the insured will buy full insurance coverage (i.e., case (1)). If the quotient \(\delta_I/\delta\) is low, but not too low (i.e., case (2)), inequality (15) will hold for small values of \(X\). However, as the losses \(X\) become sufficiently large, the marginal loss \(u'_I(c_{1I})\) for the insurer is too high. In this case, insurance is expensive and it is optimal for the insured to retain a part of the exposure to losses in the upper tail of \(X\). Finally, if the quotient \(\delta_I/\delta\) is very high (i.e., cases (3) and (4)), insurance is too expensive and it is optimal for the insured to retain full exposure to \(X\), at least for low values of \(X\).

It is important to contrast the result of Theorem 4.1 with the classical result of Raviv (1979). The key difference between our setting and that of Raviv is that our problem is *intertemporal* and optimal insurance design involves a decision on substituting between consumption today and tomorrow. By contrast, Raviv considers a static (zero-period) model where both the premium payment and the claim payout occur at the same time instant. In addition, Raviv assumes that claims settlement is costly for the insurer, with the cost being a function of insurance payment (i.e., \(C(F)\)).\(^{13}\) Raviv shows that the optimal contract is of the same form as in Theorem 4.1: either full coverage up

\(^{12}\) Here, the factor \((1 + \alpha)\) appears in the denominator because of the insurance cost. However, we still refer to this quantity as the MRIS for convenience.

\(^{13}\) These costs can be easily incorporated into our model by a simple modification of the insurer’s utility \(u_i\) at time 1, if we set \(u_{i1}(F_i(X)) = u_i(w_{1i} - F_i(X) - C(F_i(X)))\).
to a threshold, or a deductible. Furthermore, he shows that the deductible is equal to zero if and only if $C' = 0$, that is, $C$ is a constant. Theorem 4.1 shows that the introduction of intertemporal substitution into the insurance design problem changes the result completely. Namely, the structure of the optimal contract is determined by the marginal rates of intertemporal substitution. In particular, Raviv’s result does not hold true in our setting: Even with non-trivial settlement costs, deductible is always equal to zero if the difference between the MRIS of the insured and that of the insurer is sufficiently small. Furthermore, we obtain explicit characterization of optimality of both deductible and full coverage.

4.3 Heterogeneous Insurers

In this section we characterize the optimal allocation for the case of multiple heterogeneous insurers. To understand why heterogeneity is important, let us first examine the case when all insurers are risk neutral. In that case insurer $i$ is willing to accept the premium

$$P_i(F_i) = \delta_i E[F_i(X)]$$

for an indemnity $F_i$. Therefore, diversifying between different insurers is never optimal for the insured. The insurer with the smallest discount factor $\delta_{\text{min}}$ will always be the one to provide the cheapest insurance, and the insured will always buy insurance against the total indemnity $F = \sum_i F_i$ from this insurer because the premium is linear.\(^{14}\) Thus, with risk neutrality, heterogeneity does not play any role in optimal insurance design. However, when insurers are risk averse, the situation is completely different: Every insurer provides insurance against a non-zero part of $X$ if the the maximal loss $\bar{X}$ is sufficiently large.

\(^{14}\)Indeed, $\sum_i P_i(F_i) \geq \delta_{\text{min}} \sum_i E[F_i]$. 

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Proposition 4.2 The following are true:

- If all insurers are risk neutral, then only the insurer with the smallest discount factor will participate in a trade.

- If insurers are risk averse and $\bar{X}$ is sufficiently large, then all insurers will participate in a trade.

The drastic difference between the risk-neutral case and the risk-averse case arises because the marginal premium that an insurer requires for providing insurance against an additional unit of $X$ is monotone increasing with the level of $X$. Suppose, for example, that there are two insurers, 1 and 2, and the discount factor $\delta_1$ of insurer 1 is much lower than $\delta_2$. Then, clearly, insurer 1 will be ready to provide cheaper insurance than insurer 2. However, as the level of $X$ becomes sufficiently high, insurer 1’s consumption decreases. Therefore, his MRIS increases with $X$ and eventually becomes higher than that of insurer 2. It thus becomes optimal for the insured to buy (partial) insurance against the high-level portion of $X$ from insurer 2.

Since the optimal allocation for the insurance design problem is necessarily efficient, it can be derived from the general result of Theorem 3.3. In order to establish a direct link with the general setting of Section 3, let

$$F_{N+1} = X - \sum_i F_i(X)$$

and $u_{N+1}(c) = U(c)$. Then, the optimal allocation solves the social planner’s problem

$$\max \left\{ \sum_{i=1}^{N+1} \mu_i \left( u_i(c_{0i}) + \delta_i E[u_i(w_{1i} - F_i(X))] \right) \right\}$$

under budget constraints (7) and (8) and the participation constraints (6). In particular, insurers’ participation constraints determine the endogenous Pareto weights $\mu_i$. The
Intertemporal budget constraint implies that the Pareto weights are given by

\[ \mu_i = \frac{\delta_i}{u'_i(c_{0i})}, \quad i = 1, \cdots, N, \quad \text{and} \quad \mu_{N+1} = \frac{\delta}{(1 + \alpha)U'(c_0)}. \]

Consequently, by Theorem 3.3, the ranks of all agents and the optimal deductibles are determined by the minimal marginal rates of intertemporal substitution (MMRIS)\(^{15} \quad ^{16}\)

\[ Y_i = \frac{\delta_i u'_i(w_{1i})}{u'_i(c_{0i})}, \quad i = 1, \cdots, N, \]

of the insurers and the insured’s MMRIS

\[ Y = \frac{\delta U'(w_1)}{(1 + \alpha)U'(c_0)}. \]

Here, the only difference is that agent \(N + 1\) is special and assigning a rank to him does not make much sense. For this reason we will need a slight modification of Definition 3.1.

**Definition 4.3** For an insurer \(i\), we denote by \(\text{rank}(i)\) the number that insurer \(i\) will have when all insurers are reordered so that smaller \(\text{rank}(i)\) implies larger \(Y_i\). Furthermore, we denote by \(J\) the number of insurers for which \(Y_i\) is larger than \(Y\).

Note that the number \(J\) plays an important role because, in the sense of Definition 3.1, the rank of the insurer (agent \(N + 1\) in our notation) is \(\text{rank}(N + 1) = J + 1\).

Having defined the ranking order, we now restate Definition 3.2 using the new notation.

\(^{15}\)It is minimal because \(c_{1i} = w_{1i} - F_i(X) \leq w_{1i}\) and therefore \(\frac{\delta_i u'_i(c_{1i})}{u'_i(c_{0i})} \geq \frac{\delta_i u'_i(w_{1i})}{u'_i(c_{0i})}\).

\(^{16}\)Note that fixing \(Y_i\) is equivalent to fixing the insurance premia \(P_i(F_i)\) because \(c_{0i} = w_{0i} + P_i(F_i)\) and \(c_0 = w_0 - (1 + \alpha) \sum_i P_i(F_i)\).
Definition 4.4 For each \( i = 1, \cdots, N \), let:

\[
a_i \equiv \mu_i \mu_{N+1}^{-1} = \frac{\delta^{-1} (1 + \alpha) U'(c_0)}{\delta_i^{-1} u'_i(c_0)}.
\] (17)

Fix \( k \in \{0, 1, \cdots, N, N+1\} \).

- For \( k = 0 \) we define
  \[
  Z_0 = \bar{X}.
  \]

- For \( 1 \leq k \leq J \), let \( K = \text{rank}^{-1}(k) \) be the insurer whose rank is equal to \( k \) and

\[
\tilde{Z}_k = w_1 - Q(a_K u'_K(w_{1K})) + \sum_{i : \text{rank}(i) \geq k+1} (w_{1i} - q_i \left(a_i^{-1} a_K u'_K(w_{1K})\right))
\] (18)

and

\[
Z_k = \min\{\bar{X}, \tilde{Z}_k\}.
\]

- For \( k = J + 1 \) we define:

\[
\tilde{Z}_{J+1} = \sum_{i : \text{rank}(i) \geq J+1} (w_{1i} - q_i \left(U'(w_1) a_i^{-1}\right))
\] (19)

and

\[
Z_{J+1} = \min\{\bar{X}, \tilde{Z}_{J+1}\}.
\]

- For \( J + 2 \leq k \leq N \), let \( K = \text{rank}^{-1}(k - 1) \) and

\[
Z_k = \sum_{i : \text{rank}(i) \geq k} (w_{1i} - q_i \left(a_i^{-1} a_K u'_K(w_{1K})\right))
\] (20)

and

\[
Z_k = \min\{\bar{X}, \tilde{Z}_k\}.
\]
• For \( k = N + 1 \), we define \( Z_{N+1} = 0 \).

We are now ready to state the main result of this section.

**Theorem 4.5** There always exists a unique optimal allocation \( \{F_i\}_{i=1}^N \). It is non-zero (i.e., \( F(X) \neq 0 \)) if and only if:

\[
\frac{\delta U'(w_1 - \bar{X})}{(1 + \alpha) U'(w_0)} > \min_i \frac{\delta_i u'_i(w_{1i})}{u'_i(w_{0i})}.
\]

(21)

If (21) holds, then the following is true:

1. Optimal indemnities \( F_i \) and the uninsured part \( X - F(X) \) are continuous and (weakly) monotone increasing in \( X \);

2. If \( Y_i > Y \), then the insurer \( i \) only participates in tranches \( \text{Tranche}_j \) with indices \( j \leq \text{rank}(i) - 1 \);

3. If \( Y_i \leq Y \), then the insurer \( i \) only participates in tranches \( \text{Tranche}_j \) with indices \( j \leq \text{rank}(i) \);

4. The insured buys full insurance against the part of \( X \) below \( Z_{J+1} \) and retains a partial exposure to \( X \) (co-insurance) for \( X > Z_{J+1} \). That is,

\[
F(X) = \sum_i F_i(X) = X
\]

if \( X \leq Z_{J+1} \) and \( F(X) < X \) otherwise;

5. For each \( \text{Tranche}_j \), there exists a function \( \xi_j(X) \) such that:

\[
\frac{\delta_i u'_i(c_{1i})}{u'_i(c_{0i})} = \xi_j(X)
\]

(22)

for each insurer \( i \) participating in \( \text{Tranche}_j \). Furthermore,
(a) If $j \geq J + 1$ (full insurance region), then:

$$\xi_j(X) < \frac{\delta U'(c_1)}{(1 + \alpha) U'(c_0)} \quad \text{and}$$

(b) If $j < J + 1$, then:

$$\xi_j(X) = \frac{\delta U'(c_1)}{(1 + \alpha) U'(c_0)}. \quad (23)$$

First, we note that Equation (22) and Equation (23) uniquely determine the allocation. Indeed, substituting $c_{1i} = w_{1i} - F_i(X)$ into (22) gives $F_i = w_{1i} - q_i(\xi_j(X) u'_i(c_{0i}) \delta_i^{-1})$. Then, for $j \leq J + 1$, the function $\xi_j$ is uniquely determined by the constraint $\sum_i F_i(X) = X$, and for $j \geq J$, the function $\xi_j$ is uniquely determined by (23) and the insured’s budget constraint $c_1 = w_0 - X + \sum_i F_i(X)$.

As we have explained above, the structure of the optimal insurance basket is determined by the insurers’ MRIS (22). Since both the insured’s consumption $c_1 = w_1 - X + F(X)$ and the insurers’ consumption $c_{1i} = w_{1i} - F_i(X)$ are monotone decreasing in $X$, the marginal values $U'(c_1)$ and $u'_i(c_{1i})$ of the insured’s and insurers’ consumption are monotone increasing in $X$. If the MMRIS of all insurers are larger than that of the insured, then there exists a strictly positive deductible $Z_N$ and the tranche Tranche$_N$ is not insured at all. Consequently, for $X \in [0, Z_N]$, the MRIS of each insurer $i$ stays constant (equal to $Y_i$), whereas the MRIS of the insured is monotone increasing with the value of $X$. By contrast, if there is at least one insurer $i$ whose MMRIS is smaller than that of the insured, then the insured buys full insurance coverage against low levels of $X$ (i.e., Tranche$_N = [0, Z_N]$) from the insurer with the highest rank $N$. Furthermore, since $F(X) = X$ for all $X \in [0, Z_{J+1}]$, the MRIS of the insured stays constant (equal to $Y$).

---

17 That is, $\min_j Y_j > Y$

18 That is, $\min_j Y_j < Y$. 

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when $X$ varies in this interval, whereas the MRIS of insurers with ranks higher than $J$ are monotone increasing with the value of $X$. Finally, for $X \geq Z_{J+1}$, both the MRIS of the insured and the MRIS of all insurers that provide co-insurance against $X$ are monotone increasing in $X$. As the level of $X$ increases and hits the next deductible level $Z_i$, insurance from insurers participating in $[0, Z_i]$ gets too expensive, and it becomes optimal to buy co-insurance from the next insurer in the hierarchy structure determined by the ranks.

We summarize this situation in the following important result.

**Corollary 4.6** There exists an increasing sequence of insurer-specific deductibles such that each insurer provides (co-)insurance only above the corresponding deductible level.

As an illustration for the structure of optimal indemnities, consider the following example.

**Example.** Suppose there are three insurers with $Y_1 > Y > Y_2 > Y_3$.

Then, $J = 1$ and

$$\bar{X} = Z_0 > Z_1 > Z_2 > Z_3 > Z_4 = 0$$

if $\bar{X}$ is sufficiently large. In this case, optimal indemnities have the following structure:

- For $x \leq Z_3$, $F_3(x) = x$, so insurer 3 provides full insurance against $X$ for losses $X \leq Z_3$;

- For $x \in [Z_2, Z_3]$, $F_2, F_3 > 0$ and $F_2 + F_3 = X$, so insurers 2 and 3 provide full insurance against $X \in [Z_2, Z_3]$ and share the risk in an efficient way;
• For \( x \in [Z_1, Z_2] \), insurers 2 and 3 provide partial insurance against \( X \in [Z_1, Z_2] \), and the insured retains partial exposure to \( X \). That is, on that interval \( F_1(x) = 0 \), \( F_2(x), F_3(x) > 0 \) and \( F_2(x) + F_3(x) < x \); and

• Finally, for \( x > Z_1 \), \( F_1(x), F_2(x), F_3(x) > 0 \) and all three insurers provide partial insurance, so that \( F_1 + F_2 + F_3 < X \).

In particular, the insurer-specific deductibles are \( Z_3 \) and \( Z_1 \). The threshold \( Z_2 \) is not a deductible, but serves as a boundary between the full insurance regime \([0, Z_2]\) and the partial co-insurance regime \([Z_2, \bar{X}]\). Insurer 1 starts participating “with delay,” only after the intermediate tranche \([Z_2, Z_1]\) is (partially) insured by insurers 2 and 3.

5 Fixed-Point Equation and Comparative Statics

By Theorem 4.5, the optimal allocation is uniquely determined as soon as we know the rank of every insurer, as well as the thresholds \( Z_k \). By Definitions 4.3 and 4.4, both the ranks and the thresholds are uniquely determined by the \( N \)-tuple of numbers \((a_i)\). Given the \( N \)-tuple \((a_i)\), we denote \( b_i = a_i^{-1} \) as their reciprocals and denote by \( \mathbf{b} = (b_i) \) the vector of these reciprocals. We denote by \((Z_i(b), i = 0, \ldots, N+1)\) the corresponding thresholds and by \((F_i(b), i = 1, \ldots, N)\) the corresponding allocation. By definition (see (17)), the optimal allocation must satisfy:

\[
\begin{align*}
    b_i & = \frac{\delta_i^{-1} u'_i(c_{0i})}{(1 + \alpha) \delta^{-1} U'(c_0)} = \frac{\delta_i^{-1} u'_i(w_{0i} + P_i(F_i(b)))}{(1 + \alpha) \delta^{-1} U' \left( w_0 - (1 + \alpha) \sum_j P_j(F_j(b)) \right)} \quad (24)
\end{align*}
\]

for all \( i = 1, \ldots, N \). This is a highly non-linear system of equations for vector \( \mathbf{b} \). It is by no means clear how to solve it analytically or even numerically and how the solution would depend on the microeconomic characteristics of the model.

In this section we prove that this \( N \)-tuple is the unique fixed point of a contraction
mapping defined on an explicitly given compact set and can therefore be easily calculated by successive iterations.

We use the common notation $b_{-i}$ to denote the vector of all coordinates of $b$ except for $b_i$.

**Lemma 5.1** For each $i = 1, \cdots, N$, there exists a unique, piecewise $C^1$ function

$$H_i = H_i(C, b_{-i})$$

solving

$$H_i(C, b_{-i}) = \delta_i^{-1} C u'_i (w_{0i} + P_i (F_i(X, (H_i(C, b_{-i}), b_{-i})))) .$$

(25)

The function $H_i$ is monotone increasing in $C$ and $b_{-i}$, and $C^{-1} H_i$ is decreasing in $C$.

Now, we are ready to formulate the main result of this section. To this end, we need some definitions. Let

$$P_{i}^{\text{max}} = -w_{0i} + v_i (L_i - \delta_i E[u_i(w_{1i} - X)])$$

be the premium that the insurer $i$ is asking for providing full insurance against $X$.

$$C_{\text{max}} = (\delta^{-1} U'(w_0))^{-1} , \quad C_{\text{min}} = \left(1 + \alpha\right) \delta^{-1} U'(w_0 - (1 + \alpha) \sum_i P_{i}^{\text{max}}) \right)^{-1}$$

and

$$\beta_{i}^{\text{min}} = \log\left(C_{\text{min}} \delta^{-1} u'_i (w_0 + P_{i}^{\text{max}})\right) , \quad \beta_{i}^{\text{max}} = \log\left(C_{\text{max}} \delta^{-1} u'_i (w_0)\right) .$$

For simplicity, we always assume that the price $P_{i}^{\text{max}}$ is well defined for any insurer $i$. This assumption is only necessary when dealing with utilities that are either defined on a half-line or are bounded from above. It can be relaxed at the cost of more technicalities, and we omit it for the reader’s convenience.
We denote:
\[ \Omega = \times_i [\beta_i^{\min}, \beta_i^{\max}] \, . \]

Also, let
\[ \|x\|_{l^\infty} = \max_i |x_i| \]
be the \(l_\infty\)-norm of a finite sequence, equal to the maximal absolute value of its elements.

The following lemma is the main technical result of this section.

**Lemma 5.2 (contraction lemma)** For any \(C > 0\), the mapping \(G_C\) defined via
\[ (G_C)_i(d) = \log H_i(C, e^{d_{i-1}}) \]
maps the compact set \(\Omega\) into itself and is a strict contraction with respect to \(\|\cdot\|_{l_\infty}\).

Consequently, there exists a unique fixed point \(d^*(C) \in \Omega\) of this map, solving:
\[ d^*(C) = G_C(d^*(C)) \, . \]

For any \(d_0 \in \Omega\), we have:
\[ d^*(C) = \lim_{n \to \infty} (G_C)^n(d_0) \, . \]

The result of Lemma 5.2 is quite surprising because it holds under absolutely no restrictions on model parameters. In particular, we do not need to impose any smallness conditions typically used in economic applications of the contraction mapping theorem.

The last technical result we need is the following lemma.

**Lemma 5.3** There exists a unique number \(C^* \in (C_{\min}, C_{\max})\) solving:
\[ C = \left( (1 + \alpha) \delta^{-1} U \left( w_0 - (1 + \alpha) \sum_i \left( q_i(e^{d_i^*(C)} \delta_i C^{-1}) - w_{0i} \right) \right) \right)^{-1} \, . \] (26)
Now we are ready to state the main result of this section.

**Theorem 5.4** Let $b^*(C^*) = e^{d^*(C^*)}$. The optimal allocation is given by $(F_i)(b^*(C^*))$.

Theorem 5.4 provides a directly implementable algorithm for calculating the optimal allocation: Vector $d(C)$ can be calculated by successive iterations using Lemma 5.2, and then $C^*$ can be found using any standard numerical procedure for solving (26).

The characterization of the optimal allocation provided by Theorem 5.4 is perfectly suited for studying comparative statics. We need the following lemma.

**Lemma 5.5 (comparative statics lemma)** If the right-hand sides of (25) and (26) are monotone increasing in some parameter, then so are $C^*$ and $d^*(C^*)$.

By (18)-(20) and Theorem 4.5, all deductibles and other characteristics of the optimal indemnities can be expressed in terms of $a_i = e^{-d_i}$, thus we can use Lemma 5.5 to study the dependence of the optimal allocation on various model parameters. Let:

$$Z_{\text{full coverage}} \overset{\text{def}}{=} \max\{X : F(X) = X\}.$$ 

By Theorem 4.5, we have that $Z_{\text{full coverage}} = Z_{J+1}$ is positive if and only if $\min_i Y_i < Y$.

We will refer to the insurance tranches that are fully insured as senior tranches. We also let:

$$\#\{\text{senior}\} = \#\{i : Y_i < Y\}$$

be the number of insurers participating in the tranches that are fully insured. By construction, $\#\{\text{senior}\}$ is also the number of senior insurance tranches.

If $Z_{\text{full coverage}} = 0$, we define:

$$Z_{\text{deductible}} = \max\{x : F(x) = 0\}.$$
to be the minimal deductible $Z_N$; i.e., the threshold below which the insured is fully exposed to losses.

Finally, for each insurer $i$ we define:

$$\text{index}(i) = \begin{cases} 
1, & \text{if } \text{rank}(i) > J \\
0, & \text{if } \text{rank}(i) \leq J 
\end{cases}.$$ 

That is, an insurer’s index is one if the insurer participates in the tranches that are fully insured and zero otherwise.

**Definition 5.6** We say that a change in the parameters of the model leads to more insurance coverage if it leads to an increase (in the weak sense) in:

- $\#\{\text{senior}\}$,
- $Z_{\text{full coverage}}$,
- $\text{index}(i)$ for each $i$,

and to a decrease (in the weak sense) in $Z_{\text{deductible}}$.

That is, more insurance implies that a larger part of $X$ is fully insured and more insurers participate in the fully insured senior tranches.

The next proposition describes the effect of a first-order stochastic dominant (FOSD) shift in the distribution of $X$, as well as the effect of changes in the insured’s initial wealth and discount factor on the optimal allocation.

**Proposition 5.7** A decrease in the distribution of $X$ in the FOSD sense, an increase in $w_0$, or an increase in $\delta$ lead to more insurance. In particular, there exists a threshold value for $\delta$ such that $Z_{\text{deductible}}$ is positive if and only if $\delta$ is below this threshold,\(^{20}\) and similarly for $w_0$.

\(^{20}\)Here, we allow $\delta$ to vary and keep the rest of the parameters fixed.
The fact that larger initial endowment and larger discount factor lead to more insurance is very intuitive: Indeed, if \( \delta \) is large, future shocks are more important for the insured, forcing the insured to acquire more insurance. Similarly, a larger initial capital \( w_0 \) leads to a smaller marginal loss from buying insurance at time zero, again making it optimal to buy more insurance. By contrast, the fact that larger losses\(^{21}\) lead to less insurance is quite surprising. The reason is that, with larger risk, insurance gets more expensive on average. Therefore, it is optimal for the insured to increase exposure to low levels of \( X \) (achieved by raising the deductible level) and, simultaneously increase insurance coverage for higher levels on \( X \), reducing the probability of large losses.

We close this section with an analogous comparative statics result for insurance cost \( \alpha \). It turns out that the situation is more complex. The following is true:

**Proposition 5.8** If the relative risk aversion \(-\frac{cU''(c)}{U'(c)}\) of the insured is below 1 for \( c \) in the attainable consumption interval \([w_0 - (1 + \alpha) \sum_i p_{i,\max}, w_0]\) then an increase in cost \( \alpha \) leads to less insurance. By contrast, if \(-\frac{cU''(c)}{U'(c)}\) of the insured is above 1 in the attainable consumption interval and \( w_0 \) is sufficiently small, then an increase in the cost \( \alpha \) leads to more insurance.

Thus, the effect of an increase in costs of the optimal allocation depends on the position of the insured’s relative risk aversion with respect to 1. If the insured is relatively risk tolerant (risk aversion below one), an increase in the costs will make it optimal for the insured to retain a larger exposure to losses because insurance is too expensive. In contrast, if the insured’s risk aversion is relatively high (above one) and the marginal value of payment is high, the incentive to insure against \( X \) will go up, leading to more insurance.

\(^{21}\)In the sense of first order stochastic dominance.
6 CARA Preferences

6.1 General Results

In this section we consider the benchmark case when all insurers, as well as the insured, have exponential (CARA) utilities:

\[ u_i(c) = A_i^{-1}(1 - e^{-A_i c}) , \quad U(c) = A^{-1}(1 - e^{-Ac}) . \]

In this case, the optimal indemnities take a particularly simple form. Since absolute risk tolerance is constant for CARA utilities, Proposition 3.4 yields the following:

**Proposition 6.1** For each \( i \), insurer \( i \) provides linear co-insurance inside each tranche. Namely, for \( \text{rank}(i) \geq J + 1 \),

\[ F_i = \sum_{k=0}^{\text{rank}(i)} \kappa_{ik} \text{Tranche}_k, \]

with

\[ \kappa_{ik} = \begin{cases} A_i^{-1} \frac{A_i^{-1}}{\sum_{j : \text{rank}(j) \geq k} A_j^{-1}} , & \text{for } k \geq J + 1 \\ A^{-1} + \sum_{j : \text{rank}(j) \geq k+1} A_j^{-1} , & \text{for } k \leq J \end{cases} \]

For \( \text{rank}(i) \leq J \),

\[ F_i = \sum_{k=0}^{\text{rank}(i)-1} \kappa_{ik} \text{Tranche}_k, \]

with

\[ \kappa_{ik} = \frac{A_i^{-1}}{A^{-1} + \sum_{j : \text{rank}(j) \geq k+1} A_j^{-1}} . \]

The simple piece-wise linear structure of optimal indemnities simplifies the numerical procedure. Therefore, all numerical examples in the paper assume that all agents have CARA preferences. We use the following agents’ profiles in our examples:
Table 1: Agent Profiles

<table>
<thead>
<tr>
<th>Agent</th>
<th>$\delta$</th>
<th>$A$</th>
<th>$w_0$</th>
<th>$w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insured</td>
<td>1.00</td>
<td>1.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Insurer 1</td>
<td>1.25</td>
<td>0.4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Insurer 2</td>
<td>1.20</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Insurer 3</td>
<td>1.15</td>
<td>0.6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Insurer 4</td>
<td>1.10</td>
<td>0.7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Insurer 5</td>
<td>1.05</td>
<td>0.8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We first consider the case when $X$ takes either value 0 with probability $1 - p$ or a random value, uniformly distributed on $[0, 2]$, with probability $p$. Thus, $p$ is the probability that the insurance event (a disaster) occurs, and, conditional on its occurrence, losses are uniformly distributed. Figure 1(a) shows a plot of the tranche thresholds $Z_1, \cdots, Z_5$ against the disaster probability $p$.

Then, we fix the probability $p = 0.3$ and vary the distribution of losses. Conditioned on the insurance event, we allow $X$ to be distributed with the density $n x^{n-1}$ on $[0, 2]$ for some $n > 0$. Clearly, increasing $n$ makes the distribution of $X$ positively skewed and increases the losses in the FOSD sense. Figure 1(b) shows a plot of the tranche thresholds $Z_1, \cdots, Z_5$ against the skewness parameter $n$.

To illustrate the effect of the insured’s discount factor $\delta$ on the optimal allocation, we fix $p = 0.3$ and assume as above that $X$ is uniformly distributed on $[0, 2]$. Figure 2 shows a plot of the deductibles $Z_1, \cdots, Z_5$ against $\delta$. Figures 1 and 2 clearly show that the minimal deductible $Z_5$ is monotone increasing in the size of losses and decreasing in the insured’s discount factor $\delta$, in complete agreement with Proposition 5.7.

In order to understand the dependence on other model parameters, we have to understand the exact interplay between the benefits of insurance and the effective losses that the insured suffers from paying the premia at time zero. The joint effect of these two mechanisms on the insured’s MRIS may be generally ambiguous. However, when the insurer’s risk aversion is sufficiently small, his MRIS is almost insensitive to changes in
Figure 1: The Effect of Riskiness
wealth and therefore stronger comparative statics results can be obtained. The following is true.

**Proposition 6.2** If the insured’s risk aversion is sufficiently small then adding an insurer to the population of insurers, decreasing the initial endowment $w_{0i}$ of some insurer $i$, or decreasing an insurer’s discount factor $\delta_i$ always leads to more insurance.\(^{22}\)

Figures 3 and 4 show the optimal insurance design as we increase the number of insurers from two to five. In complete agreement with Proposition 6.2, we see that increasing the number of insurers always leads to more insurance. However, it is important to note that Proposition 6.2 does not generally hold when the insured’s risk aversion is sufficiently large. The reason is that, when the insured is risk averse, the marginal value of an additional unit of consumption at time zero increases when there are more

\(^{22}\)See Definition 5.6.
opportunities to buy insurance. This effect may overcome the incentive to acquire insurance against larger part of $X$ and drive the deductible down. Figure 5 illustrates this phenomenon. It shows that the decrease of the first insurer’s discount factor does not necessarily lead to more insurance because the minimal deductible $Z_{\text{deductible}}$ is not monotonic in $\delta_1$.

6.2 An Aggregation Result

In general, the ranks that the insured assigns to the insurers may depend in a non-trivial way on insurers’ preferences and endowments. It turns out, however, that when all insurers have CARA preferences, ranks can be characterized explicitly. The following is true:

**Proposition 6.3** The ranks of the insurers follow the order of their pre-trade MRIS. That is $\text{rank}(i) > \text{rank}(j)$ if and only if

$$\frac{\delta_i u'_i(w_{1i})}{u'_i(w_{0i})} < \frac{\delta_j u'_j(w_{1j})}{u'_j(w_{0j})}.$$

Suppose for simplicity that the insurers’ endowments satisfy $w_{0i} = w_{1i}$. In this case, Proposition 6.3 implies that the rank of an insurer is determined solely by his discount factor $\delta_i$ and is independent of his risk aversion $A_i$. The reason is that, when an insurer’s risk aversion is constant, the risk premium *per unit of risk* that the insurer is charging is independent of the level of $X$. By Proposition 6.1, for any insurer $i$, the insured optimally chooses the fraction of the total coverage $F(X)$ that insurer $i$ covers to be proportional to his risk tolerance $A_i$, thereby equalizing marginal risk premia across the insurers. Therefore, only discount factors $\delta_i$ matter for ranking. In particular, if several insurers have identical pre-trade MRIS, the tranches in which they participate will be the same, and their coverage functions will differ only by constant multiples. This situation leads
Figure 3: The Effect of Adding More Insurers (2 insurers and 3 insurers)
Figure 4: The Effect of Adding More Insurers (4 insurers and 5 insurers)
The Minimal Deductible

Insurer 5's Discount Rate $\delta_5$

Figure 5: A Counter Example

to the following interesting aggregation result. Denote by $A_I^{-1}$ the sum of insurers' risk tolerances:

$$A_I^{-1} = \sum_{j=1}^{N} A_j^{-1}. \quad (27)$$

The following is true:

**Proposition 6.4** Suppose that $w_{0i} = w_{1i}$\(^{23}\) for all $i$, and $\delta_i = \delta_I$ is independent of $i$.

Then, the risk-sharing is linear:

$$F_i(x) = \frac{A_i^{-1}}{A_I^{-1}} F(x)$$

and $F(x)$ coincides with the optimal indemnity schedule that the insured would choose with a single representative insurer with risk aversion $A_I$.

---

\(^{23}\)Due to translation invariance of CARA preferences, optimal allocation depends only on the differences $w_{1i} - w_{0i}$ and $w_1 - w_0$ of endowments at times zero and one.
The result of Proposition 6.4 has a natural interpretation in the framework of the theory of syndicates, developed by Wilson (1968). Namely, Proposition 6.4 implies that insurers with identical pre-trade MRIS effectively form a syndicate with the group (syndicate) utility given by that of the representative insurer. It is interesting to note that, without limited liability constraints, the syndicate result and the CARA linear risk sharing rule of Proposition 6.4 always holds, independent of the MRIS of the agents. However, this aggregation result does not generally hold under limited liability constraints.

7 Extensions

7.1 Allowing Agents to Save

In this section we show that the result of Theorem 4.5 is robust and holds even when all agents are allowed to save. For the case when all agents have CARA preferences, we also provide a detailed analysis of the optimal tranche thresholds. We also allow for heterogeneity in the insurer-specific interest rates $r_i$ and the insured’s interest rate $R$. Even though we assume for simplicity that these rates are risk-free, we view them as proxies for the expected rate of return on the investment opportunities, available to a particular agent. We also assume that the agents are potentially borrowing constrained and impose an upper bound $\tilde{K}_i \geq 0$ on the amount that insurer $i$ can borrow at time zero. Similarly, the insured can borrow at most $\tilde{K}$ at time zero. This means that, for agent $i$, the endowment stream of insurer $i$ is given by

\[
(\tilde{w}_0i, \tilde{w}_{1i}) \equiv (w_0i - s_i, w_{1i} + (1 + r_i)s_i)
\]

\footnote{In fact, the risk-sharing rule will generally be affine linear: indemnities may also differ by additive constants).}
where the pre-trade $\tilde{s}_i \geq -\tilde{K}_i$ is optimally chosen by insurer $i$. Similarly, the endowment stream of the insured is

$$(\tilde{w}_0, \tilde{w}_1) = (w_0 - s, w_1 + (1 + R)s).$$

The following is true:

**Proposition 7.1** Suppose that the agents can reallocate wealth between time periods using a riskless investment with (agent-specific) interest rates $R$ for the insured and $r_i, i = 1, \ldots, N$ for insurers. Then, the result of Theorem 4.5 still holds, but with endogenously determined endowments $\tilde{w}_0, \tilde{w}_1$ and $\tilde{w}_0, \tilde{w}_1$.

Indeed, let $s_i, i = 1, \ldots, N$ and $s$ be the optimal savings of insurers and the insured respectively. Then, the optimal allocation maximizes the social welfare function and is therefore also optimal if we prohibit the agents to save and replace their endowments by $(\tilde{w}_0, \tilde{w}_1)$ and $(\tilde{w}_0, \tilde{w}_1)$ respectively.

Proposition 7.1 implies that the general structure of the optimal allocation is robust to savings. The insurer-specific deductible levels (tranche thresholds), however, do depend on the exact nature of savings opportunities in a very non-trivial fashion. Suppose that all agents have CARA preferences. The first order condition implies that the pre-trade MRIS satisfies

$$M_i \equiv \frac{\delta_i e^{-A_i \tilde{w}_1}}{e^{-\tilde{w}_0}} \leq \frac{1}{1 + r_i}.$$  (28)

where the inequality turns into an identity if the borrowing constraint is not binding. Here,

$$(\tilde{w}_{0i}, \tilde{w}_{1i}) = (w_{0i} - \tilde{s}_i, w_{1i} + (1 + r_i)\tilde{s}_i)$$

and $\tilde{s}_i$ is insurer $i$’s optimal saving before entering an insurance agreement. The case when the constraint is binding before trade (i.e., when $M_i < \frac{1}{1+r_i}$) is especially inter-
esting in our setting. In this case, insurers can use insurance contracts to re-allocate consumption across periods and effectively undo or soften the borrowing constraints. Borrowing constraints may also have a significant effect on the insured’s decision: If she cannot borrow from the future, she may not have enough capital to acquire sufficient insurance at time zero. Denote by $s_i$ the optimal saving of insurer $i$ after entering the insurance agreement net the pre-trade saving $\bar{s}_i$. The quantity $s$ for the insurer is defined analogously. Then, the borrowing constraints take the form $s_i \geq -K_i \equiv -\bar{K}_i - \bar{s}_i$ and $s \geq -K \equiv -\bar{K} - \bar{s}$.

**Lemma 7.2** The optimal saving is given by

$$s_i = \frac{1}{A_i(1 + r_i)} \left( \log E[e^{A_iF_i(X)}] + \log \left( \frac{2 + r_i}{1 + M_i} \right) \right)$$

if the borrowing constraint is not binding, and $s_i = -K_i$ otherwise.

The corresponding price is given by

$$P_i = \frac{1}{A_i(1 + r_i)} \left( \log E[e^{A_iF_i(X)}] + (2 + r_i) \log \left( \frac{2 + r_i}{1 + M_i} \right) - (1 + r_i) \log (M_i(1 + r_i)) \right)$$

if the borrowing constraint is not binding, and

$$P_i = -A_i^{-1} \log \left( e^{-A_i(\bar{w}_{0i} + K_i)} + \bar{\delta}_i e^{-A_i(\bar{w}_{1i} - (1 + r_i)K_i)}(1 - E[e^{A_iF_i(X)}]) \right) - \bar{w}_{0i} - K_i$$

otherwise.

We can now define an analog of the map of Lemma 5.1.

**Lemma 7.3** For each $i = 1, \cdots, N$, there exists a unique, piecewise $C^1$ function

$$H_i = H_i(C, b_{-i})$$

43
satisfying
\[ H_i(C, b_{-i}) = \delta_i^{-1} C e^{-A_i \tilde{w}_0 i} \frac{(1 + r_i)(1 + M_i)}{(2 + r_i)} \] (30)
if the borrowing constraint is not binding for the insurer \( i \) with the indemnity
\[ F_i(X, (H_i(C, b_{-i}), b_{-i})) \]

and \( H_i \) is defined as the unique solution to
\[ H_i(C, b_{-i}) = \delta_i^{-1} C u'_i (\tilde{w}_0 i + K_i + P_i(F_i(X, (H_i(C, b_{-i}), b_{-i})))) \] . (31)
otherwise. The function \( H_i \) is monotone increasing in \( C \) and \( b_{-i} \), and \( C^{-1} H_i \) is decreasing in \( C \).

A straightforward modification of the arguments for the case without savings implies that the map \( G_C \) of Lemma 5.2 is still a contraction in our setting and therefore Theorem 5.4 still holds. To complete this section, we discuss in more detail the case when the borrowing constraints do not bind after trade. In this case, the constants \( b_i \) determining the structure of the optimal allocation are given fully and explicitly in (30), up a constant factor \( C \). In particular, these constants are completely independent on the nature of the insured risks \( X \). This is a very intriguing result, that should be fully attributed to special multiplicative properties of CARA preferences. Namely, to the fact that both the risk aversion and the elasticity of intertemporal substitution of each insurer is independent of his wealth. The MMRIS of each insurer is given by
\[ Y_i = \frac{\delta_i e^{-A_i \tilde{w}_1 i}}{e^{-A_i \tilde{w}_0 i} \frac{(1 + r_i)(1 + M_i)}{(2 + r_i)}} = \frac{M_i(2 + r_i)}{(1 + r_i)(1 + M_i)} \]
and we get the following analog of Proposition 6.4.
Proposition 7.4 Suppose that the borrowing constraints are not binding after trade. Then, all insurers \( j \) with identical \( \frac{M_i(2+r_i)}{(1+r_i)(1+M_i)} \) have the same deductible and share the risk linearly, as in Proposition 6.4.

If the pre-trade borrowing constraint is not binding, the identity \( M_i = (1 + r_i)^{-1} \) immediately implies that \( Y_i = (1 + r_i)^{-1} \). Therefore, the ranking of the insurers follows that of the return rates on their alternative investment opportunities. This result is very intuitive: Insurers with better investment opportunities will be more interested in raising more capital today, investing it and then using the high returns to provide insurance coverage tomorrow. Note however that the deductibles \( Z_i \) will still depend on the distribution of \( X \). As an illustration, consider the case where the insured is risk neutral and the borrowing constraints are not binding after trade. Then, \( J = 0 \), \( F(X) = \min(X, Z_1) \), \( C = \delta \) and

\[
a_i = \delta_i \delta^{-1} e^{-A_i \tilde{w}_{0i}} \frac{(1 + r_i)(1 + M_i)}{(2 + r_i)}.
\]

Therefore, by formula (19),

\[
Z_1 = \sum_i \left( \tilde{w}_{1i} + (1 + r_i)s_i - A_i^{-1} \log(a_i) \right)
\]

with \( s_i \) defined in (29) and, similarly, an analog of formula (20) also holds.

**7.2 Fixed Costs of Insurance**

In this section we discuss the effects of fixed insurance costs on the optimal allocation. Namely, we assume that acquiring insurance from any given insurer costs a fixed amount \( C \) in addition to the already included proportional costs. In this case, the optimal insurance design problem contains a discrete component. The insured has to decide how many insurance policies to enter and also optimally select a subset of insurers. Then,
given the chosen insurers and the number $K$ of the insurance contract the insured decides to enter, the problem reduces to the one studied in the previous sections, but with the initial endowment of the insured given by $w_0 - KC$. We summarize the basic implications of fixed costs in the following proposition:

**Proposition 7.5** We have:

1. The optimal number $K = K(C)$ of insurance contracts is monotone decreasing in cost $C$;

2. There exist thresholds

\[ 0 = C_{N+1} \leq C_N \leq C_{N-1} \leq \cdots \leq C_1 \leq C_0 = 0, \]

such that it is optimal to enter $K$ contracts if $C \in (C_{K+1}, C_K)$;

3. If the insured is risk neutral and a decrease in the cost does not force the insured to drop any insurers, then this decrease in cost $C$ leads to more insurance;

4. If the insured is risk averse, then for $C \in (C_{K+1}, C_K)$, an increase in $C$ leads to less selling if this increase does not lead to changing insurers.

Items (1) and (2) follow directly from the definition of the problem: The higher the cost of insurance, the lower the incentive for the insured to buy insurance. To explain the meaning of items (3) and (4), we note that a decrease in the fixed cost might force the insured to choose another group of insurers rather than add new insurers to already existing ones. For example, suppose the cost is so high that the insured buys insurance only from insurer 1. If the cost decreases, it may be optimal for the insurer to trade only with insurers 2 and 3 rather than with 1 and 2 or 1 and 3. In this case, insurer 1 is dropped. But if none of the insurers gets dropped, a decrease in the cost simply
induces the insured to add new insurers to already existing ones, which, by Proposition 6.2, always leads to more insurance and (3) follows. If the insured is risk averse, an increase in the cost reduces her initial consumption, thereby decreasing her incentives to pay for insurance. As long as the set of insurers does not change, (4) immediately follows from Proposition 5.7.

In Table 3, we numerically show how the optimal number of insurers depends on the fixed cost $C$ using the profiles specified in Table 2. The proportional cost level is again set to be 15%, and the disaster probability $p$ and the skewness parameter $n$ are set to be 0.3 and 0 respectively.

We see here that a surprising phenomenon occurs: At first, as the fixed cost increases, insurers with low ranks are gradually dropped. However, when the fixed cost is sufficiently large ($0.1 < C < 0.4$ for this example), insurer 4, who has the lowest rank among the four insurers (by Proposition 6.3), is suddenly picked again by the insured while insurers 1 and 2 are dropped. Apparently, the advantage of insurer 4 as being the most risk tolerant insurer becomes very attractive when the fixed cost is high. This pheno-

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25By Proposition 6.3, these are the insurers with high discount factors.
nomenon illustrates that, when the fixed insurance cost is high, the risk premia that the insurers are charging play a crucial role in the insured’s decision about which insurers to choose. This example also demonstrates that the problem of optimally selecting a group of insurers is very nontrivial.

7.3 Reinsurance

In real world, most insurance companies are themselves suffering from large disasters and buy insurance against extreme losses from reinsurance market. This more general frameworks can be easily incorporated into our general risk sharing model.

Namely, suppose that the insured (agent $N+1$) can contact $N$ insurers $i = 1, \cdots, N$, and suppose that each of these insurers can contact $M$ reinsurers $i = N+2, \cdots, N+M+1$. In this case, the indemnities still satisfy the limited liability constraints

$$F_i(X) \geq 0, \quad i = 1, \cdots, N+1$$

where, as above, we have defined $F_{N+1}(X) \equiv X - \sum_{i=1}^{N} F_i(X)$. However, each insurer $i$ will acquire additional insurance against losses $F_{ij}(X)$ from reinsurer $j$. We will denote the final allocation by $\bar{F}_k(X), \ k = 1, \cdots, N + M + 2$. Then, $\bar{F}_{N+1} = F_{N+1}$, and

$$\bar{F}_i(X) = F_i(X) - \sum_{j=N+2}^{N+M+1} F_{ij}(X), \quad i = 1, \cdots, N$$

$$\bar{F}_j(X) = \sum_{i=1}^{N} F_{ij}(X), \quad j = N+2, \cdots, N+M+1. \quad (32)$$

By construction, the optimal allocation is constrained Pareto efficient and can therefore be characterized as in Theorem 3.3. However, the endogenously determined Pareto weights satisfy a complicated system of non-linear equations and the general analysis of this system is beyond the scope of this paper. However, as an illustration, we consider
here the case of a single insurer and a single re-insurer. Let $I$ denote the single insurer and $R$ the single reinsurer. Denote by $a_I$ and $a_R$ the Pareto weights, such that the optimal allocation maximizes

$$E[U(w_1 + F_I(X) + F_R(X) - X)] + a_I E[u_I(w_{1I} - F_I(X))] + a_R E[u_R(w_{1R} - F_R(X))].$$

If $\text{rank}(R) = 1$, $\text{rank}(I) = 2$, and the rank of the insured is 3, then the allocation resembles real-world insurance/reinsurance allocations: Insurer $I$ provides insurance above the deductible level $Z_2$ and the reinsurer provides insurance against large losses, exceeding the deductible $Z_1$.26

It would be interesting to compare the optimal allocation in which the insured can directly acquire insurance from $I$ and $R$ with the insurance/reinsurance allocation. We leave this important problem for future research.

8 Conclusions

We solve the problem of optimal risk sharing with limited liability for a finite number of agents with heterogeneous risk preferences. We then apply our general risk sharing rule to solve the problem of optimal insurance design for an insured facing multiple insurers with heterogeneous risk attitudes, discount factors, and endowments, and without asymmetric information. We show that optimal indemnities can be characterized by insurer-specific deductibles and a hierarchical structure: The insured optimally assigns ranks to insurers depending on their MMRIS, and based on these ranks, the insured determines the

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26There is recently a very tragic event where the Italian cruise ship Costa Concordia partially sank off the Italian coast and more than 30 people lost their lives. The discussion in the press seems to suggest that the insurance allocation matches the hierarchical structure we proved. For example, one Reuters article said: “P&I clubs typically pick up liability claims triggered by shipping disasters. Individual P&I losses exceeding a certain threshold are pooled between the biggest P&I clubs, who in turn buy reinsurance in the event of losses exceeding a set ceiling.” Source: http://www.reuters.com/article/2012/01/16/us-costaconcordia-insurance-idUSTRE80F19620120116
optimal deductible level for each insurer. She then either fully insures all risks below an endogenous threshold with several (≥ 1) insurers having the highest ranks, or chooses a strictly positive minimal deductible. Afterwards, she gradually insures subsequent tranches with insurers of lower ranks, so that every subsequent tranche is co-insured by multiple insurers in a Pareto-efficient way. In particular, the last tranche is never fully insured.

When the insured and all insurers have exponential (CARA) utilities, linear co-insurance is optimal inside each tranche. The model generates a number of comparative statics results that would be interesting to test with real-world insurance contracts. Since our paper is the first to study and solve the problem of optimal insurance with multiple insurers, we believe our results are of both theoretical and practical importance and can be used in the insurance industry to achieve more efficient risk sharing.

Finally, we note that the model could also be viewed as surplus extraction through price discrimination when the insured has complete information. It would be interesting to extend our model to the case where the insured has incomplete information about insurer types, as in the model of Cremer and McLean (1985). It would also be interesting to extend our model to a dynamic, multi-period setting and allow for asymmetric information. Our techniques for analyzing constrained efficient allocations could also be applied to models outside of insurance design, such as equilibrium models with participation constraints. See Alvarez and Jermann (2000). We leave these explorations for future research.

27 Corresponding to highest levels of losses.
Appendix

A One Insurer

Proof of Theorem 4.1. The first-order Kuhn-Tucker conditions are

\[ \delta U'(w_1 - X + F(X)) - (1 + \alpha) U'(w_0 - (1 + \alpha) P) v_I' (L_I - \delta I E[u_I(w_{1I} - F)]) \delta I u_I'(w_{1I} - F) = 0 \quad (33) \]

if the constraints 0 \leq F(X) \leq X are not binding; the form

\[ \delta U'(w_1 - X + F(X)) - (1 + \alpha) U'(w_0 - (1 + \alpha) P) v_I' (L_I - \delta I E[u_I(w_{1I} - F)]) \delta I u_I'(w_{1I} - F) > 0 \quad (34) \]

with \( F(X) = X \) if the constraint \( F(X) = X \) is binding; and the form

\[ \delta U'(w_1 - X + F(X)) - (1 + \alpha) U'(w_0 - (1 + \alpha) P) v_I' (L_I - \delta I E[u_I(w_{1I} - F)]) \delta I u_I'(w_{1I} - F) < 0 \quad (35) \]

with \( F(X) = 0 \) if the constraint \( F(X) = 0 \) is binding.

Since the maximization problem for the insured is strictly concave, to prove the theorem it suffices to check that the first-order conditions, (33) through (35), hold true.\textsuperscript{28}

We consider all four cases identified in the theorem.

(1) and (4). We only prove (1). Case (4) is analogous. To show that \( F(X) = X \), we

\textsuperscript{28}Even though the problem is infinite dimensional, sufficiency of the Kuhn-Tucker conditions can be verified directly by standard methods due to the strict concavity of the problem. See, e.g., Seierstad and Sydsaeter (1977) and Mitter (2008).
need to show that the constraint is binding. That is, (34) holds with \( F(X) = X \). Using the identity
\[
v'_I(x) = 1/u'_I(v_I(x)),
\]
we get
\[
\delta U'(w_1) - (1 + \alpha) \frac{U'(w_0 - (1 + \alpha) P_{max})}{u'_I(w_0 + P_{max})} \delta_I u'_I(w_{1I} - X) > 0
\]
for all \( X \in [0, \bar{X}] \). Since \( u'_I \) is decreasing, it suffices to check it for \( X = \bar{X} \). This is equivalent to the condition \( \delta - \delta_I < K_{min} \).

(2) and (3). For simplicity, we only prove (3). Case (2) is analogous. First, we need to show that the equation for \( a \) does have a solution. To this end, by the Brower fixed-point theorem, it suffices to show that the right-hand side of (12) maps a compact interval into itself. This is clear because:
\[
u_I(w_{1I}) \geq E[u_I(w_{1I} - F_a(X))] \geq E[u_I(w_{1I} - X)].
\]

Now, we claim that the right-hand side of (12) is monotone decreasing in \( a \). Indeed, to prove it, it suffices to show that
\[
E[u_I(w_{1I} - F_a(X))]
\]
is monotone increasing in \( a \), since the numerator is decreasing in \( E[u_I(w_{1I} - F_a(X))] \) and the denominator is decreasing in it. Differentiating (10) with respect to \( a \), we get:
\[
\frac{\partial g}{\partial a} = \frac{u'_I(w_{1I} - g)}{a u''_I(w_{1I} - g) + U''(w_1 - x + g)},
\]
and therefore \( g \) is monotone decreasing in \( a \). Also, \( g(a, Z(a), w_{1I}) = 0 \). Therefore,

\[
\begin{align*}
\frac{\partial}{\partial a} (E[u_I(w_{1I} - F_a(X))]) &= \frac{\partial}{\partial a} \left( u_I(w_{1I}) \int_0^{Z(a)} p(x) dx + \int_{Z(a)}^X u_I(w_{1I} - g(a, X)) p(x) dx \right) \\
&= u_I(w_{1I}) \frac{\partial}{\partial a} \int_0^{Z(a)} p(x) dx - u_I(w_{1I}) \frac{\partial}{\partial a} \int_0^{Z(a)} p(x) dx \\
&\quad - \int_{Z(a)}^X u_I'(w_{1I} - g(a, x)) \frac{\partial g}{\partial a}(a, x) p(x) dx \\
&= - \int_{Z(a)}^X u_I'(w_{1I} - g(a, x)) \frac{\partial g}{\partial a}(a, x) p(x) dx > 0, \quad (37)
\end{align*}
\]

and the claim follows. 

**B Kuhn-Tucker First-Order Conditions for Multiple Insurers**

By strict concavity, an allocation is optimal if and only if it satisfies the first-order Kuhn-Tucker conditions. They are:

\[
\begin{align*}
\delta U'(w_1 - X + F(X)) &= (1 + \alpha) U' \left( w_0 - (1 + \alpha) \sum_i P_i \right) v_i'(L_i - \delta_i E[u_i(w_{1i} - F_i)]) \delta_i u_i'(w_{1i} - F_i) = 0 \quad (38)
\end{align*}
\]

if the constraints \( F_i \geq 0 \) and \( \sum_j F_j \leq X \) are not binding, and

\[
\begin{align*}
\delta U'(w_1 - X + F(X)) &= (1 + \alpha) U' \left( w_0 - (1 + \alpha) \sum_i P_i \right) v_i'(L_i - \delta_i E[u_i(w_{1i} - F_i)]) \delta_i u_i'(w_{1i} - F_i) < 0 \quad (39)
\end{align*}
\]
if the constraint $F_i \geq 0$ is binding but the constraint $\sum_j F_j \leq X$ is not binding.

Finally, if the constraint $\sum_j F_j \leq X$ is binding, there will be a Lagrange multiplier \(\nu(X)\) for this constraint, and the first-order condition will be

\[
\delta U' (w_1 + X - F(X)) \\
- (1 + \alpha) U' \left( w_0 - (1 + \alpha) \sum_i P_i \right) v'_i \left( L_i - \delta_i E[u_i(w_1 - F_i)] \right) \delta_i u'_i (w_{1i} - F_i) \\
= \nu(X) > 0 \quad (40)
\]

if the constraint $F_i \geq 0$ is not binding, and

\[
\delta U' (w_1 - X + F(X)) \\
- (1 + \alpha) U' \left( w_0 - (1 + \alpha) \sum_i P_i \right) v'_i \left( L_i - \delta_i E[u_i(w_1 - F_i)] \right) \delta_i u'_i (w_{1i} - F_i) \\
< \nu(X) \quad (41)
\]

if the constraint $F_i \geq 0$ is binding. By (36),

\[
v'_i \left( L_i - \delta_i E[u_i(w_{1i} - F_i)] \right) = \frac{1}{u'_i (w_{0i} + P_i)} = \frac{1}{u'_i (c_{0i})},
\]

and therefore, by (17),

\[
a_i = \delta^{-1} \delta_i \left( 1 + \alpha \right) U' \left( w_0 - (1 + \alpha) \sum_i P_i \right) \frac{u'_i (v'_i \left( L_i - \delta_i E[u_i(w_{1i} - F_i)] \right) \delta_i u'_i (w_{1i} - F_i) \right)}{u'_i (c_{0i})} \\
= \delta^{-1} \delta_i \left( 1 + \alpha \right) U' \left( w_0 - (1 + \alpha) \sum_i P_i \right) v'_i \left( L_i - \delta_i E[u_i(w_{1i} - F_i)] \right). \quad (42)
\]
Thus, we can rewrite (38) through (41) in the form:

\[ a_i u_i'(w_{1i} - F_i(X)) = U'(w_1 - X + F(X)) \]  

(43)

when none of the constraints is binding, and

\[ a_i u_i'(w_{1i} - F_i(X)) > U'(w_1 - X + F(X)) \]  

(44)

when the constraint \( F_i \geq 0 \) is binding but the constraint \( \sum_j F_j \leq X \) is not.

When the constraint \( \sum_j F_j(X) \leq X \) is binding, we have \( \sum_j F_j(X) = X \). If we set:

\[ \lambda(X) \overset{\text{def}}{=} -\delta^{-1} \nu(X) + U'(w_1), \]

then, (40) and (41) take the form

\[ a_i u_i'(w_{1i} - F_i(X)) = \lambda(X) < U'(w_1) \]  

(45)

when \( F_i \geq 0 \) is not binding and

\[ a_i u_i'(w_{1i} - F_i(X)) > \lambda(X) \]  

(46)

when it is binding.

By abuse of notation, we from now on reorder the insurers in the increasing order of their rank. In other words, insurer \( i \) means from now on the insurer whose rank is equal to \( i \).

By the uniqueness of optimal allocation, it suffices to show that the allocation, described in Proposition 4.5 and Theorem 3.4, indeed satisfies the first-order conditions (43) through (46). This is done in subsequent lemmas.
Lemma B.1 Let $k \geq J + 1$. Then, for all $X \in [Z_{k+1}, Z_k]$ (= Tranche$ _k$), the constraint $\sum_j F_j(X) \leq X$ is binding and the constraint $F_j(x) \geq 0$ is binding for all $j < k$. The optimal allocation for $X \in$ Tranche$ _k$ is uniquely determined via

$$F_j(X) = \begin{cases} w_{1j} - q_j(\lambda_k(X)a_j^{-1}), & j \geq k \\ 0, & j < k \end{cases}.$$  \hspace{1cm} (47)

Here, $\lambda_k(X)$ is the unique solution to

$$X = \sum_{j \geq k} \left( w_{1j} - q_j(\lambda_k(X)a_j^{-1}) \right).$$ \hspace{1cm} (48)

The slope of $F_j(X)$, $j \geq k$ satisfies

$$\frac{d}{dx} F_j(X) = \frac{R_j(c_{ij})}{\sum_{i \geq k} R_i(c_{1i})}.$$ \hspace{1cm} (49)

**Proof.** By construction, the conjectured optimal allocation satisfies

$$\sum_j F_j(X) = X$$

for all $X \leq Z_{J+1}$. Thus, we need to verify that (45) and (46) hold in this case. Here, the connection between $\xi_k(X)$ from Proposition 4.5 and $\lambda_k(X)$ is given by:

$$\xi_k(X) = \frac{\lambda_k(X) \delta}{(1 + \alpha) U''(c_0)}.$$ \hspace{1cm} (50)

By (47) and (48), $F_i(X)$ satisfies

$$a_i u'_i(w_{1i} - F_i(X)) = \lambda_k(X) \quad \text{and} \quad \sum_i F_i(X) = X,$$
and it remains to check that equation (48) has a solution $\lambda_k(X)$ such that

$$\lambda_k(X) \leq U'(w_1)$$  \hspace{1cm} (49)

(constraint $\sum_j F_j \leq X$ is binding) and

$$F_j = w_{1j} - q_j(\lambda_k(X) a_j^{-1}) \geq 0 \text{ for all } j \geq k$$  \hspace{1cm} (50)

(constraint $F_j \geq 0$ is not binding for $j \geq k$) and (46) holds, that is,

$$a_j u'_j(w_{1j}) \geq \lambda_k(X)$$  \hspace{1cm} (51)

for all $j < k$ (constraint $F_j \geq 0$ is binding for $j < k$). First, let $k > J + 1$. Recall now that

$$\tilde{Z}_{k+1} = \sum_{i=k+1}^{N} (q_i(a^{-1}_i a_k u'_k(w_{1k})) - w_{1i}) = \sum_{i=k}^{N} (q_i(a^{-1}_i a_k u'_k(w_{1k})) - w_{1i}),$$

and therefore $X \in [\tilde{Z}_{k+1}, \tilde{Z}_k]$ if and only if

$$\sum_{i=k}^{N} (w_{1i} - q_i(a^{-1}_i a_k u'_k(w_{1k}))) \leq X \leq \sum_{i=k}^{N} (w_{1i} - q_i(a^{-1}_i a_{k-1} u'_{k-1}(w_{1k-1}))).$$

Recalling that

$$X = \sum_{i=k}^{N} (w_{1i} - q_i(a^{-1}_i \lambda_k(X))),$$

we get that

$$\lambda_k(X) \in [a_k u'_k(w_{1k}), a_{k-1} u'_{k-1}(w_{1k-1})].$$  \hspace{1cm} (53)
If \( k = J + 1 \), the same argument implies that

\[
\lambda_{J+1}(X) \in [a_{J+1} u'_{J+1}(w_{1,J+1}), U'(w_1)].
\] (54)

Recall that the insurers are ordered in such a way that the sequence

\[
Y_i = \frac{\delta_i u'_i(w_{1i})}{u'_i(c_{0i})} = \frac{a_i u'_i(w_{1i}) \delta}{U'(c_0)}
\]

is monotone decreasing in \( i \), and the inequality

\[
Y_i < Y
\]

only holds true if \( i \geq J + 1 \). Consequently,

\[
a_N u'_N(w_{1N}) \leq \cdots \leq a_{J+1} u'_{J+1}(w_{1,J+1}) \leq U'(w_1) \leq a_J u'_J(w_{1J}) \leq \cdots \leq a_1 u'_1(w_{11}).
\] (55)

Inequalities (53), (54), and (55) immediately yield (49) and (51). Finally, for \( j \geq k \),

\[
a_j u'_j(w_{1j}) \leq a_k u'_k(w_{1k}) \iff a_j^{-1} \geq u'_j(w_{1j}) (a_k u'_k(w_{1k}))^{-1}
\]

and, using that \( \lambda_k(X) \geq a_k u'_k(w_{1k}) \), we get

\[
F_j = w_{1j} - q_j(\lambda_k(X) a_j^{-1}) \geq w_{1j} - q_j(a_k u'_k(w_{1k}) u'_j(w_{1j}) (a_k u'_k(w_{1k}))^{-1}) = 0,
\]

and (50) follows.

It remains to prove the identity for the derivative. Differentiating (47), we get

\[
F'_j(X) = -(u''_j(c_{ij}))^{-1} a_j^{-1} \lambda'_k(X),
\]
and, differentiating (48), we get

$$\lambda_k'(X) = -\frac{1}{\sum_{i \geq k} q_i' a_i^{-1}}.$$ 

Differentiating $$u'_i(q'_i(z)) = z$$ at $$z = a_i^{-1} \lambda_k(X),$$ we get

$$(u''_i(c_{1i}))^{-1} = q'_i(a_i^{-1} \lambda_k(X)).$$

Thus,

$$F'_j(X) = \frac{(u''_j(c_{1j}))^{-1} a_j^{-1}}{\sum_{i \geq k} q_i' a_i^{-1}} = \frac{(u''_i(c_{1i}))^{-1} a_i^{-1}}{\sum_{i \geq k} (u''_i(c_{1i}))^{-1} a_i^{-1}} = \frac{(u''_j(c_{1j}))^{-1} u'_j(c_{1j})}{\sum_{i \geq k} (u''_i(c_{1i}))^{-1} a_i^{-1}}.$$ (56)

which is what had to be proved. ■

It remains to cover the case when the constraint $$\sum_i F_i(X) \leq X$$ is not binding. This is done in the following lemma.

**Lemma B.2** Let $$k \leq J.$$ Then, for all $$X \in [Z_{k+1}, Z_k] (= \text{Tranche}_k),$$ the constraint $$\sum_j F_j(X) \leq X$$ is not binding, and the constraint $$F_j(x) \geq 0$$ is binding for all $$j \leq k.$$ The optimal allocation for $$X \in \text{Tranche}_k$$ is uniquely determined via

$$F_j(X) = \begin{cases} 
  w_{1j} - q_j(U'(w_1 - X + F(X)) a_j^{-1}), & j > k \\
  0, & j \leq k.
\end{cases}$$ (57)

Here, $$F(X)$$ is the unique solution to:

$$F(X) - \sum_{j \geq k+1} (w_{1j} - q_j(U'(w_1 - X + F(X)) a_j^{-1})) = 0.$$ (58)
The slope of $F_j(X)$, $j \geq k + 1$ satisfies

$$\frac{d}{dx} F_j(X) = \frac{R_j(c_{1j})}{R(c_1) + \sum_{i>k} R_i(c_{1i})}.$$

**Proof.** We need to show that the allocation (57) and (58) satisfy the Kuhn-Tucker conditions:

$$a_i u'_i(w_{1i} - F_i) = U'(w_1 - X + F(X)),$$

with $F_i \geq 0$ for all $i > k$ and

$$a_i u'_i(w_{1i}) - U'(w_1 - X + F(X)) > 0$$

for all $i \leq k$.

For simplicity let $k < J$. By assumption, $X \in [\tilde{Z}_{k+1}, \tilde{Z}_k]$; that is,

$$w_1 - Q(a_k u'_k(w_{1k})) + \sum_{i \geq k+1} (w_{1i} - q_i (a_i^{-1} a_k u'_k(w_{1k})))$$

$$> X > w_1 - Q(a_{k+1} u'_{k+1}(w_{1k+1})) + \sum_{i \geq k+1} (w_{1i} - q_i (a_i^{-1} a_{k+1} u'_{k+1}(w_{1k}))).$$

(59)

We show that the unique solution $F$ to (58) satisfies

$$X - (w_1 - Q(a_k u'_k(w_{1k}))) \leq F \leq X - (w_1 - Q(a_{k+1} u'_{k+1}(w_{1k+1}))).$$

(60)
Indeed,

\[
X - (w_1 - Q(a_{k+1} u'_{k+1}(w_{1,k+1}))
\]

\[
- \sum_{j \geq k+1} (w_{1j} - q_j(U'(w_1 - X + (X - w_1 + Q(a_{k+1} u'_{k+1}(w_{1,k+1})))) a_j^{-1}))
\]

\[
= X - Z_{k+1} \geq 0, \quad (61)
\]

and, similarly,

\[
X - (w_1 - Q(a_k u'_k(w_{1,k})))
\]

\[
- \sum_{j \geq k+1} (w_{1j} - q_j(U'(w_1 - X + (X - w_1 + Q(a_k u'_k(w_{1,k})))) a_j^{-1}))
\]

\[
= X - Z_k \leq 0. \quad (62)
\]

Consequently, by continuity and monotonicity, the right-hand side of (58) crosses zero at a single point \(F\), satisfying (60). Hence, for \(j \geq k+1\), by (55), we get:

\[
F_j(X) = w_{1j} - q_j(U'(w_1 - X + F(X)) a_j^{-1})
\]

\[
\geq w_{1j} - q_j(a_{k+1} u'_{k+1}(w_{1,k+1}) a_j^{-1}) \geq w_{1j} - q_j(a_j u'_j(w_{1j}) a_j^{-1}) = 0. \quad (63)
\]

It remains to be shown that the constraint \(F_j(X) \geq 0\) is binding for \(j \leq k\). By (60) and (55),

\[
a_j u'_j(w_{1j}) - U'(w_1 - X + F(X)) \geq a_j u'_j(w_{1j}) - a_k u'_k(w_{1k}) \geq 0,
\]

and the claim follows.

\[\blacksquare\]

To complete the proof of Theorem 4.5, we only need to show that there is no trade if and only if (21) is violated. That is, the allocation \(F_i = 0, i = 1, \cdots, N\) satisfies the
first-order Kuhn-Tucker conditions if and only if (21) does not hold. Since in this case
$Y_i$ and $Y$ coincide with the pre-trade MRIS, we need to show that
\[
\frac{\delta_i u'_i(w_{1i})}{u'_i(w_{0i})} \geq \frac{\delta U'(w_1 - X)}{(1 + \alpha)U'(w_0)}
\]
for all $i = 1, \cdots, N$ and all $X \in [0, \bar{X}]$. Since $U'(c)$ is monotone decreasing in $c$, this holds if and only if
\[
\min_i \frac{\delta_i u'_i(w_{1i})}{u'_i(w_{0i})} \geq \frac{\delta U'(w_1 - \bar{X})}{(1 + \alpha)U'(w_0)},
\]
and the claim follows.

C Contraction Mapping

We prove here the following extended version of Lemma 5.1.

For each $i = 1, \cdots, N$, let:
\[
\Omega^{-i} \overset{def}{=} \times_{j \neq i} [\beta_{i_{\min}}, \beta_{i_{\max}}] .
\]

**Lemma C.1** Fix a constant $C > 0$ and let
\[
H_i(C, b_{-i})
\]
be the unique solution to
\[
H_i(C, b_{-i}) - \delta_i^{-1} C u'_i (v_i (L_i - \delta_i E[u_i(w_{1i}) - F_i(X, H_i(C, b_{-i}), b_{-i}))])) = 0 .
\]

Then, $H_i$ is monotone increasing in $C \in [C_{\min}, C_{\max}]$ and $b_{-i} \in e^{\Omega^{-i}}$ and takes values
in $[e^{\beta_{i\min}}, e^{\beta_{i\max}}]$. Furthermore, there exists an $\eta < 1$ such that

$$\sum_{j \neq i} b_j \frac{\partial H_i}{\partial b_j} \leq \eta H_i$$

for all $b_{-i} \in \Omega^{-1}$ except for points in a finite union of hyperplanes, for which the derivatives do not exist.

Proof of Lemma 5.1. Consider the function:

$$\psi_i(y, b_{-i}, C) \overset{\text{def}}{=} \delta_i^{-1} \frac{C}{u_i} \left( v_i(L_i - \delta_i E[u_i(w_{1i} - F_i(X, (y, b_{-i})))]) \right).$$

Then, the defining equation for $H_i$ can be rewritten as

$$H_i = \psi_i(H_i, b_{-i}, C).$$

To complete the proof of the first part of the lemma, it remains to be shown that (1) $\psi_i$ is monotone decreasing in $y$; (2) for each fixed $C \in [C_{\min}, C_{\max}]$ and each fixed $b_{-i}$, it maps the whole $\mathbb{R}$ into the compact interval $[e^{\beta_{i\min}}, e^{\beta_{i\max}}]$; and (3) it is monotone increasing in $b_{-i}$, $C$ and is piecewise $C^1$ with respect to all variables.

By definition, the form of the function $F_i$ depends on the relative ranking of insurers, which, in turn, is determined by the ordering of the numbers $b_i/u_i'(w_{1i})$ (see (55)). For each permutation $\pi$ of $\{1, \cdots, N\}$, define the corresponding “sector”: the subset of $\mathbb{R}_+^n$ such that, for all $b$ in this sector, the sequence $b_{\pi(i)}/u_{\pi(i)}'(w_{1\pi(i)})$ is monotone increasing in $i$. The borders of these sectors belong to hyperplanes for which $b_i u_i'(w_{1i}) = b_j u_j'(w_{1j})$ for some $i \neq j$.

Clearly, since the function $\psi_i$ is continuous, it suffices to prove the required result for each fixed sector.$^{29}$

$^{29}$Here, one should in general take additional care of the situation when $H_i$ hits the boundaries of
As above, by abuse of notation, we reorder the insurers for each fixed sector so that (55) holds, and thus insurer $i$ will mean the insurer whose rank is equal to $i$.

First, the fact that the image of the function $\psi_i$ is inside the interval $[e^{\beta_{i_{\min}}}, e^{\beta_{i_{\max}}}]$ follows directly from the definition and the inequality:

$$0 \leq F_i(X) \leq X.$$

Now, we need the following auxiliary.

**Lemma C.2** For any $X$ inside a tranche, $F_i$ is a piecewise $C^1$-function of $b$ and satisfies

$$\frac{\partial F_i}{\partial b_i} \geq 0$$

and

$$\frac{\partial F_i}{\partial b_j} \leq 0$$

for all $j \neq i$. Furthermore,

$$b_i \frac{\partial F_i}{\partial b_i} \geq - \sum_{j \neq i} b_j \frac{\partial F_i}{\partial b_j}.$$

**Proof.** Suppose first that we are in the regime $F(X) < X$. Then, by (57),

$$F_i(X) = w_{1i} - q_i(b_i U'(w_1 - X + F(X)))$$

and

$$F(X) = F(b, x)$$

the sectors for an open set of parameters. Clearly, this cannot happen for generic values of parameters (discount factors and endowments), and we therefore ignore it. The proof can be easily modified to cover this non-generic situation.
solves

\[ F(X) - \sum_j (w_{1j} - q_j (b_j U'(w_1 - X + F(X)))) = 0. \]

Here, the summation is only over those insurers \( j \) that participate in the tranche. Thus,

\[ \frac{\partial F}{\partial b_j} = -\frac{q'_j(b_j U'(c_1)) U'(c_1)}{1 + \sum_k q'_k(b_k U'(c_1)) b_k U''(c_1)}; \]

and, hence, for \( j \neq i \),

\[ \frac{\partial F_i}{\partial b_j} = q'_i(b_i U'(c_1)) b_i U''(c_1) \frac{q'_j(b_j U'(c_1)) U'(c_1)}{1 + \sum_k q'_k(b_k U'(c_1)) b_k U''(c_1)} < 0 \]

if insurer \( j \) participates in the tranche, and the derivative is zero otherwise. Consequently,

\[ \sum_{j \neq i} b_j \frac{\partial F_i}{\partial b_j} = (q'_i(b_i U'(c_1))) b_i U''(c_1) \frac{\sum_{k \neq i} q'_k(b_k U'(c_1)) b_k U''(c_1)}{1 + \sum_k q'_k(b_k U'(c_1)) b_k U''(c_1)} \]

and

\[ \frac{\partial F_i}{\partial b_i} = -q'_i(b_i U'(c_1)) U'(c_1) \]

\[ + q'_i(b_i U'(c_1)) b_i U''(c_1) \frac{q'_i(b_i U'(c_1)) U'(c_1)}{1 + \sum_k q'_k(b_k U'(c_1)) b_k U''(c_1)} \]

\[ = -q'_i(b_i U'(c_1)) U'(c_1) \left( 1 - \frac{q'_i(b_i U'(c_1)) b_i U''(c_1)}{1 + \sum_k q'_k(b_k U'(c_1)) b_k U''(c_1)} \right) \]

\[ = -q'_i(b_i U'(c_1)) U'(c_1) \frac{1 + \sum_{k \neq i} q'_k(b_k U'(c_1)) b_k U''(c_1)}{1 + \sum_k q'_k(b_k U'(c_1)) b_k U''(c_1)}. \]

Therefore,

\[ b_i \frac{\partial F_i}{\partial b_i} > -\sum_{j \neq i} b_j \frac{\partial F_i}{\partial b_j}. \]

Suppose now that the constraint \( F(X) \leq X \) is binding, so that \( F(X) = X \). Then, by
\[ F_i(X) = w_i - q_i(\lambda(X) b_i), \]

with \( \lambda(X) \) solving

\[ X - \sum_i (w_i - q_i(\lambda(X) b_i)) = 0 \]

where the summation is only over insurers \( i \) participating in the tranche. Differentiating, we get

\[ \frac{\partial \lambda(X)}{\partial b_j} = -\frac{q'_j(b_j \lambda(X)) \lambda(X)}{\sum_k q'_k(b_k \lambda(X)) b_k} \]

and, hence,

\[ \frac{\partial F_i}{\partial b_j} = b_i q'_i(\lambda(X) b_i) \frac{q'_j(b_j \lambda(X)) \lambda(X)}{\sum_k q'_k(b_k \lambda(X)) b_k} < 0 \]

if the insurer \( j \neq i \) participates in the tranche and the derivative is zero otherwise. Similarly,

\[ \frac{\partial F_i}{\partial b_i} = -q'_i(\lambda(X) b_i) \lambda(X) + b_i q'_i(\lambda(X) b_i) \frac{q'_i(b_i \lambda(X)) \lambda(X)}{\sum_k q'_k(b_k \lambda(X)) b_k} \]

\[ = -q'_i(\lambda(X) b_i) \lambda(X) \frac{\sum_{k \neq i} q'_k(b_k \lambda(X)) b_k}{\sum_k q'_k(b_k \lambda(X)) b_k} > 0. \quad (64) \]

if \( F_i(X) \neq 0 \) (that is, if insurer \( i \) participates in the tranche), and is zero otherwise. A direct calculation implies that

\[ -b_i \frac{\partial F_i}{\partial b_i} = \sum_{j \neq i} b_j \frac{\partial F_i}{\partial b_j}. \]

Note that the function \( F_i(x) \) is continuous and is a smooth function of all \( b_i \) as long
as \( b \) varies inside a fixed sector. Therefore,

\[
\frac{\partial}{\partial b_k} E[u_i(w_{1i} - F_i(b, X))]
= \frac{\partial}{\partial b_k} \sum_j \int_{Z_{j+1}} u_i(w_{1i} - F_i(b, x))p(x)dx \\
= -\sum_j \int_{Z_{j+1}} u'_i(w_{1i} - F_i(b, x)) \left( \frac{\partial}{\partial b_k} F_i(b, x) \right) p(x)dx \\
= -E \left[ u'_i(w_{1i} - F_i(b, X)) \left( \frac{\partial}{\partial b_k} F_i(b, X) \right) \right].
\]

The derivatives of \( Z_i(b_j) \) do not appear on the right-hand side of (65) because the boundary terms cancel, as in (37).

Denote

\[
\bar{c}_{i0} = v_i(L_i - \delta_i E[u_i(w_{1i} + F_i(X, (H_i(C, b_{-i}), b_{-i})))].
\]

Then, using the identity \( v'_i(x) = (u'_i(v(x)))^{-1} \), we get

\[
\frac{\partial H_i}{\partial b_j} = \frac{\frac{C A_i(\bar{c}_{i0})}{1 + C A_i(\bar{c}_{i0})} \left[ u'_i(w_{1i} - F_i(X)) \left( -\frac{\partial}{\partial b_j} F_i(X) \right) \right]}{1 + C A_i(\bar{c}_{i0}) \left[ u'_i(w_{1i} - F_i(X)) \left( \frac{\partial}{\partial b_i} F_i(X) \right) \right]} \]

where

\[
A_i(x) = -\frac{u''_i(x)}{u'_i(x)}.
\]

Lemma C.2 implies that

\[
\sum_{j \neq i} b_j \frac{\partial H_i}{\partial b_j} = \frac{C A_i(\bar{c}_{i0}) \left[ u'_i(w_{1i} - F_i(X)) \sum_j b_j \left( -\frac{\partial}{\partial b_j} F_i(X) \right) \right]}{1 + C A_i(\bar{c}_{i0}) \left[ u'_i(w_{1i} - F_i(X)) \left( \frac{\partial}{\partial b_i} F_i(X) \right) \right]} \leq \eta b_i = \eta H_i
\]

(66)
where we have defined
\[
\eta = \max_{e\Omega} \frac{C A_i(\tilde{c}_0) E \left[ u' (w_{1i} - F_i(X)) \left( \frac{\partial}{\partial u_i} F_i(X) \right) \right]}{1 + C A_i(\tilde{c}_0) E \left[ u' (w_{1i} - F_i(X)) \left( \frac{\partial}{\partial u_i} F_i(X) \right) \right]}. 
\]

It follows from the proof of Lemma C.2 that the derivative \( \frac{\partial}{\partial u_i} F_i(X) \) stays uniformly bounded when \( b \) varies on the compact subset \( e\Omega \) and therefore \( \eta < 1 \). The proof is complete. \( \blacksquare \)

**Lemma C.3** Consider a map \( G = (G_i) : \Omega \to \Omega \) with coordinate maps \( G_i(b_1, \cdots, b_N) \), such that the following is true:

- The map \( G \) is continuous;
- There exists a finite set \( S \) of smooth hyper-surfaces such that \( G \) is \( C^1 \) on \( \Omega \setminus S \); and
- There exists a constant \( \eta < 1 \) such that

\[
\sum_j \left| \frac{\partial G_i}{\partial d_j} \right| \leq \eta
\]

for all \( i \) and all \( d = (d_j) \in \Omega \setminus S \).

Then, the map \( G \) is a contraction in the \( l_\infty \) norm \( \|d\|_{l_\infty} = \max_i |d_i| \), so that

\[
\|G(d^1) - G(d^2)\|_{l_\infty} \leq \alpha \|d^1 - d^2\|_{l_\infty}. 
\]

In particular, \( G \) has a unique fixed point \( d^* \) that satisfies

\[
d^* = \lim_{n \to \infty} G^n(d^0)
\]

for any \( d^0 \in \Omega \).
Proof of Lemma C.3. With continuity, we may assume that the two points \( d_1 \) and \( d_2 \) are in a generic position, so that the segment,

\[
d(t) = d_1 + t(d_2 - d_1), \; t \in [0, 1]
\]

connecting \( d_1 \) and \( d_2 \), intersects the hyperplanes from \( S \) for a finite set

\[
t_1 < t_2 < \cdots < t_{m+1}.
\]

Then,

\[
|G_i(d_1) - G_i(d_2)| = \left| \sum_{k=1}^{m} \int_{t_k}^{t_{k+1}} \sum_j \frac{\partial G_i}{\partial d_j}(d(t)) (d_j^2 - d_j^1) \, dt \right| \leq \max_j |d_j^2 - d_j^1| \eta = \eta \|d^1 - d^2\|_{l_\infty}.
\]

The last claim follows from the contraction mapping theorem (see Lucas and Stockey (1989), Theorem 3.2 on p. 50). 

Proof of Lemma 5.2. Let \( b_j = e^{d_j} \). By Lemma C.1,

\[
\sum_j \frac{\partial (G_C)_i}{\partial d_j} = \sum_{j \neq i} (H_i)^{-1} \frac{\partial H_i}{\partial b_j} b_j \leq \alpha,
\]

and the claim follows from Lemma C.3. 

Proof of Lemma 5.5. Pick a parameter \( \zeta \) and suppose that

\[
G_C(d, \zeta_1) \geq G_C(d, \zeta_2)
\]

for all \( d \) and, for any fixed \( d = (d_i) \), the expression

\[
\left( (1 + \alpha) \delta^{-1} \, U' \left( w_0 - (1 + \alpha) \sum_i \left( g_i (e^{d_i} \delta_i C^{-1}) - w_0 \right) \right) \right)^{-1}
\]
is larger for $\zeta_1$ than for $\zeta_2$. Pick a point $d_0 \in \Omega$. Then, since $G_C$ is monotone increasing in $d$, we get:

$$G^2_C(d_0, \zeta_1) = G_C(G_C(d_0, \zeta_1), \zeta_1) \geq G_C(G_C(d_0, \zeta_2), \zeta_1) \geq G_C(G_C(d_0, \zeta_2), \zeta_2) = G^2_C(d_0, \zeta_2). \quad (68)$$

Repeating the same argument, we get:

$$G^n_C(d_0, \zeta_1) \geq G^n_C(d_0, \zeta_2)$$

for any $n \in \mathbb{N}$. Sending $n \to \infty$ and using Lemma 5.2 and Lemma C.3, we get:

$$d^*(C, \zeta_1) \geq d^*(C, \zeta_2)$$

for any $C$. This immediately yields that $C^*(\zeta_1) \geq C^*(\zeta_2)$, and therefore

$$d^*(C^*(\zeta_1), \zeta_1) \geq d^*(C^*(\zeta_2), \zeta_1) \geq d^*(C^*(\zeta_2), \zeta_2)$$

and the claim follows.

**Lemma C.4** Suppose that an increase in a parameter $\zeta$ leads to an increase in the optimal $d^*$. Then, this also leads to more insurance.

**Proof of Lemma C.4.** An increase in $d_i$, $i = 1, \ldots, N$ is equivalent to a decrease in all coordinates of $a = (a_i) = (e^{-d_i})$. Consequently, the number of the coordinates of $a$ for which $a_i u'_i(w_{1i}) < U'(w_1)$ increases. This is precisely $\#\{\text{senior}\}$. Similarly, by definition, $Z_{\text{full coverage}} = Z_{J+1}$ is monotone decreasing in all $a_i$ (see (19)), and $Z_{\text{deductible}}$ is monotone increasing in all coordinates of $a$. Finally, the participation index is equal to 1 if $a_i u'_i(w_{1i}) < U'(w_1)$ and therefore stays equal to 1 if $a_i$ decreases.
Proof of Proposition 5.7. By the definition of FOSD dominance, an increase in the distribution of $X$ in the FOSD sense leads to a decrease of

$$E[u_i(w_{1i} - F_i(b, X))],$$

for all $i = 1, \ldots, N$ and, consequently, to a decrease in the right-hand side of (25) for any fixed $a$. Therefore, the solution $H_i$ to (25) also decreases in response to this change in the distribution of $X$. By Lemma 5.5, this leads to decrease of all coordinates of vector $b$. The claims follow now from Lemma C.4.

Similarly, an increase in $w_0$ and an increase in $\delta$ lead to an increase in the right-hand side of (26). This leads to an increase in $C$, and therefore, by Lemma 5.5, all coordinates of vector $b$ increase. ■

Proof of Proposition 5.8. Let $\gamma(c) \equiv -\frac{e U''(c)}{U'(c)}$. Differentiating the equation for $C$ with respect to $\alpha$, we get that it is monotone increasing in $\alpha$ if and only if

$$1 - \gamma(c_0) + w_0 A(c_0) < 0$$

and the claim follows. ■

Proof of Proposition 6.3. A direct calculation shows that, under the CARA assumption, the vector $b = (b_i)$ solves

$$b_i = \delta_i^{-1} C \left( e^{-A_i w_{0i}} + e^{-\delta_i - A_i w_{1i}} E[1 - e^{A_i F_i(X)}] \right), \; i = 1, \ldots, N. \quad (69)$$

Suppose that

$$\frac{\delta_i e^{-A_i w_{1i}}}{e^{-A_i w_{0i}}} > \frac{\delta_j e^{-A_j w_{1j}}}{e^{-A_j w_{0j}}}$$

(70)
for some insurers $i$ and $j$, but $\text{rank}(i) < \text{rank}(j)$. By definition, this means that

$$b_i e^{A_i w_{1i}} \leq b_j e^{A_j w_{1j}}. \quad (71)$$

We now claim that the inequality $\text{rank}(i) < \text{rank}(j)$ implies

$$A_i F_i \leq A_j F_j. \quad (72)$$

Indeed, for all tranches in which insurer $i$ participates, the slopes of $A_i F_i$ and $A_j F_j$ coincide by Proposition 3.4. Since $j$ has a higher rank, $A_i F_i(X) = 0$ for all $X$ for which $A_j F_j(X) = 0$. The claim (72) follows now by continuity of $F_i$ and $F_j$. Consequently,

$$E[1 - e^{A_i F_i(X)}] \geq E[1 - e^{A_j F_j(X)}],$$

and therefore (69) and (70) together yield

$$b_i e^{A_i w_{1i}} = \delta_i^{-1} C e^{A_i (w_{1i} - w_{0i})} + C E[1 - e^{A_i F_i(X)}] \geq \delta_j^{-1} C e^{A_j (w_{1j} - w_{0j})} + C E[1 - e^{A_j F_j(X)}] = b_j e^{A_j w_{1j}}, \quad (73)$$

which contradicts (71). The proof is complete. ■

**Proof of Proposition 6.2.** Since the insured is risk neutral, $C = \delta$. An increase in $w_{0i}$ leads to an decrease in all coordinates of the map $G_C$, and the claim follows from Lemmas 5.5 and C.4.

Now, consider the optimal allocation with $N + 1$ insurers and let $(\tilde{\mathbf{d}}, d_{N+1})$ be the corresponding vector, with the coordinate $d_{N+1}$ corresponding to the new, $(N + 1)$th, insurer. Let $\tilde{G}_C$ be the map of Lemma 5.2, corresponding to the case of $N + 1$ insurers. Further, let $\tilde{G}_C^{(N)}$ be the “submap” of $\tilde{G}_C$, consisting of the first $N$ coordinates of $\tilde{G}_C$. Finally, let $G_C$ be the map corresponding to the $N$ insurer case. Then, the vector $\tilde{\mathbf{d}}$
satisfies

\[ \tilde{d} = \tilde{G}^{(N)}(\tilde{d}, d_{N+1}) . \]

Similarly, the vector \( d \) corresponding to the \( N \) insurer case solves

\[ d = G_C(d) . \]

Now, let \( \tilde{d}_{N+1} = \min\{d_{N+1}, \log(u'_{N+1}(w_{1,N+1})) \} \). By definition, \( \tilde{d}_{N+1} \leq d_{N+1} \). Furthermore, such a small \( \tilde{d}_{N+1} \) will correspond to a large \( \tilde{a}_{N+1} \), satisfying

\[ \tilde{a}_{N+1} u'_{N+1}(w_{1,N+1}) \geq 1 = U'(w_1). \]

Therefore, by (55), such a \( \tilde{d}_{N+1} \) will not change the tranche structure\(^{30} \) and, consequently,

\[ \tilde{G}^{(N)}(x, \tilde{d}_{N+1}) = G_C(x) \]

for any \( x \). Thus, we have:

\[ \tilde{d} = \tilde{G}^{(N)}(\tilde{d}, d_{N+1}) \geq \tilde{G}^{(N)}(\tilde{d}, \tilde{d}_{N+1}) = G_C(\tilde{d}) . \]

Applying \( G_C \) to this inequality repeatedly and using monotonicity of \( G_C \), we get:

\[ \tilde{d} \geq G_C(\tilde{d}) \geq G_C(G_C(\tilde{d})) \geq \cdots \geq (G_C)^n(\tilde{d}) . \]

Sending \( n \to \infty \), we get, by Lemma 5.2, that

\[ \tilde{d} \geq d . \]

\(^{30}\)Since the insured is risk neutral, tranches with indices smaller than \( J + 1 \) are not sold.
The required result follows now from Lemma C.4.

Let \( x = \delta_i^{-1} \). Consider the function:

\[
f(\delta) = x u'_i \left( v_i (u_i(w_{0i}) - x^{-1} E[u_i(w_{1i}) - \delta_i(w_{1i})]) \right)
\]

for any fixed \( 0 \leq \phi(X) \leq X \). Then,

\[
\frac{\partial}{\partial x} f(x) = u'_i \left( v_i (u_i(w_{0i}) - x^{-1} E[u_i(w_{1i}) - \phi(X)) - u_i(w_{1i})]) \right)
+ x^{-1} E[u_i(w_{1i}) - u_i(w_{1i})] u''_i \left( v_i (u_i(w_{0i}) - x^{-1} E[u_i(w_{1i}) - \phi(X)) - u_i(w_{1i})]) \right)
\times v'_i (u_i(w_{0i}) - x^{-1} E[u_i(w_{1i}) - \phi(X)) - u_i(w_{1i})]) . \tag{74}
\]

Denote \( z = v_i (u_i(w_{0i}) - x^{-1} E[u_i(w_{1i}) - \phi(X)) - u_i(w_{1i})]) \). Then, using the identity \( v'_i(x) = (u'_i(v_i(x)))^{-1} \), we get:

\[
\frac{\partial}{\partial x} f(x) = u'_i(z) + x^{-1} E[u_i(w_{1i}) - \phi(X)) - u_i(w_{1i})] \frac{u''_i(z)}{u'_i(z)}
= u'_i(z) - (u_i(w_{0i}) - u_i(z)) \frac{1}{R_i(z)} . \tag{75}
\]

The right-hand side is nonnegative. Hence, \( f(\delta) \) is monotone increasing. Therefore, by Lemma 5.5, the coordinates of the vector \( d'(C) \) are decreasing in \( \delta_i \), and the claim follows since the risk neutrality of the insured implies \( C = \delta \). \( \blacksquare \)

**Proof of Lemma 7.3.** The first order condition for \( s_i \) is

\[
e^{-A_i(\bar{w}_{0i}+P_i-s_i)} = \delta_i (1 + r_i) E[e^{-A_i(\bar{w}_{1i}+s_i(1+r_i)-F_i(X))}] \tag{76}
\]

if the borrowing constraint \( s_i \geq -K_i \) is satisfied and

\[
e^{-A_i(\bar{w}_{0i}+P_i+K_i)} > \delta_i (1 + r_i) E[e^{-A_i(\bar{w}_{1i}-K_i(1+r_i)-F_i(X))}]
\]
otherwise. In the case when the constraint is not binding, we get

\[ e^{A_i(2+r_i)s_i} = e^{A_i(\tilde{w}_0+P_i-\tilde{w}_1)} \delta_i(1+r_i)E[e^{A_iF_i(X)}] = e^{A_iP_i}M_i(1+r_i)E[e^{A_iF_i(X)}] \] (77)

The indifference price satisfies

\[ e^{-A_i(\tilde{w}_0+P_i-s_i)} + \delta_iE[e^{-A_i(\tilde{w}_1+s_i(1+r_i)-F_i(X))}] = e^{-A_i\tilde{w}_0i} + \delta_i e^{-A_i\tilde{w}_1i} \] (78)

Substituting, we get

\[ \delta_i(1+r_i)E[e^{-A_i(\tilde{w}_1+s_i(1+r_i)-F_i(X))}] + \delta_iE[e^{-A_i(\tilde{w}_1+s_i(1+r_i)-F_i(X))}] = e^{-A_i\tilde{w}_0i} + \delta_i e^{-A_i\tilde{w}_1i} \]

that is

\[ e^{s_iA_i(1+r_i)} = E[e^{A_iF_i(X)}] \frac{(2+r_i)M_i}{1+M_i} \iff s_i = \frac{1}{A_i(1+r_i)} \left( \log E[e^{A_iF_i(X)}] + \log \frac{(2+r_i)M_i}{1+M_i} \right) \]

Substituting this expression into (77), we get

\[ P_i = \frac{1}{A_i(1+r_i)} \left( \log E[e^{A_iF_i(X)}] + (2+r_i)\log \frac{(2+r_i)M_i}{1+M_i} - (1+r_i)\log(M_i(1+r_i)) \right) \]

Consequently,

\[ P_i - s_i = \frac{1}{A_i} \left( \log \frac{(2+r_i)}{(1+r_i)(1+M_i)} \right) \] (79)

and the claim follows.

References


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